STATIC OUTPUT FEEDBACK ROBUST CONTROLLER DESIGN VIA LMI APPROACH

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The paper addresses the problem of the robust output feedback controller design with a guaranteed cost and parameter dependent Lyapunov function quadratic stability for linear continuous time polytopic systems. The proposed design methods lead to an iterative LMI based algorithm. Numerical examples are given to illustrate the design procedure.

K e y w o r d s: quadratic stability, parameter dependent quadratic stability, robust controller, output feedback

1 INTRODUCTION

During the last decades numerous papers dealing with the design of static robust output feedback control schemes to stabilize uncertain systems have been published (Benton and Smith, 1999; Crusius and Trofino, 1999; El Ghaoui and Balakrishnan, 1994; Geromel, De Souza and Skelton, 1998; Henrion, Tarboriech and Garcia, 1999; Kose and Jabbari, 1999; Li Yu and Jian Chu, 1999; Mehdi, Al Hamid and Perrin, 1996; Pogyeon, Young Soo Moon and Wook Hyun Kwon, 1999; Tuan, Apkarian, Hosoe and Tuy, 2000; Veselý, 2002). Various approaches have been used to study the two aspects of the robust stabilization problem, namely conditions under which the linear system described in state space can be stabilized via output feedback and the respective procedure to obtain a stabilizing or robustly stabilizing control law.

The necessary and sufficient conditions to stabilize the linear continuous time invariant system via static output feedback can be found in (Kučera, and De Souza, 1995; Veselý, 2001). In the above and other papers, the authors basically conclude that despite the availability of many approaches and numerical algorithms the static output feedback problem is still open.

Recently, it has been shown that an extremely wide array of robust controller design problems can be reduced to the problem of finding a feasible point under a Biaffine Matrix Inequality (BMI) constraint. The BMI has been introduced in (Goh, Safonov and Papavassilopoulos, 1995). In this paper, the BMI problem of robust controller design with output feedback is reduced to a LMI problem (Boyd *et al*, 1994).The theory of Linear Matrix Inequalities has been used to design robust output feedback controllers in (Benton and Smith, 1999; Crusius and Trofino, 1999; El Ghaoui and Balakrishnan, 1994; Henrion, Tarboriech and Garcia, 1999; Li Yu and Jian Chu, 1999; Tuan, Apkarian, Hosoe, and Tuy, 2000; Veselý, 2001). Most of the above works present iterative algorithms in which a set of LMI problems are repeated until certain convergence criteria are met. The V-K iteration algorithm proposed in (El Ghaoui and Balakrishnan, 1994) is based on an alternative solution of two convex LMI optimization problems obtained by fixing the Lyapunov matrix or the gain controller matrix. This algorithm is guaranteed to converge, but not necessarily, to the global optimum of the problem depending on the starting conditions.

The main criticism formulated by control engineers against modern robust analysis and design methods for linear systems concerns the lack of efficient easy to use and systematic numerical tools. This is especially true when analyzing robust stability as affected by highly structured uncertainty with BMI, for which no polynomial-time algorithm has been proposed so far (Henrion, Alzelier and Peaucelle, 2002).

This paper is concerned with the class of uncertain linear systems that can be described as

$$\dot{x}(t) = (A_0 + A_1\theta_1 + \cdots + A_p\theta_p)x(t) \tag{1}$$

where $\theta = [\theta_1 \dots \theta_p] \in R^p$ is a vector of uncertain and possibly time varying real parameters.

The system represented by (1) is a polytope of linear affine systems which can be described by a list of its vertices

$$\dot{x}(t) = A_{ci}x(t), \quad i = 1, 2, \dots, N \tag{2}$$

where $N = 2^p$.

The system represented by (2) is quadratically stable if and only if there is a common Lyapunov matrix $P > 0$ such that

$$A_{ci}^{\top} P + P A_{ci} < 0, \quad i = 1, 2, \dots, N\,. \tag{3}$$

A weakness of quadratic stability is that it guards against arbitrary fast parameter variations. As a result, this test tends to be conservative for constant or slow-varying parameters θ, for polytopic systems. To reduce conservatism when (1) is affine in θ and the parameters of system are time invariant, in (Gahinet, Apkarian and Chilali,

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1996) the parameter-dependent Lyapunov functions $P(\theta)$ has been used in the form

$$P(\theta) = P_0 + \theta_1 P_1 + \dots + \theta_p P_p \,. \tag{4}$$

Robust controller design with guaranteed cost and affine quadratic stability has been proposed in [14]. Other types of parameter-dependent Lyapunov functions have been proposed in [2] for the stability analysis of linear discrete time systems and for the analysis and the design of continuous time systems with affine type uncertainties in (Shaked, 2001; Henrion, Alzelier and Peaucelle, 2002; Takahashi, Ramos, and Peres, 2002). In this paper, we pursue the idea of (Takahashi, Ramos, and Peres, 2002) and introduce a new robust controller LMI design procedure with less conservative results and guaranteed cost. The proposed approach allows to reduce important class of BMI problems to LMI. For guaranteed cost and system (2) this leads to an iterative LMI based algorithm. The proposed design procedure guarantees with sufficient conditions parameter dependant Lyapunov function quadratic stability (PDQS) for closed loop systems.

The paper is organized as follows. In Section 2 the problem formulation and some preliminary results are brought. The main results are given in Section 3. In Section 4 the obtained theoretical results are applied. We have used the standard notation. A real symmetric positive (negative) definite matrix is denoted by $P > 0$ ($P < 0$). Much of the notation and terminology follows the references of (Kučera, and De Souza, 1995; and Gahinet, Apkarian and Chilali, 1996).

2 PRELIMINARIES AND PROBLEM FORMULATION

We shall consider the following affine linear time invariant continuous time uncertain systems

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t)\,, \\ y(t) &= C(\theta)x(t)\,, \quad x(0) = x_0 \end{aligned} \tag{5}$$

where $x(t) \in R^n$ is the plant state; $u(t) \in R^m$ is the control input; $y(t) \in R^l$ is the output vector of system; $A(\theta), B(\theta), C(\theta)$ are matrices of appropriate dimensions and

$$\begin{aligned} A(\theta) &= A_0 + A_1\theta_1 + \dots + A_p\theta_p\,, \\ B(\theta) &= B_0 + A_1\theta_1 + \dots + B_p\theta_p\,, \\ C(\theta) &= C_0 + C_1\theta_1 + \dots + C_p\theta_p\,. \end{aligned}$$

Note that, in order to keep the polytope affine property, the matrix $B(\theta)$ or $C(\theta)$ must be precisely known. In the following we assume that $C(\theta)$ is known and equal to matrix C. In general, a polytope description of uncertainties results in a less conservative controller design than other characterizations of uncertainty [4]. However, as the number of uncertain parameters increases, the number of vertices increases exponentially, and the design time increases exponentially too. The system represented by (5) is a polytope of linear systems. The linear matrix inequality approach requires that system (5) be described by a list of its vertices, *ie*, in the form

$$\{(A_{v1}, B_{v1}, C_{v1}), \dots, (A_{vN}, B_{vN}, C_{vN})\}\,. \tag{6}$$

Consequently, the system (6) is static output feedback quadratically stabilizable if and only if there is a Lyapunov matrix $P > 0$ and a feedback matrix F such that

$$(A_{vi} + B_{vi}FC_{vi})^\top P + P(A_{vi} + B_{vi}FC_{vi}) < 0, \qquad i = 1, 2, \dots, N\,. \tag{7}$$

If (7) holds for $P > 0$ and some F, then the vertices of the polytope (6) are said to be simultaneously quadratically stabilized by F. It is well known [4] that if P is a common Lyapunov matrix for the vertices of the polytope (6), it serves as a common Lyapunov function for the uncertain system (5) for all admissible uncertainties $\theta_i \in <\underline{\theta_i}, \overline{\theta_i}>$, $i = 1, 2, \dots, p$. In (6), each vertex is computed for a different permutation of the p variables θ_i, alternatively taken at maximum and minimum values.

Recently, new type of parameter dependent Lyapunov function (PDLF) (Shaked, 2001; Henrion, Alzelier and Peaucelle, 2002; Takahashi, Ramos, and Peres, 2002) has been introduced in the form

$$P(\alpha) = \sum_{i=1}^{N} P_i\alpha_i\,, \quad \sum_{i=1}^{N} \alpha_i = 1\,, \;\; \alpha_i \geq 0\,. \tag{8}$$

The robust stability conditions of closed-loop polytopic system with output feedback algorithm

$$u = FCx \tag{9}$$

$$A_c = \sum_{i=1}^{N} A_{ci}\alpha_i = \sum_{i=1}^{N} (A_{vi} + B_{vi}FC)\alpha_i \tag{10}$$

is given by the following lemma (Takahashi, Ramos, and Peres, 2002).

LEMMA 1. *Suppose there exist positive definite Lyapunov matrices P_j, $j = 1, 2, \dots, N$ such that*

$$A_{ci}^\top P_{vi} + P_{vi}A_{ci} < -I\,, \tag{11}$$

$$A_{ck}^\top P_{vj} + P_{vj}A_{ck} + A_{cj}^\top P_{vk} + P_{vk}A_{cj} < \frac{2}{N-1}I\,; \tag{12}$$

$$i = 1, 2, \dots, N\,,\; k = 1, 2, \dots, N-1\,;\; j = k+1, \dots, N$$

then

$$P(\alpha) = \sum_{i=1}^{N} P_{vi}\alpha_i\,, \;\; \sum_{i=1}^{N} \alpha_i = 1\,, \; \alpha_i \geq 0 \tag{13}$$

is a parameter dependent Lyapunov function for any

$$A_c = \sum_{i=1}^{N} A_{ci}\alpha_i\,. \tag{14}$$

The following performance index is associated with the system (5)

$$J = \int_0^\infty (x(t)^\top Qx(t) + u(t)^\top Ru(t))\mathrm{d}t \tag{15}$$

where $Q = Q^\top \geq 0, R = R^\top > 0$ are matrices of compatible dimensions.

The problem studied in this paper can be formulated as follows: For a continuous time system described by (5) design a static output feedback controller with the gain matrix F and control algorithm

$$u(t) = Fy(t) = FCx(t) \tag{16}$$

so that the closed loop system

$$\dot{x} = (A(\alpha) + B(\alpha)FC)x(t) = A_c x(t) \tag{17}$$

where

$$A(\alpha) = \sum_{i=1}^{N} A_{vi}\alpha_i$$

is PDQS with guaranteed cost.

DEFINITION 2. Consider the system (5). If there exists a control law u^* and a positive scalar J^* such that closed loop system (17) is stable and the closed loop value cost function (15) satisfies $J \le J^*$, then J^* is said to be the guaranteed cost and u^* is said to be the guaranteed cost control law for system (5).

3 THE MAIN RESULTS

In this paragraph we present a new procedure to design a static output feedback controller for polytopic continuous time linear systems (5) which ensure the guaranteed cost and PDQS of closed loop system. The one of the main result can be summarized in the following theorem.

THEOREM 1. *Consider the closed loop polytopic system (17). Then the following statements are equivalent.*
a. The polytopic system (17) is static output feedback simultaneously PDQS stabilizable with a guaranteed cost

$$\int_0^\infty (x^\top Qx + u^\top Ru)dt \le \max_i x_0^\top P_{vi}x_0 = J^* \tag{18}$$

and $P > 0$.
b. There exist symmetric matrices $P_{vj} > 0$, $j = 1,2,\dots,N$, $R > 0$, $Q > 0$, $M > 0$ *and a matrix* F *such that the following inequalities hold*

$$(A_{vi} + B_{vi}FC)^\top P_{vi} + P_{vi}(A_{vi} + B_{vi}FC) + Q + C^\top F^\top R_1 FC \le -M \quad \text{for } i = 1,2,\dots,N. \tag{19}$$

$$A_{ck}^\top P_{vj} + P_{vj}A_{ck} + A_{cj}^\top P_{vk} + P_{vk}A_{cj} + C^\top F^\top R_1 FC \le \frac{2}{N-1}M \tag{20}$$

where $i = 1,2,\dots,N$; $k = 1,2,\dots,N-1$; $j = k+1,\dots,N$, $R_1 = \frac{N+1}{2N}R$ *and* M *is positive definite matrix which provides at design procedure less conservative results.*
c. There exist matrices $P_{vj} > 0$, $R > 0$, $Q > 0$, $M > 0$ *and a matrix* F *that the following inequalities hold*

$$(A_{vi} + B_{vi}FC + P_{vi})^\top (A_{vi} + B_{vi}FC + P_{vi}) - (A_{vi} + B_{vi}FC)^\top (A_{vi} + B_{vi}FC)P_{vi}P_{vi} + Q + C^\top F^\top R_1 FC \le -M \tag{21}$$

and $i = 1,2,\dots,N$,

$$\begin{aligned}(A_{vk} &+ B_{vk}FC + P_{vj})^\top (A_{vk} + B_{vk}FC + P_{vj}) \\ &+ (A_{vj} + B_{vj}FC + P_{vk})^\top (A_{vj} + B_{vj}FC + P_{vk}) \\ &- \big[A_{vk}^\top A_{vk} + A_{vj}^\top A_{vj} + (A_{vk}^\top B_{vk} + A_{vj}^\top B_{vj})FC \\ &+ \big((A_{vk}^\top B_{vk} + A_{vj}^\top B_{vj})FC\big)^\top\big] - P_{vk}P_{vk} - P_{vj}P_{vj} \\ &+ C^\top F^\top (R - B_{vk}^\top B_{vk} - B_{vj}^\top B_{vj})FC \le \frac{2}{N-1}M\end{aligned} \tag{22}$$

with condition

$$R - B_{vk}^\top B_{vk} - B_{vj}^\top B_{vj} > 0 \tag{23}$$

where $k = 1,2,\dots,N-1$; $j = k+1,\dots,N$.

Proof. For polytopic system (17) and Lyapunov function $V = x^\top P(\alpha)x$ it is well known that, cost is guaranteed if the following inequality holds

$$A_c^\top P(\alpha) + P(\alpha)A_c + Q + C^\top F^\top RFC \le 0\,. \tag{24}$$

Assume, the inequalities (19) and (20) hold, than using (14) one can obtains

$$\begin{aligned}&\sum_{i=1}^{N}(A_{ci}^\top P_{vi} + P_{vi}A_{ci} + Q + C^\top F^\top R_1 FC)\alpha_i^2 \\ &+ \sum_{k=1}^{N-1}\sum_{j=k+1}^{N}(A_{ck}^\top P_{vj} + P_{vj}A_{ck} + A_{cj}^\top P_{vk} + P_{vk}A_{cj} \\ &+ C^\top F^\top RFC)\alpha_k\alpha_j\,.\end{aligned} \tag{25}$$

Hence,

$$\sum_{i=1}^{N}\alpha_i^2 \in < \frac{1}{N} 1 >, \qquad \sum_{k=1}^{N-1}\sum_{j=k+1}^{N}\alpha_k\alpha_j \le \frac{N-1}{2N}\,,$$

for the worst case we obtain

$$\begin{aligned}&\sum_{i=1}^{N}(A_{ci}^\top P_{vi} + P_{vi}A_{ci})\alpha_i^2 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N}(A_{ck}^\top P_{vj} + P_{vj}A_{ck} \\ &+ A_{cj}^\top P_{vk} + P_{vk}A_{cj})\alpha_k\alpha_j + Q + C^\top F^\top RFC \\ &= A_c^\top P(\alpha) + P(\alpha)A_c + Q + C^\top F^\top RFC\,.\end{aligned} \tag{26}$$

Substitute (21) and (22) to (25) we obtain

$$\sum_{i=1}^{N}(-M)\alpha_i^2 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N}\frac{2}{N-1}M\alpha_k\alpha_j$$

or

$$-M\Big(\sum_{i=1}^{N}(N-1)\alpha_i^2 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N}2\alpha_k\alpha_j\Big) = -M\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(\alpha_i - \alpha_j)^2 < 0\,. \tag{27}$$

Inequality (27) proves the stability of closed loop polytopic systems (17) with PDLF and equality (26) proves the equivalence of the first and second statement with sufficient conditions. For proof the equivalence of the second and third statement (19) and (21)) the following equality has been used

$$
\begin{aligned}
(A_{vi} + B_{vi}FC + P_{vi})^\top (A_{vi} + B_{vi}FC + P_{vi})& \\
= (A_{vi} + B_{vi}FC)^\top (A_{vi} + B_{vi}FC) + (A_{vi}& \\
+ B_{vi}FC)^\top P_{vi} + P_{vi}(A_{vi} + B_{vi}FC) + P_{vi}P_{vi}\,.& \quad (28)
\end{aligned}
$$

Substitute the first and second term of (19) from (28) the inequality (21) has been obtained. The equivalence of the second and third statement is then evident.

For LMI solution of inequalities (21) and (22) with (23) the term $-PP$ has to be replaced. In this note two possibilities are proposed.

(1) $$-P_{vi}P_{vi} \le -P_{vi}D_{vi} - D_{vi}P_{vi} + D_{vi}D_{vi} = F_{vi} \quad (29)$$

where $D_{vi} = D_{vi}^\top > 0$ is the initial value of matrix P_{vi}.

(2) $$-P_{vi}P_{vi} \le (\varrho^2 - \varrho_1^2)I - Z_i = F_{zi}(Z) \quad (30)$$

where $\varrho_1 I < P_{vi} < \varrho I\,,\quad i = 1, 2, \ldots, N\,,$

$$\begin{bmatrix} Z_i & P \\ P & I \end{bmatrix} > 0\,, \quad 0 \le Z_i \le \varrho^2\,.$$

$Z_i = Z_i^t > 0$ is any positive definite matrix, a new LMI variable that satisfies the condition (30).

From (21) and (29) the following LMI condition results

$$
\begin{bmatrix} L_i & C^\top F^\top & (A_{ci} + P_{vi})^\top \\ FC & -(R_{1i})^{-1} & 0 \\ A_{ci} + P_{vi} & 0 & -I \end{bmatrix} < 0\,, \qquad (31)
$$
$$0 < P_{vi} < \varrho I\,, \quad i = 1, 2, \ldots, N$$

where $R_{1i} = R_1 - B_{vi}^\top B_{vi} > 0\,,\quad L_i = -\big[A_{vi}^\top A_{vi} + A_{vi}^\top B_{vi}FC + (A_{vi}^\top B_{vi}FC)^\top\big] + Q + M + F_{vi}\,,$

$i = 1, 2, ..., N$. Equation (22) results to following LMI problem

$$
\begin{bmatrix} L_{kj} & C^\top F^\top & S_{kj} & S_{jk} \\ FC & -(R_{kj})^{-1} & 0 & 0 \\ S_{kj}^\top & 0 & -I & 0 \\ S_{jk}^\top & 0 & -I & 0 \end{bmatrix} < 0
$$

$$
\begin{aligned}
L_{kj} = -\big[A_{vk}^\top A_{vk} + A_{vj}^\top A_{vj} + (A_{vk}^\top B_{vk} + A_{vj}^\top B_{vj})FC& \quad (32) \\
+ ((A_{vk}^\top B_{vk} + A_{vj}^\top B_{vj})FC)^\top\big] + F_{vk} + F_{vj} - \frac{2}{N-1}M\,,& \\
S_{kj} = (A_{vk} + B_{vk}FC + P_{vj})^\top\,,& \\
S_{jk} = (A_{vj} + B_{vj}FC + P_{vk})^\top&
\end{aligned}
$$

with condition

$$R_{kj} = R - B_{vk}^\top B_{vk} - B_{vj}^\top B_{vj} > 0$$

and

$$k = 1, 2, \ldots, N-1\,, \quad j = 1, 2, \ldots, N$$

f the LMI solutions of (6) and (32) are feasible with respect to matrices $F, P_{vi}\,,\ i = 1, 2, \ldots, N$ and M then the uncertain system (5) is parameter dependent quadratically stable with a guaranteed cost control algorithm

$$u = Fy \quad \text{and} \quad J^* = \max_i x_0^\top P_{vi} x_0$$

is the guaranteed cost for the uncertain closed loop system.

4 EXAMPLES

In this example we consider the linear model of two cooperating DC motors. The problem is to design two PI controllers for a laboratory MIMO system which will guarantee PDQS of a closed loop uncertain system with guaranteed cost. The system model is given by (5) with a time invariant matrix affine type uncertain structure, where

$$
A_0 = \begin{bmatrix}
0 & -.2148 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1.014 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -.2605 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -.9107 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -.1639 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -.8137 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -.2279 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -.8251 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

$$
A_1 = \begin{bmatrix}
0 & -.025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -.1395 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -.0938 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -.2911 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .0188 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .0208 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -.0333 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -.1173 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
A_2 = \begin{bmatrix}
0 & .0125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & .0594 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .0116 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .0308 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -.0188 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -.0156 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & .0208 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -.0333 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
B_0 = \begin{bmatrix}
.3148 & 0 \\
.0478 & 0 \\
0 & -.1028 \\
0 & -.0091 \\
-.0841 & 0 \\
-.0287 & 0 \\
0 & .3676 \\
0 & .2448 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\qquad
B_1 = \begin{bmatrix}
.0625 & 0 \\
-.0798 & 0 \\
0 & -.0462 \\
0 & -.0449 \\
.0016 & 0 \\
.0072 & 0 \\
0 & .077 \\
0 & -.005 \\
0 & 0 \\
0 & 0
\end{bmatrix}
$$

$$B_2 = \begin{bmatrix} -.0094 & 0 \\ .0151 & 0 \\ 0 & .0019 \\ 0 & -.003 \\ -.0121 & 0 \\ -.03 & 0 \\ 0 & -.064 \\ 0 & .0189 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C^\top = \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$$

The number of polytope systems is equal to 4 and the polytope vertices are computed for different permutations of the two variables θ_1, θ_2 alternatively taken at their maximum $\overline{\theta_i}$ and minimum $\underline{\theta_i}$, $i = 1, 2$. The decentralized control structure for the two PI controllers can be obtained by the choice of the static output feedback gain matrix F structure. It is given as follows

$$F = \begin{bmatrix} f11 & 0 & f13 & 0 \\ 0 & f22 & 0 & f24 \end{bmatrix}$$

The results of calculation of a static output feedback gain matrix F for PDQS for different $Q = qI$, $R = rI$, $\theta_m = |\theta_1| = |\theta_2|$, ρ and ρ_1 are summarized as follows.

- Eqs. (30), (31) and (32).
  For $\theta_m = 1$, $q = .0001$, $r = 1.15$ and $\rho = 75$ after 100 repeated calculations, that is $D_{vi}^{j+1} = P_{vi}^{j}$, $i = 1, 2, 3, 4$ and $j = 1, 2, \dots$ gain matrix F, maximal closed-loop eigenvalue of all four polytopic systems $maxeig(CL)$ and the value of guaranteed cost $\lambda_M(P_{vi})$
  $$\max_i x_0^\top P_{vi} x_0 \le \max_i \|x_0\|^2 \lambda_M(P_{vi})$$
  are equal
  $$F = \begin{bmatrix} -2.5498 & 0 & -.5549 & 0 \\ 0 & -6.124 & 0 & -2.3094 \end{bmatrix},$$
  $maxeig(CL) = -.1882$ and $\lambda_M(P_{vi}) = 69.2702$. For the same case, V-K iterative method (El Ghaoui and Balakrishnan, 1994) gives $maxeig(CL) = -.173$,
  $$F = \begin{bmatrix} -2.0672 & 0 & -.4318 & 0 \\ 0 & -2.0108 & 0 & -.4594 \end{bmatrix}.$$
- For the parameters $\theta_m = 1.8$, $q = .0001$, $r = 1$ and $\rho = 75$ after 100 repeated procedures the results of calculation are as follows.
  $$F = \begin{bmatrix} -2.1272 & 0 & -.4462 & 0 \\ 0 & -4.6625 & 0 & -1.3408 \end{bmatrix},$$
  $maxeig(CL) = -.1278$ and $\lambda_M(P_{vi}) = 71.2643$. V-K iterative procedure gives maximal closed-loop eigenvalue of all four polytopic systems $maxeig(CL) = -.0075$ and $\lambda_M(P_{vi}) = 71.9258$.
- For the parameters $\theta_m = 1$, $r = .57$ and $\rho = 100$ and different q after 100 repeated procedures the results of calculation are as follows.
  a. $q = .1$,
  $$F = \begin{bmatrix} -3.51 & 0 & -.8393 & 0 \\ 0 & -10.9463 & 0 & -5.1024 \end{bmatrix},$$
  $maxeig(CL) = -.1915$ and $\lambda_M(P_{vi}) = 91.7532$. V-K iterative procedure gives maximal closed-loop eigenvalue of all four polytopic systems $maxeig(CL) = -.1659$ and $\lambda_M(P_{vi}) = 91.799$.
  b. $q = 1$,
  $$F = \begin{bmatrix} -4.6478 & 0 & -1.1418 & 0 \\ 0 & -12.8322 & 0 & -5.6151 \end{bmatrix},$$
  $maxeig(CL) = -.192$ and $\lambda_M(P_{vi}) = 94.8176$. V-K iterative procedure gives infeasible solution. All LMI solutions are feasible.

The second example has been borrowed from (Benton and Smith, 1999) to demonstrate the use of the algorithm given by (6) and (32). It is known that the presented system is static output feedback stabilizable. Let (A, B, C) in (1) be defined as

$$A = \begin{bmatrix} -0.036 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.010 & 0.0024 & -4.0208 \\ 0.1002 & q_1(t) & -0.707 & q_2(t) \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ q_3(t) & -7.59222 \\ -5.520 & 4.490 \\ 0 & 0 \end{bmatrix}, \quad C = [0 \quad 1 \quad 0 \quad 0]$$

with parameters bounds $-0.6319 \le q_1(t) \le 1.3681$, $1.22 \le q_2(t) \le 1.420$, and $2.7446 \le q_3(t) \le 4.3446$. Find a stabilizing output feedback matrix F. The four vertices are calculated. The nominal model of (A, B) is given by the above matrices when we substitute for the entries $A(3,2) = 0.3681, A(3,4) = 1.32$ and $B(2,1) = 3.5446$. The affine model uncertainty (5) (A_1, A_2, B_1, B_2) are matrices with the following entries $A_1(3,2) = 1, A_2(3,4) = 0.1$ and $B_1(2,1) = 0.8, B_2 = 0$ with $\theta_i \in < -1, 1 >, i = 1, 2$. Other entries of the above uncertain matrices are equal to zero. The nominal model is unstable with eigenvalues:

$$eig\{-2.0516, 0.2529 \pm 0.3247i, -0.2078\}.$$

Let the structure of F be defined as

$$F^\top = [F(1,1) \quad F(2,1)].$$

- Eqs. (30), (31), (32).
  For $Q = q * I$, $q = .00001$, $R = r * I$, $r = 246.2$, $\varrho = 100$ and $\theta_m = 1$ one obtains the following results.
  $$F^\top = [-1.502 \quad 2.726],$$
  $$maxeig(CL) = -.0683, \quad \lambda_M(P_{vi}) = 98.02.$$
- For $\theta_m = 2$, $r = 264.2$ results are as follows.
  $$F^\top = [-1.3167 \quad 2.6816]$$
  $$\text{maxeig}(CL) = -.0676, \quad \lambda_M(P_{vi}) = 98.72.$$
- For $\theta_m = 2.5$, $r = 273.86$ results are as follows.
  $$F^\top = [-1.2879 \quad 3.0857],$$
  $$maxeig(CL) = -.0684, \quad \lambda_M(P_{vi}) = 99.83.$$
- For $\theta_m = 1$, $r = 246.2$, $\varrho = 100$ and different q the following results are obtained.
  a. $q = .1$
  $$F^\top = [-1.1071 \quad 2.699],$$

$$maxeig(CL) = -.0719\,;\ \lambda_M(P_{vi}) = 98.16\,.$$

Note, that eigenvalues of matrix M are $eig(M) = [.9984; .8902; .6963; .1222]$. Hence, they are different and are not equal to 1 as supposed in Lemma 1 (Takahashi, Ramos, and Peres, 2002). We can conclude that using the matrix M in the design procedure the less conservative results could be obtained.

b. $q = 1$

$$F^\top = [-1.5033 \quad 2.8373]\,,$$
$$maxeig(CL) = -.0689\,;\ \lambda_M(P_{vi}) = 99.346\,.$$

c. $q = 6$

$$F^\top = [-2.3635 \quad 2.8861]\,,$$
$$maxeig(CL) = -.0616\,;\ \lambda_M(P_{vi}) = 99.926\,.$$

All LMI solutions are feasible. Note that for the above parameters the V-K iterative method does not give any reasonable solutions. Usually, the repeated procedure generates less conservative results than first one. The convergence of the above special repeated procedure has not been proven yet, however if the reasoning of (El Ghaoui and Balakrishnan, 1994) is taken account we can conclude that the proposed algorithm is guaranteed to converge but not necessarily to the global optimum of the problem, depending on the starting conditions.

5 CONCLUSIONS

In this paper, we have proposed a new procedure for robust output feedback controller design for linear systems with affine parameter uncertainty. The feasible solution of the proposed output feedback controller design procedure with sufficient conditions guarantees the parameter dependent Lyapunov function quadratic stability and guaranteed cost. The proposed design procedure pursue the idea of (Takahashi, Ramos, and Peres, 2002). Examples show that proposed approach gives the less conservative results as could be obtained from quadratic design procedure.

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