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COMPLEX INDUCTANCE AND ITS COMPUTER MODELLING

Daniel Mayer — Bohuš Ulrych *

The paper introduces the concept of the complex inductance as a parameter of a coil loaded by sinusoidal current and placed near an electrically well conductive body in which the magnetic field of the coil induces eddy currents. Studied will be only axi-symmetric cylindrical coils and the paper shows how easily their complex inductances can be determined by means of professional programs for electromagnetic field analysis. The suggested methodology is illustrated on computation of the inductance of a ring coil placed near a conductive slab. Investigated is also the dependence of the complex inductance on frequency.

K e y w o r d s: coil inductance, eddy currents, quasistationary electromagnetic field

1 INTRODUCTION

Methods of computation of the self- as well as mutual inductances of coils and lines of various geometries can be found in a lot of references. Analytical solutions, for example, are dealt with in [1–4], while numerical solutions in [5–8]. The paper is aimed at the influence of near massive electrically conductive bodies on the coil inductance. It is well known that the presence of a near permeable body increases the coil inductance while the presence of a conductive non-magnetic body decreases it the more the higher is the frequency of the field current. The physical backgrounds of these changes are obvious. Ferromagnetic parts lower the reluctance of paths of the magnetic flux linked with the coil, which leads to an increase of the magnetic flux linked with particular turns of the coil and, therefore, to a growth of its inductance. In case the coil carries alternating current, its magnetic field in a near electrically conductive body induces eddy currents that (by the Lenz rule) reduce the flux linked with the coil and lower its inductance.

Building a mathematical model for an electric circuit containing a coil with near metal bodies requires investigation of the magnetic field in the neighbourhood of the coil, which is not quite easy. In order to simplify the circuit equations we can characterize the coil by a parameter — let us call it a *complex self-inductance* — with which we will work in the same manner as with the usual self-inductance defined in non-conductive and non-magnetic media. Difficulty of solution of the problem will be then transferred to the question of investigation of this complex self-inductance, whose determination assumes computation of the magnetic field of the coil. Such a task may easily be processed by means of any suitable professional code for the magnetic field analysis.

These considerations may analogously be extended to a couple of coils with consequent introduction of the concept of a *complex mutual inductance*. This, however, will not be dealt with in this paper.

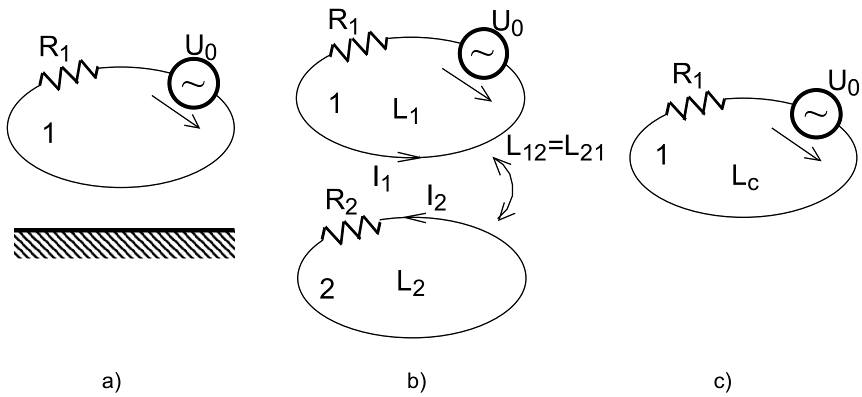

Fig. 1. a) loop above the conductive half-space, b) two inductively linked loops, c) equivalent loop in the free space.

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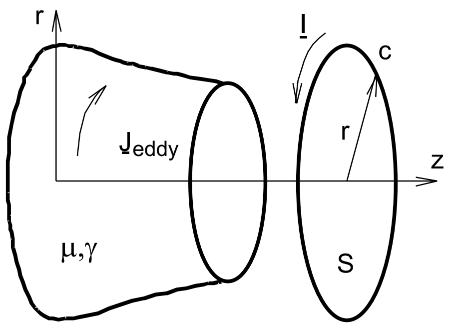

Fig. 2. A loop near an electrically conductive body.

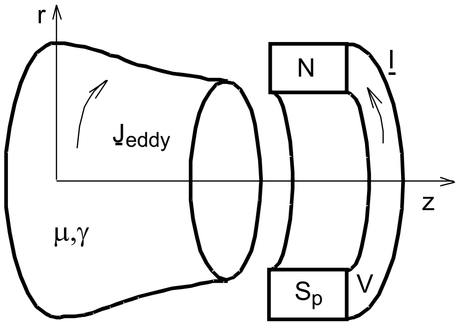

Fig. 3. A coil with N turns near an electrically conductive body.

2 MOTIVATION FOR INTRODUCTION OF THE COMPLEX SELF–INDUCTANCE

A thin loop **1** carrying alternating current is placed near an electrically conductive plane (Fig. 1a). The influence of this body may be replaced by another short-circuited loop **2** (Fig. 1b). Loop **1** will be characterized by only one complex parameter $\underline{L}_c$, see Fig. 1c. The system in Fig. 1b is described by matrix equations

$$\begin{bmatrix} R_1 + j\omega L_1 & j\omega L_{12} \\ j\omega L_{21} & R_2 + j\omega L_2 \end{bmatrix} \cdot \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix} = \begin{bmatrix} \underline{U}_0 \\ 0 \end{bmatrix} . \tag{1}$$

Elimination of current $\underline{I}_2$ provides

$$(R_1 + j\omega \underline{L}_c)\underline{I}_1 = \underline{U}_0 \tag{2}$$

where

$$\underline{L}_c = L_1 - \frac{\omega L_{12} L_{21}}{R_2^2 + \omega^2 L_2^2} (\omega L_2 + jR_2) . \tag{3}$$

Quantity $\underline{L}_c$ will be denoted as the *complex self-inductance*. The voltage on it can be expressed as

$$\underline{U}_c = j\omega \underline{L}_c \underline{I}_1 \tag{4}$$

This consideration (but not the corresponding equations) holds even in the case that the induced currents are not concentrated in loop **2** but distributed in a near conductive body. The complex self-inductance depends on angular frequency ω of the field current ($\underline{L}_c = \underline{L}_c(\omega)$); with higher frequency the module of $\underline{L}_c$ decreases and argument of $\underline{L}_c$ reaches negative values.

3 MATHEMATICAL FORMULATION OF THE PROBLEM

3.1 Starting assumptions

Solution to the task starts from the following assumptions:

- The coil as well as the conductive bodies are axisymmetric (cylindrical or ring coils), see Figs. 2 and 3. The magnetic field is, therefore, of 2D character.
- The coil has N turns and carries sinusoidal current.
- The coil and its neighbourhood represent a magnetically linear and hysteresis-free medium ($\mu = \text{const}$) that is also electrically linear ($\gamma = \text{const}$).
- Only one coil is considered. Extension of the methodology to two inductively linked coils would be, however, relatively easy.

The last assumption is a good professional program that solves the harmonically variable electromagnetic field in a linear medium described by the Helmholtz equation (see [8] or [9]) for phasor $\underline{\boldsymbol{A}}$

$$\Delta \underline{\boldsymbol{A}} + \underline{k}^2 \underline{\boldsymbol{A}} = -\mu_0 \underline{\boldsymbol{J}} \tag{5}$$

where $\underline{k}^2 = -j\omega\gamma\mu$, $\underline{\boldsymbol{A}} = \boldsymbol{\alpha}_0 \underline{A}_\alpha(r, z)$ and $\underline{\boldsymbol{J}} = \boldsymbol{\alpha}_0 \underline{J}_\alpha$ is the phasor of the current density in the field coil. The magnetic flux density phasor may be determined from relation $\underline{\boldsymbol{B}} = \operatorname{rot} \underline{\boldsymbol{A}}$ that provides two component equations

$$\underline{B}_r = -\frac{\partial \underline{A}_\alpha}{\partial z} , \quad \underline{B}_z = \frac{1}{r} \frac{\partial}{\partial r} (r \underline{A}_\alpha) . \tag{6}$$

Information about the available programs for the magnetic field analysis is given in [10].

3.2 Calculation of $\underline{L}_c$ from the static definition

Let an electrically conductive body be placed in the neighbourhood of a thin loop, forming circle c with radius r. Its complex self-inductance may be expressed by means of the static definition extended on phasors:

$$\underline{L}_c = \frac{\underline{\phi}}{\underline{I}_1} = \frac{\underline{\phi} \underline{I}_1^*}{I_1^2} , \tag{7}$$

where $\underline{\phi}$ is the phasor of magnetic flux $\phi(t)$ linked with loop c, $\underline{I}_1$ is the phasor of the current in the coil and $\underline{I}_1^*$ its complex conjugate. Phasor $\underline{\phi}$ may be expressed as

$$\underline{\phi} = \underline{\phi}_1 + \underline{\phi}_{12} , \tag{8}$$

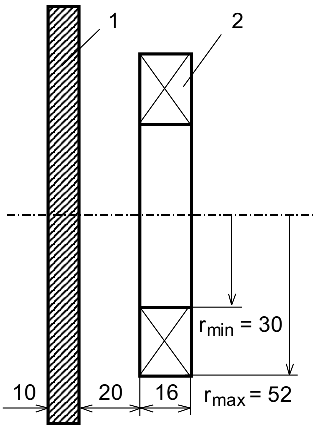

Fig. 4. To the example: a ring coil above an electrically conductive slab.

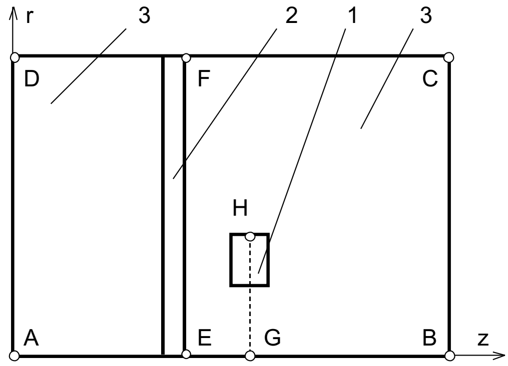

Fig. 5. The definition area of the solved boundary problem.

where $\underline{\phi}_1$ is the phasor of the magnetic flux induced by current $\underline{I}_1$ and $\underline{\phi}_{12}$ the phasor of the magnetic flux induced by eddy currents in the near conductive body. Phasors $\underline{\phi}_1$ and $\underline{I}_1$ are in phase, but phasor $\underline{\phi}_{12}$ is shifted and gives a complex character to the inductance $\underline{L}_c$ defined by (7).

Using the Stokes theorem, the total magnetic flux may be expressed as

$$\underline{\phi} = \int_S \underline{\boldsymbol{B}}\,\mathrm{d}\boldsymbol{S} = \oint_c \underline{\boldsymbol{A}}\mathrm{d}\boldsymbol{l} = 2\pi \underline{A}_\alpha\, r \tag{9}$$

where S is the area of the circle bounded by curve c and

$$\underline{\boldsymbol{B}} = \mathbf{r}_0 \underline{B}_\mathrm{r}(r,z) + \mathbf{z}_0 \underline{B}_\mathrm{z}(r,z)\,,$$
$$\mathrm{d}\boldsymbol{S} = \mathbf{z}_0 \mathrm{d}S\,, \quad \mathrm{d}\boldsymbol{l} = \boldsymbol{\alpha}_0 \mathrm{d}l\,, \quad \underline{\boldsymbol{A}} = \boldsymbol{\alpha}_0 \underline{A}_\mathrm{z}\,.$$

Let us consider a circular coil formed by N thin turns with radii r_i, $i = 1,\dots,N$. The coil has inner and outer radii R_1 and R_2 and height h. Its complex self-inductance is given by (7), where

$$\underline{\phi} = \sum_{i=1}^{N} \underline{\phi}_i = 2\pi \sum_{i=1}^{N} \underline{A}_{\alpha i}\, r_i\,. \tag{10}$$

Vector potential $\underline{A}_{\alpha i}$ is induced in the i-th thin turn by current $\underline{I}_1$ in the coil and eddy currents in the near conductive body. Integration of the vector potential over volume V_c provides

$$\int_{V_\mathrm{c}} \underline{A}_\alpha \mathrm{d}V = \int_{r=R_1}^{R_2} \int_{\alpha=0}^{2\pi} \int_{z=0}^{h} \underline{A}_\alpha\, r\mathrm{d}r\mathrm{d}\alpha\mathrm{d}z = 2\pi \int_{S_\mathrm{p}} \underline{A}_\alpha\, r\mathrm{d}S$$
$$\doteq 2\pi \sum_{i=1}^{N} \underline{A}_{\alpha i}\, r_i\, \Delta S_\mathrm{p} = 2\pi \Delta S_\mathrm{p} \sum_{i=1}^{N} \underline{A}_{\alpha i}\, r_i\,, \tag{11}$$

where ΔS_p is the cross-section of one turn. Denoting S_p the active section of the whole coil, then

$$S_\mathrm{p} = N\Delta S_\mathrm{p} \quad \text{and} \quad \underline{J}_\alpha = \frac{\underline{I}_1}{\Delta S_\mathrm{p}} = \frac{N\underline{I}_1}{S_\mathrm{p}} \tag{12}$$

so that, in accordance with (7), the complex self-inductance is given as

$$\underline{L}_\mathrm{c} = \frac{N}{\underline{I}_1\, S_\mathrm{p}} \int_{V_\mathrm{c}} \underline{A}_\alpha \mathrm{d}V\,. \tag{13}$$

Computation of integral $\int_{V_\mathrm{c}} \underline{A}_\alpha \mathrm{d}V$ is usually implemented in the postprocessor menu of most professional programs see *eg* [11]). Definition (7) has a sense in the case that the phasor of the voltage induced in the coil is expressed by common relation (4). Indeed, for definition (7) we get equation (4):

$$\underline{U}_\mathrm{c} = \mathrm{j}\omega \sum_{i=1}^{N} \underline{\phi}_i = \mathrm{j}\omega \underline{L}_\mathrm{c} \underline{I}_1\,.$$

3.3 Calculation of $\underline{L}_\mathrm{c}$ from the electromagnetic field energy

Two thin loops (Fig. 1b) carry currents $i_1 = I_{m1}\sin(\omega t + \varphi_1)$ and $i_2 = I_{m2}\sin(\omega t + \varphi_2)$. The magnetic energy of the system is

$$W_\mathrm{m}(t) = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + L_{12} i_1 i_2 = \frac{1}{2}L_1 I_{\mathrm{m}1}^2 \sin^2(\omega t + \varphi_1)$$
$$+ \frac{1}{2}L_2 I_{\mathrm{m}2}^2 \sin^2(\omega t + \varphi_2) + \frac{1}{2}L_{12} I_{\mathrm{m}1} I_{\mathrm{m}2}\big[\cos(\varphi_1 - \varphi_2)$$
$$- \cos(2\omega t + \varphi_1 + \varphi_2)\big]. \tag{14}$$

The average value of this magnetic energy may be expressed by means of phasors

$$\overline{W}_\mathrm{m} = \frac{1}{2}\operatorname{Re}\underline{W}_\mathrm{m}\,, \tag{15}$$

where $\underline{W}_\mathrm{m}$ is the complex magnetic energy. As

$$\underline{W}_\mathrm{m} = \frac{1}{2}L_1 \underline{I}_1 \underline{I}_1^* + \frac{1}{2}L_2 \underline{I}_2 \underline{I}_2^* + L_{12} \underline{I}_1 \underline{I}_2\,, \tag{16}$$

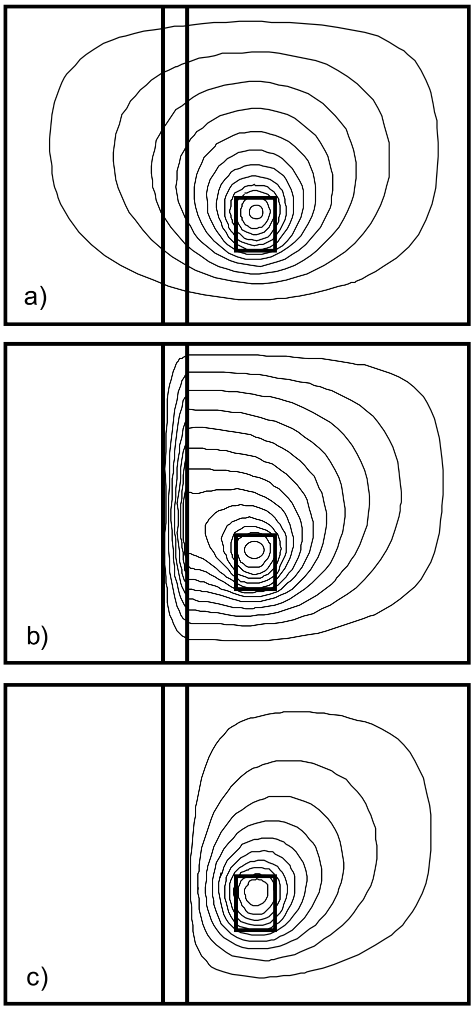

Fig. 6. Force lines of the magnetic field: a) a copper slab, $f = 1$ Hz, b) an iron slab, $f = 1$ Hz, c) a copper slab, $f = 1000$ Hz.

it is obvious that the complex character of $\underline{W}_{\rm m}$ is caused by the third term expressing the linkage energy.

We use the relation for magnetic field energy

$$W_{\rm m} = \frac{1}{2}\int_{V_c} \boldsymbol{A}\boldsymbol{J}{\rm d}V\,,$$

where integration is performed over region V_c containing the coil and near electrically conductive body. In case vectors $\boldsymbol{A}(t)$ and $\boldsymbol{J}(t)$ vary sinusoidally in time, the average value $\overline{W}_{\rm m}$ of the magnetic energy may be expressed by means of complex energy $\underline{W}_{\rm m}$ according to (15), where the complex energy of coil **1** (region V_1) and near conductive body **2** (region V_2) is given by formula

$$\underline{W}_{\rm m} = \underline{W}_{\rm m1} + \underline{W}_{\rm m2} = \frac{1}{2}\int_{V_1} \underline{\boldsymbol{A}}_1 \underline{\boldsymbol{J}}_1^* {\rm d}V + \frac{1}{2}\int_{V_2} \underline{\boldsymbol{A}}_2\, \underline{\boldsymbol{J}}_2^* {\rm d}V \quad (17)$$

where both phasors of vector potentials $\underline{\boldsymbol{A}}_1$ and $\underline{\boldsymbol{A}}_2$ (complex amplitudes) consist of two components:

$$\underline{\boldsymbol{A}}_1 = \underline{\boldsymbol{A}}_{11} + \underline{\boldsymbol{A}}_{12}\,, \quad \underline{\boldsymbol{A}}_2 = \underline{\boldsymbol{A}}_{21} + \underline{\boldsymbol{A}}_{22}\,. \quad (18)$$

Here potential $\underline{\boldsymbol{A}}_{11}$ in region V_1 is induced by current $\underline{I}_1$, potential $\underline{\boldsymbol{A}}_{22}$ in region V_2 is induced by the eddy currents, potential $\underline{\boldsymbol{A}}_{12}$ in region V_1 is also induced by the eddy currents and finally potential $\underline{\boldsymbol{A}}_{21}$ in region V_2 is induced by current $\underline{I}_1$.

The complex self-inductance of the coil may be defined from extended energetic definition

$$\underline{L}_{\rm c} = \frac{2\underline{W}_{\rm m1}}{\underline{I}_1 \underline{I}_1^*}\,. \quad (19)$$

After substituting for $\underline{W}_{m1}$ from (17) and using

$$\underline{\boldsymbol{J}}_1^* = \boldsymbol{\alpha}_0 \underline{J} = \underline{\rm const}\,, \quad (20)$$

we easily get again eq. (13).

It is not complicated to prove that phasor of the voltage on complex self-inductance $\underline{L}_{\rm c}$ is also expressed by (4). This may be performed by means of equations (19), (11), (20) and (12).

4 ILLUSTRATIVE EXAMPLE — A RING COIL ABOVE A CONDUCTIVE SLAB

The suggested algorithm of computation of the complex self-inductance is illustrated on the following example: a ring coil is placed above a conductive slab (Fig. 4). Number of turns of the coil $N = 107$ and material of the slab is a) copper ($\gamma = 5.7\times10^7$ S/m), b) aluminium ($\gamma = 3.8\times10^7$ S/m), c) steel ($\gamma = 3\times10^6$ S/m, $\mu_r = 1000$), d) non-conductive and non-magnetic material. Frequency $f \in \langle 0, 10^4\rangle$ Hz. Current $I = 20$ A and its density determined from the sizes of the coil $J = 6.11\times10^6\,{\rm Am}^{-2}$. Cross-section $S_{\rm p}$ of the coil is $3.52\times10^{-4}\,{\rm m}^2$.

4.1 Mathematical model

The continuous mathematical model is defined as a boundary problem on domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ in cylindrical co-ordinates r, α, z (Fig. 5). Distribution of the phasor of vector-potential $\underline{\boldsymbol{A}} = \boldsymbol{\alpha}_0 A_\alpha$ in particular sub-regions follows from equations

$$\begin{aligned} &\Omega_1\ \text{(coil)}:\ \Delta\underline{A}_\alpha + \underline{k}^2\underline{A}_\alpha = -\mu_0\underline{J}\,,\ \text{where}\ \underline{k}^2 = -{\rm j}\omega\gamma\mu_0\,,\\ &\Omega_2\ \text{(slab)}:\ \Delta\underline{A}_\alpha + \underline{k}^2\underline{A}_\alpha = 0\,, \qquad (21)\\ &\Omega_{3,4}\text{(air)}:\ \Delta\underline{A}_\alpha = 0\,, \end{aligned}$$

where $\underline{J}$ is the phasor of the chosen current density. As the medium is linear, its selection can be quite arbitrary. The boundary conditions read:

AB: $\underline{A}_\alpha = 0$ (antisymmetry),

BCDA: $\underline{A}_\alpha = 0$ (force line, continuity).

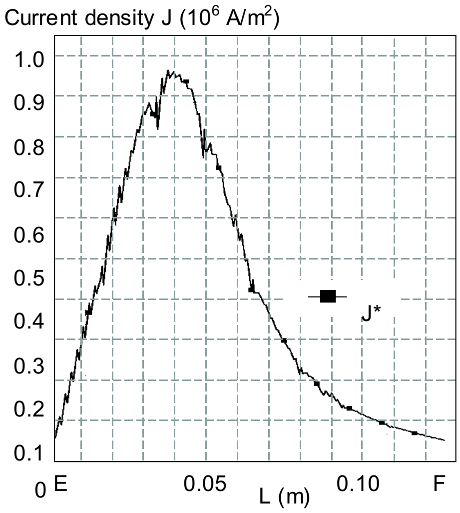

Fig. 7. Distribution of the amplitude of the eddy current density on surface EF (see Fig. 5), a copper slab, $I = 1$ A, $f = 10000$ Hz.

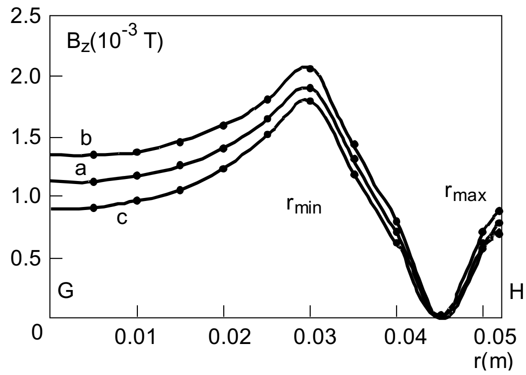

Fig. 8. Distribution of the amplitude of the magnetic flux density in plane GH (see Fig. 5), $I = 1$ A, a) a copper slab, $f = 1$ Hz, b) an iron slab, $f = 1$ Hz, c) a copper slab, $f = 10000$ Hz.

4.2 Results and their discussion

The mathematical model was solved by the FEM-based SW product QuickField [11]. Distribution of the electromagnetic field in the neighbourhood of the coil is shown in Fig. 6. Figure 7 depicts the distribution of the eddy currents on the surface of the slab and Fig. 8 an analogous magnetic flux density distribution in the plane parallel with the surface of the slab at the distance of 18 mm.

The inductance of the coil itself, without presence of any conductive slab, was calculated by means of the approximate algorithm described in [3] and its value is 9.575×10^{-4} H.

The dependence of the complex inductance on angular frequency ω (it is $\underline{L}_c = |\underline{L}_c(\omega)| \underline{/ \varphi(\omega)}$) is shown in Figs. 9 and 10. The module of $\underline{L}_c$ varies with frequency in accordance with expectations. The self-inductance of the coil without presence of a conductive slab does not depend on frequency, while the presence of such a slab leads to a monotonic decrease of the inductance with increasing frequency. The dependence on frequency f is negligible in the domain of low frequencies. The presence of an iron slab leads to a growth of the inductance.

5 CONCLUSION

The complex self-inductance was introduced as a characteristic parameter of an AC current carrying coil placed in electrically conductive media. This parameter may relatively easily be determined by means of existing professional codes for the analysis of electromagnetic fields. The methodology is fully usable even in the case that the coil is wound from massive conductors with non-negligible skin effect. Despite this fact, there exists an open question whether this conception may be generalized even for periodical, but non-harmonic waveforms of currents or for general waveforms of currents occurring in the analysis of transients. Another question is associated with using the methodology for other shapes (non-cylindrical) of the field coils.

APPENDIX

The complex inductance represents a useful tool for determining the losses due to eddy currents induced in a

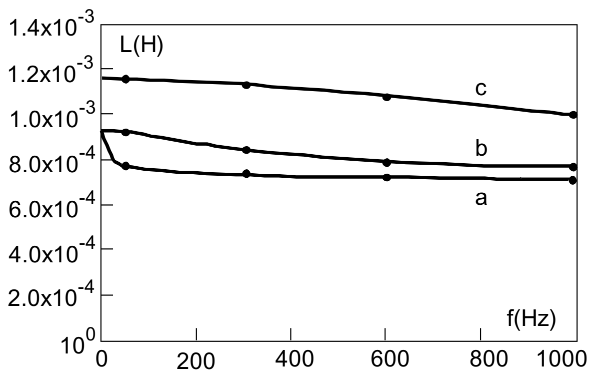

Fig. 9. Dependence of the module of $\underline{L}_c$ on frequency for a) a copper slab b) aluminium slab, c) an iron slab.

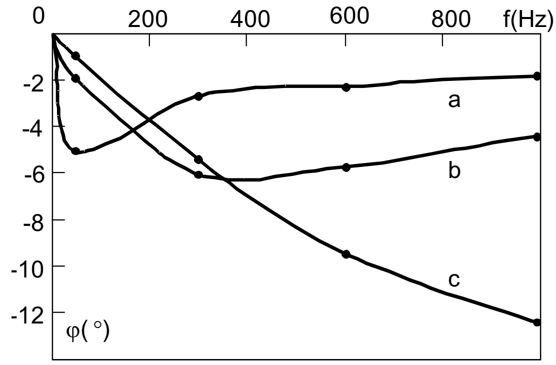

Fig. 10. Dependence of the argument of $\underline{L}_c$ on frequency for a) a copper slab, b) an aluminium slab, c) an iron slab

near conductive body. Let the coil carry sinusoidal current

$$i(t) = \sqrt{2}\, I \sin \omega t$$

which phasor (complex RMS value) $\underline{I} = I$. The phasor of voltage on the coil is

$$\underline{U} = \mathrm{j}\omega(L_\mathrm{r} + \mathrm{j}L_\mathrm{i})\underline{I}\,.$$

The complex power is

$$\underline{S} = \underline{U}\,\underline{I}^* = P + \mathrm{j}Q = \mathrm{j}\omega(L_\mathrm{r} + \mathrm{j}L_i)I^2\,.$$

The losses due to eddy currents are $P = -\omega\, L_\mathrm{i} I^2$ and as $L_\mathrm{i} < 0$, then $P > 0$.

Acknowledgement

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