Journal of ELECTRICAL ENGINEERING, VOL. 54, NO. 1-2, 2003, 22–29

ON THE REDUCTION AND CONTROL FOR A CLASS OF NONHOLONOMIC UNDERACTUATED SYSTEMS

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This paper addresses the reduction and control of underactuated mechanical systems with first order nonholonomic constraints and kinetic symmetry. It is shown that this class of system can be reduced to a well-defined reduced order Lagrangian system in Cascade with the constraint equation. Depending on whether the number of control inputs and constraint equations is less or equal to the number of configuration variables, the reduced Lagrangian system is fully actuated or underactuated, respectively. The problem of quasi-exponential stabilization for this class of nonholonomic underactuated mechanical systems is also addressed is this paper using discontinuous control of the chained-form systems. The enclosed simulation made for a car-like vehicle model shows the effectiveness of the method.

K e y w o r d s: Underactuated, Nonholonomic, Stabilization, Mobile Robots

1 INTRODUCTION

A variety of real life control systems including car-type vehicles, mobile robots, surface vessels and underwater vehicles are examples of underactuated mechanical systems with nonholonomic velocity constraints. All these examples of nonholonomic systems possess symmetry properties, *ie* their kinetic energy, potential energy, or both are independent configuration variables.

There is a significant interest in controlling the motion of nonholonomic mechanical systems. This challenging problem becomes even more difficult when the system is underactuated, that is possesses fewer actuators than configuration degrees of freedom. A particularly interesting class of underactuated systems consists of those systems for which only a proper subspace of the space of generalized velocities is accessible at each configuration. This situation occurs, for instance, when a system is subject to nonholonomic (nonintegrable) constraints on its kinematics, and can also arise in systems with symmetric dynamics.

We focus our study on the reduction of underactuated systems with symmetry and nonholonomic velocity constraints. Our main result on reduction of nonholonomic underactuated systems with symmetry is that under certain assumptions, the reduced system can be represented as a cascade of the constraint equation and a lower order underactuated (or fully actuated) Lagrangian system. The presented reduction method provides a systematic way for calculation of the reduced system of the system. Furthermore, we prove that the kinetic model of some typical underactuated nonholonomic systems is diffeomorphic. This implies that it can be transformed into a first order chained-type nonholonomic system using an explicit change of coordinates.

Based on Brockett's result (1983), the class of mechanical systems with first order nonholonomic constraints can not be stabilized to the origin using a smooth static state feedback. Hence, a considerable effort has been expended in order to find continuous time varying control laws (Bomet, 1992, M'Closkey and Murray, 1993), discontinuous ones (Bloch and Crough, 1995) and middle strategies (discontinuous and time varying) (Murray, Sodarlen and Egeland, 1993). Moreover, as pointed in [10], any C^1 periodic state feedback control law is unable to exponentially stabilize the closed loop system. The oscillatory behavior shown by many controlled nonholonomic systems (Neimark and Fuvaev, 1972) is not intrinsic to the system and is not even necessary to move from an initial configuration to the desired final one. Here we address the problem of asymptotic convergence for a class of nonholonomic control systems via discontinuous control using a change of coordinates called σ process (Astolfi, 1996). In this paper we use this approach for the quasi-exponential stabilization of the reduced system. It will be shown through some examples that systems whose reduced dynamics can be reduced to the chained-form of nonholonomic systems can achieve a quasi-exponential stability behavior via the discontinuous feedback control.

2 UNDERACTUATED MECHANICAL SYSTEMS

A controlled mechanical system with configuration vector $q \in Q$ and Lagrangian $L(q, \dot{q})$ satisfying the Euler-

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ISSN 1335-3632

Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = F(q)u \tag{1}$$

is called underactuated mechanical system if $m = \operatorname{rank} F(q) < n = \dim(Q)$. In other words, underactuated systems are mechanical systems that have fewer actuators than configuration variables. This restriction of the control authority does not allow exact feedback linearization of underactuated systems. For the special case that $F(q) = (0, I_m)^\top$ the first $(n-m)$ equations in (1) can be expressed as a second order dynamical system

$$\varphi(q, \dot q \ddot q) = 0\,. \tag{2}$$

For this case, we refer to the first $(n-m)$ equations of (1) as the unactuated subsystem and to the last m equations as the actuated subsystem. Spong (1996) has shown that the actuated subsystem of a general underactuated system with $F(q) = (0I_m)^\top$ can be linearized using an invertible change of control. This procedure is called partial feedback linearization. However, after partial linearization, the unactuated subsystem of (1) still remains as a nonlinear system that is coupled with the linearized actuated system through both the new control input and other nonlinear terms, which highly complicates the control design for the underactuated system.

3 NONHOLONOMIC MECHANICAL SYSTEMS

A Lagrangian mechanical system with $m < n$ velocity constraints

$$W^\top(q)\dot q = 0\,, \quad W \in \Re^{n\times n} \tag{3}$$

that are non-integrable, *ie* there is no $h(t)\colon \dot h = W^\top(q)\dot q$, is called a mechanical system with first order nonholonomic constraints. The control of a nonholonomic mechanical system with velocity constraints has been extensively studied in the recent years by many researchers (Ostrowski and Budrick, 1998). Following the formulation in (Bloch and Reyhanoglu, 1992 and Ostrowski and Budrick, 1998) and based on (Neimark and Fuvaev, 1972), the dynamics of nonholonomic systems with velocity constraints can be described by

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = W^\top(q)\lambda) + F(q)u\\ W(q)\dot q = 0\end{gathered} \tag{4}$$

where $\lambda \in \Re^m$ is the vector of Lagrange multipliers.

4 SYMMETRY IN MECHANICS

Formally, the symmetry in mechanics is defined as the invariance of the Lagrangian of a system under the action of a left (or right) invariant Lie group (Abraham and Mardsen, 1978). Almost all real physical systems possess certain symmetry properties. For example, the Lagrangian of a helicopter or a car is independent of their position. This gives rise to symmetries with respect to the translation. Here we only consider the symmetry of Lagrangian systems in an Euclidean space $Q = \Re^n$. We say that the Lagrangian $L(q, \dot q)$ is symmetric with respect to the configuration variable q_i iff

$$\frac{\partial L}{\partial q_i} = 0\,, \quad i = 1, \dots, n\,. \tag{5}$$

Here, we use a different notion of symmetry called kinetic symmetry that is rather similar to the classical definition of symmetry. By kinetic symmetry we mean that the Kinetic energy of the system is invariant with respect to q_i, *ie*

$$\frac{\partial K}{\partial q_i} = 0\,, \quad i = 1, \dots, n\,. \tag{6}$$

5 REDUCTION OF NONHOLONOMIC SYSTEMS WITH SYMMETRY

We address here the reduction of underactuated mechanical systems with nonholonomic firstorder constraints and symmetry. We precisely characterize a board class of underactuated nonholonomic systems of interest by giving three assumptions.

Consider a Lagrangian system with an n-dimensional configuration vector q, force matrix $F(q)$, and m nonholonomic first-order constraints as the following

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = W^\top(q)\lambda + F(q)u\\ W(q)\dot q = 0\end{gathered} \tag{7}$$

where $\lambda \in \Re^m$ is the vector of Lagrange multipliers, $u \in \Re^l$ is the control input, $l = \operatorname{rank}\big(F(q)\big)$ and $m + l \le n$. Due to the fact that $m \ge 1$ and $l \le (n-1)$, the nonholonomic system in (7) is an underactuated system. The term $W\top(q)\lambda$ represents the effect of constraint forces. This is based on the principle of virtual work, which states that the constraint forces do not work on motions allowed by the constraints. W is an $m \times n$ matrix.

Remark 1. The last condition that $m + l \le n$ is one of the main conditions that distinguishes this work from the result of Bloch et al. in [4] on the reduction of Caplygin control systems. In that work it is assumed that $m + l \ge n$.

Assumption 1. Assume $W(q)$ has a full row rank. Then q can be partitioned as (q_1, q_2) such that $W(q) = \big(W_1(q), W_2(q)\big)$ where $W_1(q)$ is an invertible matrix. Therefore the constraint equation in (1) can be rewritten as

$$W_1(q)\dot q_1 + W_2(q)\dot q_2 = 0 \tag{8}$$

ASSUMPTION 2. Assume $M(q)$, $F(q)$ and $W(q)$ are all independent of q_1.

ASSUMPTION 3. Assume the potential energy is in the form

$$V(q) = K_v^\top q_1 U(q_2) \tag{9}$$

where $k \in \Re^{n-m}$ is a constant vector.

Remark 2. Under Assumption 3 with $K_v^\top = 0$, the notions of kinetic symmetry and classical symmetry coincide. Therefore by saying "symmetry", we refer to both of these notions.

Remark 3. Broad classes of mechanical systems with nonholonomic velocity constraints including mobile robots, car type vehicles, surface vessels, rolling disks, Caplygin control systems and the snakeboard system satisfy Assumptions 1–3. Moreover, all the aforementioned examples of nonholonomic systems are underactuated.

Under the above assumptions, the dynamics of the underactuated nonholonomic system in (7) can be expressed as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} &= W^\top(q_2)\lambda + F(q_2)u \\ W_1(q)\dot q_1 + W_2(q)\dot q_2 &= 0\,. \end{aligned} \tag{10}$$

Nonholonomic systems in the form (4) with $F(q) \equiv 0$ are called Caplygin systems, which were introduced by Bloch and Reyhanoglu [4].

To eliminate λ from (10), one can multiply both sides of the forced Euler-Lagrange equation in (4) by matrix $A(q)$ that annihilates $W^\top(q)$, *ie* $A(q)W^\top(q) = 0$. For doing so, let us define

$$\omega_{12}(q_2) = -W_1^{-1}(q_2)W_2(q_2) \tag{11}$$

then

$$\dot q = D(q_2)\dot q_2 \tag{12}$$

where

$$D(q_2) = \begin{bmatrix} \omega_{12}(q_2) \\ I_{n-m} \end{bmatrix}. \tag{13}$$

By direct calculation, it can be readily shown that $A(q_2) = D^\top(q_2)$ annihilates $W^\top(q_2)$. This eventually leads to the following reduction theorem for underactuated nonholonomic systems with symmetry.

THEOREM 1. *Consider the underactuated mechanical control system with nonholonomic constraints and symmetry in* (10). *Then* (10) *with* $(2n+m)$ *first-order equations can be reduced to a system of (2n - m) first-order equations in the following cascade form:*

$$\begin{aligned} \dot q_x &= \omega_r(q_r)\dot q_r \\ M_r(q_r)\ddot q_r + C_r(q_r,\dot q_r)\dot q_r + G_r(q_r) &= F_r(q_r)u \end{aligned} \tag{14}$$

where $(q_x, q_r) = (q_1, q_2)$, $\omega_r(q_r) = \omega_{12}q_2$, *and*

$$\begin{aligned} M_r(q_r) &= D^\top(q_2)M(q_2)D(q_2) \\ G_r(q_r) &= \omega_r^\top(q_r)k + \nabla_{q_r}U(q_r) \\ F_e(q_r) &= D^\top(q_2)F(q_2)\,. \end{aligned} \tag{15}$$

Moreover, if $V(q) = U(q_2)$ *ie* $K_v = 0$, *the reduced system is a well defined Lagrangian system with configuration vector* $q_r = q_2$ *and Lagrangian function*

$$L_r(q_r,\dot q_r) = \frac{1}{2}\dot q_r^\top M_r(q_r)\dot q_r - U(q_r) \tag{16}$$

which satisfies a forced Euler -Lagrange equation in cascade with the constraint equation as the following

$$\begin{aligned} \dot q_x &= \omega_r(q_r)\dot q_r \\ \frac{\mathrm{d}}{dt}\frac{\partial L_r}{\partial \dot q_r} - \frac{\partial L_r}{\partial q_r} &= F_r(q_r)u\,. \end{aligned} \tag{17}$$

In addition, if $l+m<n$ *or* $l+m=n$, *the reduced system with configuration vector* q_r *is an underactuated (or fully actuated) mechanical system.*

Proof. The forced Euler-Lagrange equation in (10) can be rewritten as

$$M(q_2)\ddot q + C(q_2,\dot q)\dot q + G(q) = W^\top(q_2)\lambda + F(q_2)u \tag{18}$$

Where $C(q_2,\dot q)$ satisfies $\dot M = C + C^\top$ and $G(q) = \nabla_q V(q)$. Multiplying both sides of the last equation by $A(q_2) = D^\top(q_2)$ eliminates λ and gives

$$D^\top(q_2)M(q_2)\ddot q D^\top(q_2)C(q_2,\dot q)\dot q + G_r(q) = F_r(q_2)u \tag{19}$$

where $F_r(q_2)u = D^\top(q_2)F(q_2)$ and

$$\begin{aligned} G_r(q_2) = D^\top(q_2)G(q) &= [\,\omega_{12}^\top(q_2) \quad I\,]\begin{bmatrix} \nabla_{q_1}V(q) \\ \nabla_{q_2}V(q) \end{bmatrix} \\ &= \omega_{12}^\top(q_2)k + \nabla_{q_2}U(q_2) = G_r(q_2) \end{aligned}$$

on the other hand, taking the derivative of $\dot q = D(q_2)\dot q_2$ implies

$$\ddot q = D(q_2)\ddot q_2 + \dot D(q_2,\dot q_2)\dot q_2\,.$$

Substituting $\ddot q$ in (19), we get

$$\begin{aligned} D^\top(q_2)M(q_2)D(q_2)\ddot q_2 + \big[D^\top(q_2)C(q_2,\dot q)D(q_2) \\ + D^\top(q_2)M(q_2)\dot D(q_2,\dot q_2)\big]\dot q_2 + G_r(q_2) = F_r(q_2)u \end{aligned}$$

by taking $q_r = q_2$, this last equation can be rewritten as

$$M_r(q_r)\ddot q_r + C_r(q_r,\dot q_r)\dot q_r + G_r(q_r) = F_r(q_r)u \tag{20}$$

where

$$M_r(q_r) := D^\top q_2 M(q_2) D(q_2) \tag{21}$$

$$C_r(q_r, \dot q_r) := D^\top q_2 C(q_2, \dot q) D(q_2) + D^\top(q_2) M(q_2) \dot D(q_2, \dot q_2)\,. \tag{22}$$

To establish that (20) is in fact equivalent to the forced Euler-Lagrange equation for the reduced system with the Lagrangian function $L_r(q_r, \dot q_r)$ and the force matrix $F_r(q_r)$, we need to prove that $C_r(q_r, \dot q_r)$ satisfies $\dot M_r = C_r + C_r^\top$. By direct calculation we have

$$\begin{aligned}\dot M_r &= D^\top \dot M D + \dot D^\top M D + D^\top M \dot D \\ &= D^\top (C + C^\top) D + \dot D^\top M D + D^\top M \dot D \\ &= (D^\top C D + D^\top M \dot D) + (D^\top C^\top D + \dot D^\top M D) \\ &= C_r + C_r^\top\end{aligned}$$

and the result follows.

Many well-studied cases of nonholonomic systems such as the rolling disk, two-wheeled mobile robot, *etc* are special examples of underactuated nonholonomic systems considered in Theorem 1 which possess a fully actuated reduced system. We formalize this special case in the following corollary.

COROLLARY 1. *Consider the underactuated nonholonomic system in* (10). *Assume that the number of inputs and constraints add up to* n, *ie* $m + l = n$. *Suppose that* q_2 *is actuated and the system has non-interacting inputs, ie* $F(q) = col\big(0, F_2(q_2)\big)$ *where* $F_2(q_2)$ *is an invertible matrix. Then there exists a change of control input that transforms the dynamics of* (14) *into the following non-triangular normal form with a vector double-integrator linear part*

$$\begin{aligned}\dot z &= f(\xi_1, \xi_2) := \omega_r(\xi_1)\xi_2 \\ \dot\xi_1 &= \xi_2 \qquad \dot\xi_2 = v\,.\end{aligned} \tag{23}$$

Proof. Notice that $F_r(q_r) = D^\top(q_2)F(q_2) = F_2(q_2)$ and thus F_r is invertible. After renaming the variables as

$$z = q_x \qquad \xi_1 = q_r \qquad \xi_2 = \dot q_r$$

and applying the change of control

$$v = F_r^{-1}(q_r)\big(M_r(q_r)u + C_r(q_r, \dot q_r)\dot q_r + G_r(q_r)\big)$$

we get $\ddot q_r = v$ and the result follows.

7 EXAMPLE OF THE ROLLING DISK

A vertical rolling disk which is not allowed to slip is an example of mechanical system with nonholonomic velocity constraints (Fig. 1). These constraints can be expressed as

$$\dot x = r\cos(\theta)\dot\varphi \qquad \dot y = r\sin(\theta)\dot\varphi\,.$$

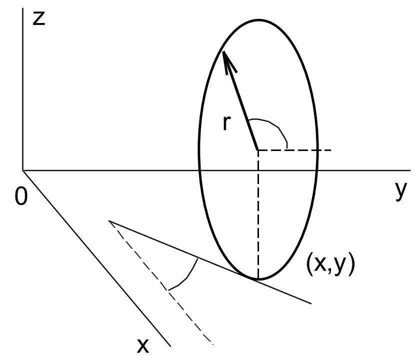

Fig. 1. The rolling disk.

To consider the rolling disk as a control system, assume that the rotation angle φ and the heading angle θ of the disk can be controlled using control torques τ_1 and τ_2, respectively. The Lagrangian of the rolling disk with configuration vector $q = (x, y, \varphi, \theta)$ is

$$L = \frac{1}{2}\dot q^\top M \dot q$$

with a constant inertia matrix

$$M = \mathrm{diag}(m, m, J_1, J_2)$$

where m is the mass of the disk and J_1 and J_2 are the inertia of the disk. Thus, the rolling disk is a flat underactuated mechanical system with actuated variable (φ, θ) and unactuated variable (x, y). The constraint equation is in the form $W(q)\dot q = 0$ with

$$W(q) = \begin{bmatrix} 1 & 0 & -r\cos(\theta) & 0 \\ 0 & 1 & -r\sin(\theta) & 0 \end{bmatrix}. \tag{24}$$

Clearly $W(q)$ is of full rank.

By setting $q_1 = (x, y)$ and $q_2 = (\varphi, \theta)$, one obtains $W_1(q_2) = I_2$ which is an invertible matrix. Since the number of control inputs and constraints add up to $n = 4$, based on theorem 1, the reduced system for this underactuated nonholonomic system is a fully actuated system. The configuration vector of the reduced system is $q_r = (\varphi, \theta)$ and the overall dynamics of the system can be expressed as

$$\begin{aligned}\dot x &= r\cos(\theta)\dot\varphi \qquad & \ddot\varphi &= u_1 \\ \dot y &= r\sin(\theta)\dot\varphi \qquad & \ddot\theta &= u_2\end{aligned} \tag{25}$$

where $u_i = \tau_i / J_i$, $i = 1, 2$. The reduced system of the rolling disk is a vector doubleintegrator $\ddot q_r = u$ ($u = (u_1, u_2)^\top$).

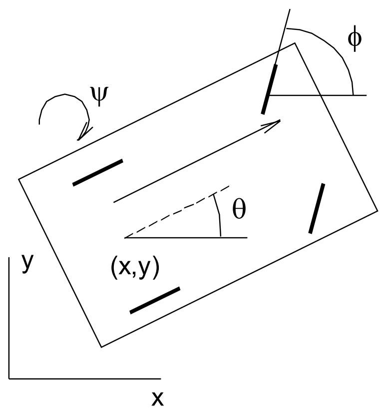

Fig. 2. A car-like vehicle.

Assume that the input controls of the system are $\dot{\varphi} = \omega_1$ and $\dot{\theta} = \omega_2$. Then after normalization of (x, y) units by r, the kinematic model of the system can be written as

$$\begin{aligned} \dot{x} &= r\cos(\theta)\omega_1 \\ \dot{y} &= r\sin(\theta)\omega_1 \end{aligned} \qquad \dot{\theta} = \omega_2\,. \tag{26}$$

Applying the following change of coordinates and control for $\theta \in (-\pi/2, /pi/2)$

$$\begin{aligned} x_1 &= x \\ x_2 &= \tan(\theta) \\ x_3 &= y \end{aligned} \qquad \begin{aligned} v_1 &= \omega_1/\cos(\theta) \\ v_2 &= \omega_2/(1+\tan^2(\theta)) \end{aligned} \tag{27}$$

the kinematics of the rolling disk in (26) transforms into a first-order chained-type nonholonomic system as the following

$$\dot{x}_1 = v_1 \qquad \dot{x}_2 = v_2 \qquad \dot{x}_3 = x_2 v_1 \tag{28}$$

which is of the form of a chained-type systems.

8 THE CAR–LIKE VEHICLE

In this section, we address the reduction of the dynamic model of a car-like vehicle as shown in Fig. 2.

The dynamic model of the car is an example of underactuated nonholonomic systems with five degrees of freedom, two control inputs and two velocity constraints. Let $q = (x, y, \theta, \psi, \phi)$ denote the configuration vector of the system. (x, y) denote the position of the center of the axle between the two rear wheels, θ is the orientation of the car body with respect to x-axis, ψ is the angle of rotation of each wheel and ϕ is the steering angle with respect to the car body.

The velocity constraints of the front and rear wheels are given by

$$\begin{aligned} &\cos(\theta+\psi)\frac{\mathrm{d}}{\mathrm{d}t}(x + l\cos\theta) - \cos(\theta+\phi)\frac{\mathrm{d}}{\mathrm{d}t}(x + l\sin\theta) = 0 \\ &\sin(\theta)\dot{x} - \cos(\theta)\dot{y} = 0 \end{aligned} \tag{29}$$

or

$$\begin{aligned} &\sin(\theta+\psi)\dot{x} - \cos(\theta+\phi)\dot{y} - l\cos\phi\dot{\theta} = 0 \\ &\sin(\theta)\dot{x} - \cos(\theta)\dot{y} = 0 \end{aligned} \tag{30}$$

which can be written as $W(q)\dot{q} = 0$ where $W = (W_1, W_2)$ is partitioned according to $q_x = (x, y)$ and $q_r = (\theta, \psi, \phi)$ such that

$$\begin{aligned} W_1(\theta,\phi) &= \begin{bmatrix} \sin(\theta+\phi) & -\cos(\theta+\phi) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \\ W_2(\theta,\phi) &= \begin{bmatrix} -l\cos\phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{31}$$

Note that W_1 is not invertible.

Now $\omega_{12}(\theta,\phi) = -W_1^{-1}W_2 = \begin{bmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{bmatrix}$ where $\alpha_1(\theta,\phi) = \frac{l\cos\theta}{\tan\phi}$, $\alpha_2(\theta,\phi) = \frac{l\sin\theta}{\tan\phi}$. And $D(q)$ can then be determined from $\omega_{12}(\theta,\phi)$ as

$$D(q) = D(\theta,\phi) = \begin{bmatrix} \alpha_1 & \alpha_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top. \tag{32}$$

The Lagrangian of the dynamic car is given by

$$\begin{aligned} L = \frac{1}{2}m\big(\dot{x} - \frac{1}{2}\sin(\theta)\dot{\theta}\big)^2 + \frac{1}{2}(J_b + 2J_v)\dot{\theta}^2 \\ + \frac{1}{2}\big[2J_h\big(1 + \frac{1}{\cos^2(\phi)}\big)\big]\dot{\psi}^2 + \frac{1}{2}2J_v(\dot{\theta}+\dot{\phi})^2 \end{aligned} \tag{33}$$

where m is the mass of the car, J_b is the inertia of the body, J_h is the inertia of each wheel along the horizontal axes and J_v is the inertia of each wheel in the vertical axes. The Lagrangian can be expressed as

$$L = \frac{1}{2}\dot{q}^\top \times \begin{bmatrix} m & 0 & -\frac{ml}{4}\sin\theta & 0 & 0 \\ 0 & m & \frac{ml}{4}\cos\theta & 0 & 0 \\ \frac{ml}{4}\sin\theta & \frac{ml}{4}\cos\theta & J_\theta & 0 & 2J_v \\ 0 & 0 & 0 & 2J_h(2+\tan^2\phi) & 0 \\ 0 & 0 & 2J_v & 0 & 2J_v \end{bmatrix} \dot{q} \tag{34}$$

where $J_\theta = J_b + 4J_v + ml^2/4$. The differential-algebraic equations of motion of the dynamic car are such as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = W^\top(q)\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0_{3\times 2} \\ I_{2\times 2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{35}$$

where $(\lambda_1, \lambda_2) \in \Re \times \Re$ are the Lagrange multipliers and $(u_1, u_2) \in \Re \times \Re$ are the torques applied to the rear wheels and the steering wheel, respectively. Based on Theorem 1 the dynamics of the car in (35) can be reduced to the cascade of the constraint equation and a reduced Lagrangian system with configuration vector $q = (\theta, \psi, \phi)$ as

$$M_r(q_r)\ddot{q}_r + C_r(q_r, \dot{q}_r)\ddot{q}_r + G_r(q_r) = F_r(q_r)u$$

where
$M_r = D^\top MD$, $C_r = D^\top CD + D^\top M\dot{D}$, $F_r = D^\top F$.
By direct calculation and after simplification we get

$$F_r = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$
$$M_r = M_r(\phi) = \begin{bmatrix} \overline{J}_\theta(\phi) & 0 & 2J_v \\ 0 & 2J_k(2+\tan^2\phi) & 0 \\ 2J_v & 0 & 2J_v \end{bmatrix} \tag{36}$$

with $\overline{J}_\theta(\phi) = J_\theta + \dfrac{ml^2}{\tan^2\phi}$.

Clearly the reduced Lagrangian is itself underactuated with three degrees of freedom and two controls. In addition, $\theta, \psi)$ are the external variables and ϕ is the shape variable of the car. The dynamics of the actuated variables (θ, ψ) of the reduced system can be linearized as

$$\ddot{\psi} = \tau_1 \qquad \ddot{\phi} = \tau_2 \tag{37}$$

using an explicit collocated change of variable in the form

$$u = \eta(\phi) + \beta(\phi, \dot{q}_r)$$

where

$$\eta(\phi) = \begin{bmatrix} 2J_h(2+\tan^2\phi) & 0 \\ 0 & 2J_v\big(1 - \frac{2J_v\tan^2\phi}{ml^2\tan^2\phi}\big) \end{bmatrix}. \tag{38}$$

$\eta(\phi)$ is well defined and positive definite for all ϕ. From the first constraint equation, we can solve for $\dot{\theta}$ to get

$$\dot{\theta} = \tan(\phi)\big(\cos(\theta)\dot{x} + \sin(\theta)\dot{y}\big)\,. \tag{39}$$

¿From this equation, the overall dynamics of the car can be written as

$$\dot{x} = r\cos\theta\cdot\dot{\psi} \qquad \dot{\theta} = \frac{r}{l}\tan\theta\cdot\dot{\psi} \qquad \ddot{\psi} = \tau_1$$
$$\dot{y} = r\sin\theta\cdot\dot{\psi} \qquad\qquad \ddot{\phi} = \tau_2 \tag{40}$$

where r is the radius of the wheel.

After normalization of the units of (x, y) by r, and taking $\dot{\psi} = \omega_1$ we get

$$\dot{x} = r\cos\theta\cdot\omega_1 \qquad \dot{\theta} = \frac{1}{l}\tan\theta\cdot\omega_1$$
$$\dot{y} = r\sin\theta\cdot\omega_2 \qquad \dot{\phi} = \omega_2 \tag{41}$$

where $\dot{\omega}_1 = \tau_1$, $\dot{\omega}_2 = \tau_2$.

Now applying the change of coordinates and control as in [63],

$$x_1 = x \qquad x_2 = \frac{\tan\phi}{l\cos^3\theta}$$
$$x_3 = \tan\theta \qquad x_4 = y$$
$$v_1 = \cos\theta\cdot\omega_1\phi \qquad v_2 = \frac{1+\tan^2\phi}{l\cos^3\theta}\omega_2 + \frac{3\tan\theta\tan^2\phi}{l^2\cos^3\theta}$$

transforms finally the nonholonomic into a chained form system as

$$\dot{x}_1 = v_1 \qquad \dot{x}_3 = x_2 v_1$$
$$\dot{x}_2 = v_2 \qquad \dot{x}_4 = x_3 v_1\,. \tag{42}$$

9 DISCONTINUOUS CONTROL OF CHAINED FORMS

Control design for broad classes of nonholonomic mechanical systems can be reduced to the control of chained-type systems as the following.

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{x}_3 &= x_2 v_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1}v_1 \end{aligned} \qquad k_i \geq 1\,,\ i = 1, \dots, n\,. \tag{43}$$

The control of chained forms occupies a special place in the theory of nonholonomic control. In fact, many nonholonomic mechanical systems such as cars pulling trailers, nonholonomic manipulators can be represented by kinematic models in chained form or are feedback equivalent to chained forms.

Applying the σ process coordinate transformation to the system yields

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= x_2 \\ \xi_3 &= \frac{x_3}{x_1} \\ &\vdots \\ \xi_{n-1} &= \frac{x_{n-1}}{x_1^{n-3}} \\ \xi_n &= \frac{x_n}{x_1^{n-2}}\,. \end{aligned} \tag{44}$$

In the new coordinates the system is described by the following set of equations

$$\begin{aligned} \xi_1 &= v_1 \\ \xi_2 &= v_2 \\ \xi_3 &= \frac{\xi_2 - \xi_3}{\xi_1}v_1 \\ &\vdots \\ \xi_n &= \frac{\xi_{n-1} - (n-2)\xi_n}{\xi_1}v_1\,. \end{aligned} \tag{45}$$

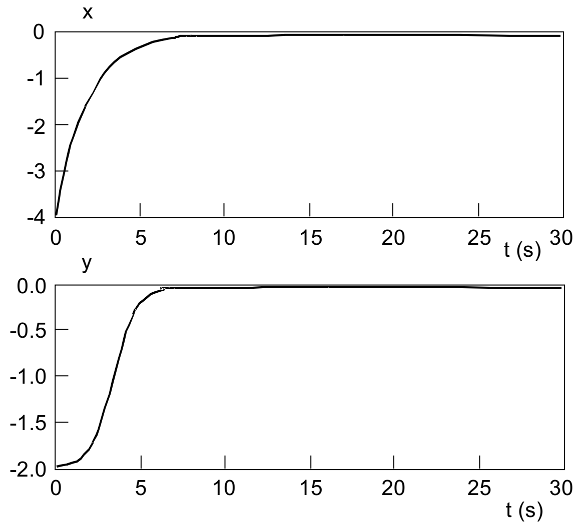

Fig. 3. Time history of $x(t)$ and $y(t)$.

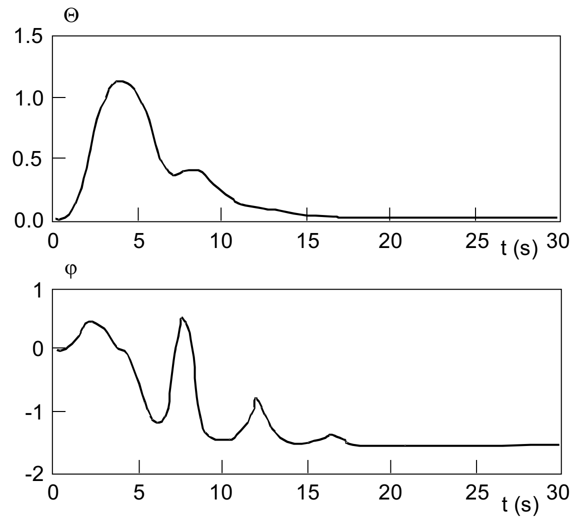

Fig. 4. Time history of $\theta(t)$ and $\varphi(t)$.

The resulting system is now described by equations of the form

$$\dot{\xi} = A\xi + B_2 v_2 \tag{46}$$

where

$$A = \begin{bmatrix} -k & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -k & k & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -k & -2k & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -k & (n-3)k & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -k & (n-2)k \end{bmatrix}$$

and $\quad B_2 = [0 \ \ 1 \ \ 0 \ \ \dots \ \ 0]$.

The linear system can be exponentially stabilized using the control linear control

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -kx_1 \\ p_2\xi_2 + p_3\xi_2 + \dots + p_{n-1}\xi_{n-1} + p_n\xi_n \end{bmatrix}. \tag{47}$$

Proposition 1 (Astolfi, 1996). *In the x coordinates the control law expressed by*

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -kx_1 \\ p_2x_2 + p_3\frac{x_3}{x_1} + \dots + p_{n-1}\frac{x_{n-1}}{x_1^{n-3}} + p_n\frac{x_n}{x_1^{n-2}} \end{bmatrix} \tag{48}$$

with $k > 0$ and the p_i such that the eigenvalues of the matrix $A + B_2p$ have negative real parts ($p = [0, p_2, p_3, \dots, p_{n-1}, p_n]$), globally exponentially stabilizes the system (45) in the open and dense set

$$\Omega_1 := \{x \in \Re^n : x_1 \neq 0\}.$$

Remark 4. It is possible to show that the discontinuous control law (48) is well defined and bounded, for all $t \geq 0$, along the trajectories of the closed loop system (43)–(48) whenever $x_1(0) \neq 0$.

Remark 5. The discontinuous control law (46) does not exponentially stabilize system (43). It only guarantees exponential convergence for all the initial conditions in the open and dense set

$$\{(x_1, \dots, x_n) \in \Re^n : x_1 \neq 0\}.$$

10 DISCONTINUOUS FEEDBACK CONTROL OF A CAR–LIKE VEHICLE

Consider now equation (40), if we apply the control transformation

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \frac{v_1}{\cos\theta} \\ -\frac{3}{l}\sin^2\phi\tan\theta\sec\theta v_1 + l\cos^2\phi\cos^3\theta v_2 \end{bmatrix}$$

and applying the s process [2]

$$\xi_1 = x \qquad \xi_3 = \tan\theta / x$$

$$\xi_2 = \frac{1}{l}\sec^3\theta\tan\theta \qquad \xi_4 = y/x^2$$

we obtain a system described, in the new input and state variables, by equations of the form

$$\xi_1 = v_1 \qquad \xi_3 = \frac{\xi_2 - \xi_3}{\xi_1}v_1$$

$$\xi_2 = v_2 \qquad \xi_4 = \frac{\xi_3 - 2\xi_4}{\xi_1}v_1.$$

By using Proposition 1, we design a state feedback law described by equations of the form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -k\xi_1 \\ p_2\xi_2 + p_3\xi_3 + p_4\xi_4 \end{bmatrix}. \tag{49}$$

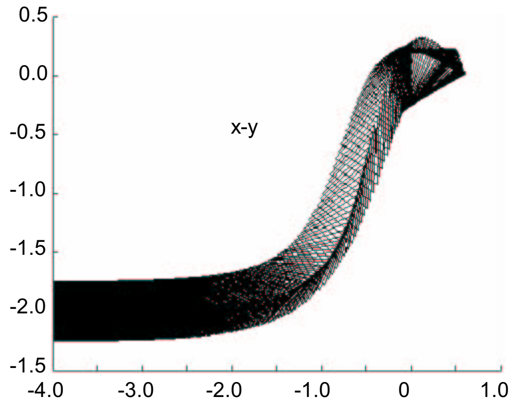

Fig. 5. $x-y$ trajectory of the robot.

In the original coordinates the feedback law is described by

$$\begin{aligned}\omega_1 &= -k\sec\theta \\ \omega_2 &= k\frac{3}{l}\sin^2\phi\tan\theta\sec^2\theta\cdot x + l\cos^2\theta\cos^3\theta \\ &\times\Big[p_2\Big(\frac{1}{l}\sec^3\theta\tan\phi\Big) + p_3\frac{\tan\theta}{x} + p_4\frac{y}{x^2}\Big]\,.\end{aligned} \tag{50}$$

To obtain almost exponential stability of the closed loop system it is necessary to have $k > 0$ and to set the coefficients p_2, p_3 and p_4 such that $\sigma(A_4) \in C^-$ where A_4 is given by

$$A_4 = \begin{bmatrix} p_2 & p_3 & p_4 \\ -k & k & 0 \\ 0 & -k & -2k \end{bmatrix}.$$

Figures 3–5 show simulation results for the car-like vehicle. We set $k = 1$ and p_2, p_3 and p_4 such that the eigenvalues of the matrix A_4 are all at $\lambda = -2$. Moreover we assume that $x_1(0) \neq 0$.

11 CONCLUSION

In this paper the problem of reduction of underactuated nonholonomic systems is addressed. It is shown that this class of system can be reduced to a well-defined reduced order Lagrangian system in Cascade with the constraint equation. The control of a class of these systems is also studied and it is shown that a board of systems can be presented in the form of chained-type nonholonomic systems for which a discontinuous control is applied to ensure global quasiexponential stability.

References

[1] ABRAHAM, R.—MARDSEN, J. E.: Foundations of Mechanics, Addison-Wesley, 1978.

[2] ASTOLFI, A.: Discontinuous Control of Nonholonomic Systems, Systems and Control Letters No. 27 (1996), 37–45.

[3] BLOCH, A. M.—P. E. CROUGH,: Nonholonomic Control Systems on Riemannian Manifolds, SIAM J. Control and Optimization **33** No. 1 (1995), 126–148.

[4] BLOCH, A. M.—REYHANOGLU, M.—McCLAMROCH, N. H.: IEEE Transactions on Automatic Control **37** No. 11 (1992), 1746–1757.

[5] BOMET, J. B.: Explicit Design of Time-Varying Stabilizing Control Laws for a Class of Controllable Systems without Drift, Systems and Control Letters **18** (1992), 147–158.

[6] BROKLETT, R. W.: Asymptotic Stability and Feedback Stabilization, in Differential Geometry Control Theory, 1983, pp. 181–191.

[7] MARDSEN, J. E.—RATIU, T. S.: Introduction to Mechanics and Symmetry, Springer, 1999.

[8] M'CLOSKEY, R. T.—MURRAY, R. M.: Nonholonomic Systems and Exponential Convergence: Some Analysis Tools, in 32nd IEEE Conference on Decision and Control, S. Antonio, Texas, 1993, pp. 943–948.

[9] NEIMARK, J. I.—FUVAEV, F.: Dynamics of nonholonomic systems, volume 33, AMS. 1972.

[10] MURRAY, R. M.: Control of Nonholonomic Systems Using Chained Form, Dynamic and Control of Mechanical Systems. The falling cat and related Problems, The Field Institute for Research in Mathematical Sciences.

[11] OSTROWSKI, J. P.—BUDRICK, J. W.: The Mechanics and Control of Undalotory Locomotion, Int. Journal of Robotics Research **177** No. 7 (1998), 683–701.

[12] SODARLEN, O. J.—EGELAND, O.: Exponential Stabilization of Chained Nonholonomic Systems, 2nd European Control Conference, Croningen, The Netherlands, 1993, pp. 1438–1443.

[13] SPONG, M. W.: Energy Based Control of Underactuated Mechanical Systems, 1996, IFAC world congress, 1996.

Received 22 November 2002

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