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COMMUNICATIONS

STATIC OUTPUT FEEDBACK ROBUST CONTROL OF UNCERTAIN LINEAR DISCRETE–TIME SYSTEMS

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The output feedback robust stabilization problem is studied in this paper for uncertain discrete-time linear system described in state space using polytopic model. The necessary and sufficient conditions for output feedback stabilizability with guaranteed cost are provided as well as the algorithm for computation of output feedback gain matrix. The LMI formulation enables to employ standard efficient computational tools (MATLAB).

K e y w o r d s: discrete-time systems, output feedback, quadratic stabilization, Riccati equations, LMI approach

1 INTRODUCTION

Considering the control design in the time domain or state space, usually not a complete state vector is measurable and available for the feedback loop but only a part of it that is covered by the output vector. Basically there are two ways to cope with this fact. One approach employs a state reconstructor or observer that brings additional dynamics to the original system. Another way is to look for output feedback where the control implementation is simple and does not require any additional dynamics when static output feedback is applied. Therefore, besides the state feedback, also the output feedback problem is attractive for control designers.

The output feedback controller problem has been thoroughly studied during past decades [1], [2], [4], [5]. Though the state feedback problem has been completely resolved, this is not the case of the output feedback. The fundamental difference between the state and output feedback stabilization problems is that while stabilization by state feedback implies a convex problem formulation, the introduction of output feedback constrains the problem and converts it into a nonconvex one. Therefore computational difficulties arise since in general the solution of nonlinear problems requires NP hard algorithms. In other words, a solution of a convex problem can be found using standard software tools such as linear matrix inequalities approach (LMI) supported in MATLAB that requires a polynomial time in general (computational time depends polynomially on the system dimension). On the other hand a computational solution of a nonconvex problem requires in general a non-polynomial time, i.e., a small increase in dimension can cause a fail of the algorithm due to an immense rise of the computational time. The output feedback problem can be solved using the bilinear matrix inequalities (BMI) approach at the expense of a not simple computational burden.

Majority of presently existing approaches to output feedback control either solve directly the nonconvex problem using iterative procedures where the question of convergence remains open or add supplementary conditions to the output feedback problem so that it is restricted to convex problem formulation. However, in the latter case the space where the solution is looked for is reduced from necessary and sufficient conditions to sufficient ones, which introduces certain conservativism of the final result. Therefore the present effort in this field concentrates on finding the ways to relax the conservativism that the supplementary conditions bring to problem formulation and to develop simple computationally efficient algorithms based on standard software tools like LMI solvers to obtain the required result - the output feedback gain matrix.

In this paper the necessary and sufficient conditions for output feedback stabilizability with guaranteed cost for uncertain discrete-time systems with polytopic model are provided based on the discrete-time counterpart to the results of [1] for continuous-time output feedback stabilization. The developed stability conditions use the notion of quadratic stability of uncertain systems. The corresponding LMI based algorithm for computation of the output feedback matrix is presented as well as an illustrative example. The developed algorithm includes the LMI iterative solution of a disrete-time Riccati inequality. In future the obtained results can be modified with the use of parameter dependent Lyapunov function to reduce the conservativism of the quadratic stability approach. The key results were presented in [3].

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2 PROBLEM FORMULATION AND PRELIMINARIES

Consider a linear discrete-time uncertain dynamic system

$$\begin{aligned} x(k+1) &= (A+\delta A)x(k) + (B+\delta B)u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x(k) \in R^n$, $u(k) \in R^m$, $y(k) \in R^l$ are state, control and output vectors, respectively; A, B, C are known constant matrices of appropriate dimensions and δA, δB are matrices of uncertainties of appropriate dimensions. Uncertainties are considered to be of the affine type

$$\delta A = \sum_{j=1}^{p} \varepsilon_j \bar{A}_j, \quad \delta B = \sum_{j=1}^{p} \varepsilon_j \bar{B}_j \tag{2}$$

where $\underline{\varepsilon}_j \leq \varepsilon_j \leq \bar{\varepsilon}_j$ are unknown parameters; $\bar{A}_j$, $\bar{B}_j$, $j = 1, 2, ..., p$ are constant matrices of the corresponding dimensions. The affine parameter dependent model (1), (2) can be readily converted into a polytopic one and described by a list of its vertices

$$\{(A_1, B_1, C), (A_2, B_2, C), ..., (A_N, B_N, C)\} \quad N = 2^p. \tag{3}$$

The considered control law is a static output feedback

$$u(k) = KCu(k). \tag{4}$$

Consider a quadratic cost function associated with the uncertain system (1), (2), (3)

$$J = \sum_{k=0}^{\infty} [x(k)^T Q x(k) + u(k)^T R u(k)], \tag{5}$$

where Q, R are symmetric positive definite matrices, $Q \in R^{n\times n}, R \in R^{m\times m}$. Our aim is to state necessary and sufficient conditions and the corresponding controller design procedure for output feedback stabilization of the uncertain system (1), (2), (3) with guaranteed cost.

The frequent approach that is used to study stabilization of uncertain systems is based on quadratic stability notion. The quadratic stability is equivalent to the existence of one Lyapunov function for the whole set of system models that describes the uncertain system. The polytopic system is quadratically stable if and only if there exists a Lyapunov function for all vertices of the respective polytope describing the uncertain system. The following lemma summarizes the stability condition for a closed loop uncertain polytopic system with output feedback.

LEMMA 1. *System (1) with uncertainties (2), or equivalently the system represented by a polytopic convex set with vertices (3) is output feedback quadratically stable if and only if there exists a symmetric positive definite matrix P such that*

$$\begin{aligned} (A_i + B_i KC)^T P (A_i + B_i KC) - P < 0 \\ i = 1, 2, ..., N. \end{aligned} \tag{6}$$

The notion of guaranteed cost J_0 represents the cost function value for the closed loop system $J \leq J_0$ for all admissible uncertainties and considered initial conditions. Our aim is to provide the necessary and sufficient conditions and the corresponding controller design procedure for output feedback quadratic stabilization of the uncertain system (1), (2) with guaranteed cost.

3 MAIN RESULT

The obtained results are presented in two stages. Firstly, the necessary and sufficient conditions are provided for output feedback quadratic stabilizability of the uncertain system. Secondly, the algorithm is proposed to calculate the corresponding stabilizing static output feedback gain matrix. The necessary and sufficient conditions presented in this paper are based on the discrete-time counterpart to the results of [1] modified for LMI and extended to the uncertain system [3]. They are summarized in the following theorem.

THEOREM 1. *Consider the uncertain system (1), (2) equivalently represented by a polytopic set with N vertices (3) and a cost function (5). This system is quadratically stabilizable with guaranteed cost $J \leq J_0 = x(0)^T P x(0)$ for a symmetric positive definite matrix P if and only if the pair (A_i, B_i) is stabilizable, the pair (A_i, C) is detectable for $i = 1, 2, ..., N$ and there exist matrices P and K such that*

$$\begin{aligned} \Phi_i = A_i^T P A_i - P + Q - \\ A_i^T P B_i (B_i^T P B_i + R)^{-1} B_i^T P A_i \leq 0 \end{aligned} \tag{7}$$

and

$$\begin{bmatrix} -I & G_i \\ G_i^T & \Phi_i \end{bmatrix} \leq 0, \tag{8}$$

where

$$\begin{aligned} G_i = (B_i^T P B_i + R)^{-\frac{1}{2}} B_i^T P A_i + \\ (B_i^T P B_i + R)^{\frac{1}{2}} KC. \end{aligned} \tag{9}$$

The proof of Theorem 1 is based on the standard LQ theory approach and the fact that (7) and (8) together provide Riccati-like inequality

$$\begin{aligned} A_i^T P A_i - P + Q - \\ A_i^T P B_i (B_i^T P B_i + R)^{-1} B_i^T P A_i + G_i^T G_i \leq 0 \end{aligned} \tag{10}$$

that is equivalent to Lyapunov quadratic stability condition for a closed loop system

$$\begin{aligned} (A_i + B_i KC)^T P (A_i + B_i KC) \\ -P + C^T K^T R K C + Q \\ \leq 0, i = 1, 2, ..., N. \end{aligned} \tag{11}$$

Details of the proof are omitted. The inequalities (7) and (8) indicate the way to computing the stabilizing output feedback gain matrix K. The computational procedure consists in principle of two steps corresponding to inequalities (14) and (15) so that P and K can be treated "separately" and their multiplication is avoided. Firstly, the simultaneous solution of Riccati-like inequalities (7) is required. Technically it is not a simple task since there is no direct non-iterative method known to the authors that would guarantee the result in one step. Therefore to obtain the LMI appropriate formulation we employ the iterative algorithm to solve the Riccati equation developed in [6] having good convergence properties and modify it for inequalities (7). In this way we obtain simultaneous solution P for all vertices (3), that completes the first step of the proposed algorithm. Secondly, having P, inequality (8) is solved with respect to K using standard LMI tools. The crucial point that still remains open is in the first step of algorithm, since the resulting P is chosen without considering condition (8), therefore certain conservativism is included into the algorithm. Our present effort is therefore focused on finding the way to solve discrete-time Riccati inequality directly through LMI tools that would allow simultaneous solution of (7) and (8). The computational procedure is summarized in the Algorithm below.

Algorithm

Step 1.

Simultaneous solution to N algebraic Riccati-like inequalites

$$\begin{gathered}\Phi_i = A_i^T P A_i - \\ P + Q - A_i^T P B_i (B_i^T P B_i + R)^{-1} B_i^T P A_i \le 0, \\ i = 1, 2, ..., N\end{gathered}$$

using the following above outlined iterative procedure.

1. $j = 1,\ P_0 = I,\ error = eps, norm = 2 * eps$
2. while $error < norm$ solve

$$A_i^T P_{j-1} A_i + Q - P_j < 0, \ \ i = 1, 2, ..., N$$

3. for unknown matrix P_j

$$Re_i = B_i^T P_j B_i + R$$

$$L_i = P_j B_i Re_i$$

4. solve for unknown P_{j+1}

$$\begin{gathered}(I - L_i B_i^T) P_j (I - L_i B_i^T)^T + \\ L_i R L_i^T - P_{j+1} < 0, \\ i = 1, 2, ..., N\end{gathered}$$

5. $norm = \|P_{j+1} - P_{j-1}\|,\ j = j + 1,\ P \leftarrow P_j$ end.

Step 2.

Simultaneous solution of N inequalities of LMI-type for unknown matrix K

$$\begin{bmatrix} -I & G_i \\ G_i^T & \Phi_i \end{bmatrix} \le 0$$

where $G_i = (B_i^T P B_i + R)^{-\frac{1}{2}} B_i^T P A_i + (B_i^T P B_i + R)^{\frac{1}{2}} K C$, Φ_i is defined by (7) as in the Step 1 for the obtained P. If there is no LMI feasible solution K, modify Q, R in step 1.

3 EXAMPLE

The example illustrates the computational results obtained using the proposed approach and demonstrates the ability to find the output feedback stabilizing control. Consider uncertain system (1), (2) with matrices

$$A = \begin{bmatrix} 0.7118 & 0.0736 & 0.1262 \\ 0.7200 & 0.6462 & 2.3432 \\ 0 & 0 & 0.6388 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0122 & 0.0412 \\ 0.3548 & 0.1230 \\ 0.2015 & 0.2301 \end{bmatrix} \text{ - nominal model}$$

$$\bar{A}_1 = \begin{bmatrix} 0.088 & 0.006 & 0.01 \\ 0.07 & 0.03 & 0.1 \\ 0 & 0 & 0.03 \end{bmatrix} \qquad \bar{B}_1 = \begin{bmatrix} 0 & 0.001 \\ 0.012 & 0 \\ 0.007 & 0.004 \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} 0 & 0.0004 & 0 \\ 0.1 & 0 & 0.12 \\ 0 & 0 & 0 \end{bmatrix} \qquad \bar{B}_2 = \begin{bmatrix} 0 & 0 \\ 0.02 & 0 \\ 0.014 & 0.02 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $-1 \le \varepsilon_i \le 1,\ i = 1, 2$. Note that the above system is unstable in one vertex. For $R = I_m$, $Q = I_n$ the results are:

$$K = \begin{bmatrix} -1.4944 & -5.269 \\ 0.7519 & 1.1244 \end{bmatrix}$$

and the value of the cost function is $J \le x(0)^T 325 x(0)$. A feasible solution was obtained. The shift of spectral radii for individual vertices is summarized in Table 1. It can be seen that the proposed static output feedback control stabilizes the unstable uncertain system and shifts spectral radii of all vertices of the model polytope towards the origin. The latter can be interpreted as an increase of stability robustness and performance quality.

Table 1. The shift of spectral radii for individual vertices

Spectral radius	vertex 1 (-1,-1)	vertex 2 (-1,1)	vertex 3 (1,-1)	vertex 4 (1,1)
open loop	0.9748	0.8499	1.007	0.8164
	↓	↓	↓	↓
closed loop	0.8675	0.7117	0.8814	0.6971

4 CONCLUSION

The output feedback control design for uncertain discrete LTI system has been studied. Some critical points in this problem have been discussed and possible improvements of the present state of the art outlined. An LMI approach has been used to find the output feedback gains from the formulated necessary and sufficient conditions. The proposed solution has been implemented in MATLAB and illustrated on an example.

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