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COMMUNICATIONS

CONVERGENCE ASPECTS OF BAND–LIMITED SIGNALS

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In Estalilla [1] convergence aspects of bandlimited signals were discussed and it was shown that convergence in the square mean implies uniform convergence. The reverse question, which asks for convergence in the square mean following from pointwise convergence, could not be answered affirmatively. This paper gives the answer to this question, the proof and an example.

Keywords: Bandlimited signals, convergence properties, uniform and pointwise convergence, convergence in the square mean.

1 INTRODUCTION

We consider square integrable signals (*ie* signals with finite energy) with a finite interval as the support of their Fourier transform. The interval of length 2α shall be centered at the origin of the real axis. Those signals we call bandlimited with the band limit α. The Hilbert space of square integrable real signals is denoted by $\mathbf{L}_2$ and the norm of a signal f in this space is given by $\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x$.

It is well known that in case of bandlimited signals with finite energy convergence in the square mean implies uniform convergence, too (see for example [1]). The question for the reverse implication, *ie* for convergence in the square mean following from pointwise convergence, yet has not been answered affirmatively. We shall demonstrate here that there is an additional requirement to be met, namely the convergence of the norm.

2 THEOREM AND PROOF

THEOREM. *A sequence $\{f_n\}$ of signals bandlimited with some fixed band limit α converges in $\mathbf{L}_2$ with some likewise bandlimited signal f if and only if it converges with f uniformly on $\mathbb{R}$ and $\lim_{n\to\infty} \|f_n\| = \|f\|$.*

Proof. First we want to prove that uniform convergence and the convergence of the norm of f_n follows from convergence in $\mathbf{L}_2$. Without limiting the generality we assume the band limit to be $\alpha = \pi$. The difference $|f(t) - f_n(t)|$ can be estimated as follows, using the Schwarz inequality to estimate the term (1) by the term (2).

$$|f(t) - f_n(t)| = \frac{1}{2\pi}\Big|\int_{-\pi}^{\pi} \big(\widehat{f}(\omega) - \widehat{f}_n(\omega)\big) \mathrm{e}^{j\omega t} \mathrm{d}\omega\Big|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|\big(\widehat{f}(\omega) - \widehat{f}_n(\omega)\big)\mathrm{e}^{j\omega t}\big|\mathrm{d}\omega$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|\widehat{f}(\omega) - \widehat{f}_n(\omega)\big|\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|\widehat{f}(\omega) - \widehat{f}_n(\omega)\big| \cdot 1\mathrm{d}\omega \tag{1}$$

$$\leq \frac{1}{2\pi}\Big(\int_{-\pi}^{\pi} \big|\widehat{f}(\omega) - \widehat{f}_n(\omega)\big|^2 \mathrm{d}\omega\Big)^{\frac{1}{2}} \Big(\int_{-\pi}^{\pi} \mathrm{d}\omega\Big)^{\frac{1}{2}} \tag{2}$$

$$= \frac{1}{\sqrt{2\pi}}\big\|\widehat{f} - \widehat{f}_n\big\| = \frac{1}{\sqrt{2\pi}}\big\|f - f_n\big\|\,. \tag{3}$$

Eq. (3) follows from the theorem of Parseval and Plancherel. With (3) we have uniform convergence:

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} |f(t) - f_n(t)| = 0\,.$$

Since the norm is continuous, $\lim_{n\to\infty} \|f_n\| = \|f\|$ holds as well. Now we prove the reverse implication. The term $\|f - f_n\|^2$ can be separated as follows.

$$\|f - f_n\|^2 = \|f\|^2 + \|f_n\|^2 - (f,\, f_n) - (f_n,\, f) \tag{4}$$

with the scalar product

$$(f, f_n) = \int_{-\infty}^{\infty} f(t) f_n(t) \mathrm{d}t =$$

$$\int_{-\infty}^{-T} f(t) f_n(t) \mathrm{d}t + \int_{-T}^{T} f(t) f_n(t) \mathrm{d}t + \int_{T}^{\infty} f_n(t) f(t)\, \mathrm{d}t\,,$$

where $T > 0$. With $\|f\|^2 = (f, f)$ we can estimate

$$\Big|\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t - \int_{-\infty}^{\infty} f(t)\, f_n(t) \mathrm{d}t\Big|$$

$$\leq \Big|\int_{-T}^{T} |f(t)|^2 \mathrm{d}t - \int_{-T}^{T} f(t) f_n(t) \mathrm{d}t\Big| + \int_{|t|\geq T} |f(t)|^2 \mathrm{d}t + \int_{|t|\geq T} |f(t)| \cdot |f_n(t)|\, \mathrm{d}t\,.$$

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Because we presupposed $\lim_{n\to\infty}\|f_n\| = \|f\|$ a real $C > 0$ exists with $\|f_n\| < C$ for all $n \in \mathbb{N}$. Now we consider an arbitrary $\varepsilon > 0$ and choose $T_0 = T_0(C)$ sufficiently large to obtain $\int_{|t|\ge T_0} |f(t)|^2 \mathrm{d}t < \varepsilon$. With this T_0 holds

$$\int_{|t|\ge T_0} |f(t)\, f_n(t)| \mathrm{d}t \le \Big(\int_{|t|\ge T_0} |f(t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \cdot \Big(\int_{|t|\ge T_0} |f_n(t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} < \sqrt{\varepsilon}\cdot C\,.$$

Therefore we have

$$\Big|\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t - \int_{-\infty}^{\infty} f(t) f_n(t)\,\mathrm{d}t\Big| \le \Big|\int_{-T_0}^{T_0} f(t)\cdot f(t) - f(t)\cdot f_n(t) \mathrm{d}t\Big| + \varepsilon + \sqrt{\varepsilon} C\,. \quad (5)$$

The integral at the left hand side of the inequality (5) can be estimated as follows.

$$\Big|\int_{-T_0}^{T_0} f(t)\cdot\big(f(t) - f_n(t)\big)\mathrm{d}t\Big| \le \Big(\int_{-T_0}^{T_0} |f(t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \cdot \Big(\int_{-T_0}^{T_0} |f(t) - f_n(t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \le \max_{|t|\le T_0} |f(t) - f_n(t)| \cdot \sqrt{2\,T_0} \cdot \|f\|\,.$$

Now we choose $n_0 = n_0(C)$ sufficiently large to obey

$$\max_{|t|\le T_0} |f(t) - f_n(t)| \cdot \sqrt{2\,T_0} \cdot \|f\| < \varepsilon$$

and thereby obtain from (5)

$$\Big|\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t - \int_{-\infty}^{\infty} f(t)\, f_n(t) \mathrm{d}t\Big| < 2\varepsilon + \sqrt{\varepsilon} C \quad (6)$$

for all $n \ge n_0$. From (6) follows $\lim_{n\to\infty}(f,\, f_n) = (f,\, f)$ and thus according to (4)

$$\lim_{n\to\infty} \|f - f_n\|^2 = \|f\|^2 + \|f\|^2 - \|f\|^2 - \|f\|^2 = 0\,,$$

showing that indeed we have convergence in $\mathbf{L}_2$.

3 EXAMPLE

As an example we consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, given by

$$f_n(t) := \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} \frac{\sin \pi(t-k)}{\pi(t-k)}\,, \quad n = 1,\, 2,\, 3,\, \ldots \quad (7)$$

Obviously we have $f_n(l) = 1/\sqrt{2n+1}$ for $l \in \mathbb{N}$, $1 \le l \le n$ and $f_n(l) = 0$ for $l \in \mathbb{N}\backslash\{1,\, 2,\, \ldots,\, n\}$. The reproducing kernel, for details see [2]. $K(x,y) := \frac{\sin\pi(x-y)}{\pi(x-y)}$ of the Hilbert space of π-bandlimited functions renders the reproduction of such a function g by means of the standard scalar product in $\mathbf{L}_2$, *ie* for every $x,\, y \in \mathbb{R}$ holds

$$g(x) = \int_{\mathbb{R}} K(x,\, y)\, g(y) \mathrm{d}y = \int_{\mathbb{R}} \frac{\sin \pi(x-y)}{\pi(x-y)}\, g(y) \mathrm{d}y\,. \quad (8)$$

Thus with (7) the energy norm of f_n for every n is determined by

$$\begin{aligned}\|f_n\|^2 &= \int_{\mathbb{R}} |f_n(t)|^2 \mathrm{d}t = \int_{\mathbb{R}} f_n(t)\cdot f_n(t) \mathrm{d}t\\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} \frac{\sin\pi(t-k)}{\pi(t-k)}\, f_n(t) \mathrm{d}t\\ &= \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} \int_{\mathbb{R}} \frac{\sin\pi(t-k)}{\pi(t-k)}\, f_n(t) \mathrm{d}t\\ &= \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} f_n(k)\\ &= \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} \frac{1}{\sqrt{2n+1}} = \frac{2n+1}{2n+1} = 1\,. \end{aligned} \quad (9)$$

In [3] the uniform convergence of the sequence $\sum_{k=-n}^{n} \frac{\sin\pi(t-k)}{\pi(t-k)}$ on the real axis with 1 for $n \to \infty$ is proved, which implies that the sequence f_n uniformly converges with zero, although because of (9) it does not so in the energy norm. Figure 1 gives an idea of how the energy of the functions f_n is distributed over the real axis with increasing n.

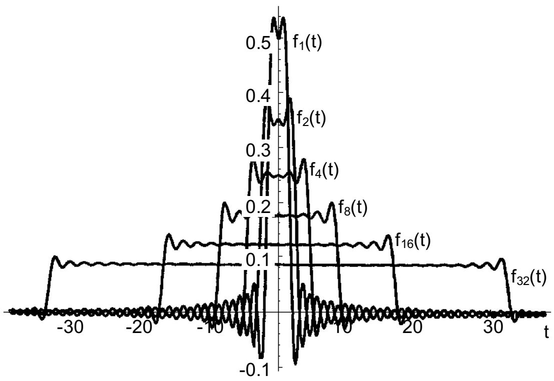

Fig. 1. The functions f_n with $n = 2^0,\, 2^1,\, \ldots,\, 2^5$.

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