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COMMUNICATIONS

NOVEL APPROACH TO DERIVATION OF DIFFERENTIAL EQUATION OF REFLECTION AT CONFORMAL ANGLE DEFINITION

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The paper presents a novel approach to deriving the differential equation of reflection which is a starting point for mathematical modelling of reflectors. The differential equation of reflection is based on well-known formulae and principles from optics, but with slight modifications for the lighting technology practice, *ie* for calculation of reflectors for luminaries. The approach presented in the paper is built on a direct expression of the situation in the meridian plane and considered under assumption of conformal angle definition for description of both incident and reflected rays, which differs from previous schemes.

Keywords: reflector, luminance, mathematical modelling, reflecting surface, forming curve, differential equation of reflection

1 INTRODUCTION

The methods for mathematical modelling of reflectors can be basically divided into two approaches. Analysis of a reflector is based upon assumption that the reflecting surface is described by a mathematical expression, the luminous intensity distribution curve (LIDC) of the light source is known and LIDC of the luminaire (exactly said — a combination of light source with reflector) is what one is searching for. In opposite to this, synthesis of a reflector gains on assumption that LIDC of a luminaire is also known because this is given by common needs of lighting technology. In this case, the shape of the reflector is unknown.

Both analysis and synthesis are theoretically based on the differential equation of reflection. It is necessary to note that some publications refer to this equation using the term "differential equation of reflector" what, in fact, is a misinterpretation. The equation does not describe reflector's shape but the relation between the angles of rays incident upon the reflector and reflected from its surface. From optics it is known that both of these angles with respect to the normal of the surface in the point where reflection takes place mus be equal. In lighting technology it is obvious to measure these angles with respect to the y-axis taking into account that the light ray is emitted from a light source placed in the origin of coordination system, which is also the origin for the reflector forming curve and is usually called the "optical middle".

Many models assume the light source to be of a point-type. The point source is an abstraction, indeed. Light bulbs or short-arc discharge lamps may be used with neglection of differences in measures between an infinitely small point source and a real filament or arc. But for lamps with tubes covered by luminophor this approach is unusable. Thus, the surface of such a lamp must be fragmented to small points with defined luminance indicatrices for every direction.

In the case when the point source cannot be used or for any reason the light source must be shifted from its origin (*eg* to modify resulting LIDCs, which is a very desirable operation), simple modifications to the basic expression of the differential equation can be easily done.

The previously published derivation of the differential equation of reflection for lighting applications (see *eg* [1]) usually begins with the situation in a general space and the law of reflection is applied to an elementary surface of the luminary reflector. The reflecting surface is assumed to be three-dimensional and of arbitrary shape, light rays are described with spatial vectors. The situation is depicted in Fig. 1.

The law of reflection for this characteristic arrangement of the reflector and light source is applied in the form

$$\boldsymbol{n}_0 \cdot \mathrm{d}\boldsymbol{r} = 0 \tag{1}$$

and after derivation it assumes a generally valid form as follows

$$\frac{\mathrm{d}r}{r} = \frac{\sin(\Psi-\beta)\mathrm{d}\Psi + \sinh\frac{\ln\operatorname{cotg}(\varphi/2)+\ln\coth(\alpha/2)}{\sin\varphi}\mathrm{d}\varphi}{\cos(\Psi-\beta) - \cosh\frac{\ln\operatorname{cotg}(\varphi/2)+\ln\coth(\alpha/2)}{\sin\varphi}\mathrm{d}\varphi}\,. \tag{2}$$

For practical application eq. (2) is too complicated, moreover, in most cases its direct application is not necessary. Shortcomings of a direct use of Eq. (2) are:

1. Reflecting surfaces used for luminaries are in most cases formed by rotation of a planar curve (in xy coordinating system) around the y-axis or by shifting of

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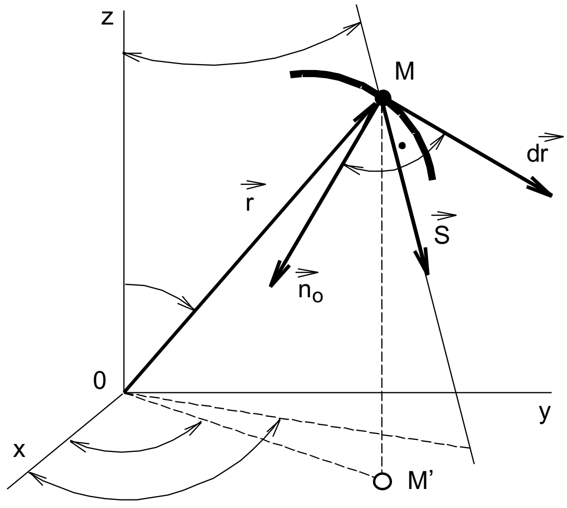

Fig. 1. Scheme of reflection in space

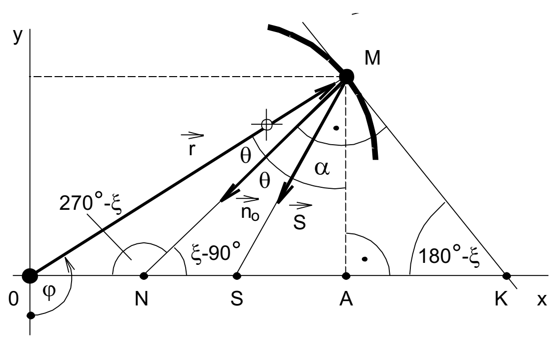

Fig. 2. Scheme of reflection in plane

a planar curve along the z-axis. The first case results in formation of rotary symmetric reflectors (*eg* for incandescent lamps, short-arc discharge lamps or special cases of discharge lamps with elliptic bulb) and the second case results in formation of rotary un-symmetric reflectors (fluorescent lamps and possibly medium-arc linear burner discharge lamps). In addition to this, some reflectors can be formed by shifting one planar curve (in xy system) along another planar curve (in yz system), usually used for discharge lamps with an elliptical bulb in a horizontal position. It is clear that when the reflecting surface can be reduced to a planar forming curve, calculation and simulation can be performed for this curve only, which may significantly simplify the necessary formulae.

2. In catalogues of luminaries it is sufficient to provide LIDCs in a chosen number of planes — one for rotary symmetric reflectors (in the so-called C 0–180° measuring plane) and at least in two planes (usually C 0–180° and C 90–270°) for rotary un-symmetric planes. Of course, for calculation of lighting parameters using LIDCs in the role of input, detailed LIDCs in all planes with 5° step must be contained in databases. But even in this case it is much simpler to modify the calculations in the plane to the situation in the space taking into account only specific ways of formation of the reflecting surface, not necessarily for arbitrary formation.
3. There is a discrepancy reading the ray angles: the ray passing from the light source to the reflector is measured from the zenith, nevertheless the reflected ray is measured from the nadir. In lighting technology LIDC is always measured from the reflector's output area, *ie* from the nadir, both for light the source and the luminary.

Thus, the classical approach leads to simplification of Eq. 2 by means of overcomplicated substitutions and hyperbolic functions to its expression in plane. The final form is then used for mathematical modelling:

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = r \cdot \mathrm{tg}\,\frac{\varphi \mp \alpha}{2}\,. \tag{3}$$

This paper aims to derive the differential equation of reflection directly for a situation in the plane, with planar curves and 2D vectors. This approach will serve for educational purposes because it is much more comprehensive. Another task is oriented to gain conformity between the angles of incident φ and reflected rays α as these must be read from the nadir anti-clockwise like LIDC. Both aims will be solved simultaneously by drawing an appropriate starting scheme of an arbitrary planar forming curve.

2 THEORETICAL PART

Situation is clearly depicted in Fig. 2. Let M be an arbitrary point of the reflector forming curve. The point source is located in the origin O of the system of coordinates. The light ray is directed along vector $\boldsymbol{r}$. This ray is then reflected in point M on the reflector surface and according to the law of reflection it is passing along vector $\boldsymbol{s}$. Assuming a curve described by function $y = f(x)$, the angle ξ is given by expression $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{tg}\,\xi$ and can be easily derived from the angles of incident and reflected rays. ¿From triangles $\triangle MNO$ and $\triangle MNA$ it follows that

$$\xi - \varphi + \alpha = 180 - \xi\,. \tag{4}$$

The angle ξ is then equal to

$$\xi = \frac{180 + \varphi - \alpha}{2}\,. \tag{5}$$

Combining (5) with the basic expression for ξ we obtain a differential equation of reflection in Cartesian system of coordinates:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{tg}\Big(\frac{\pi}{2} + \frac{\varphi \pm \alpha}{2}\Big) = -\,\mathrm{cotg}\,\frac{\varphi \pm \alpha}{2} \tag{6}$$

where $(+)$ sign is valid for rays reflected to the optical axis and sign $(-)$ is valid for rays directed away from the optical axis. The expression in brackets has been simplified when taking into account that $\mathrm{tg}(\pi/2 \pm \alpha) = \mp \mathrm{cotg}\,\alpha$. Now, using transformation equations $x = r\sin\varphi$ and $y = -r\cos\varphi$ we obtain corresponding differentials

$$\mathrm{d}x = \mathrm{d}r\sin\varphi + r\cos\varphi\mathrm{d}\varphi$$
$$\mathrm{d}y = -\mathrm{d}r\cos\varphi + r\sin\varphi\mathrm{d}\varphi$$

The ration of both differentials gives the derivative of function $y = f(x)$ with respect to x, from which we try to extract the dependance of $\frac{\mathrm{d}r}{\mathrm{d}\varphi}$ on $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\mathrm{d}r\cos\varphi + r\sin\varphi\mathrm{d}\varphi}{\mathrm{d}r\sin\varphi + r\cos\varphi\mathrm{d}\varphi} \quad \Big|\frac{1}{\mathrm{d}\varphi}\frac{1}{\cos\varphi} \tag{7}$$

$$\mathrm{d}yx = \frac{-\frac{\mathrm{d}r}{\mathrm{d}\varphi} + r\,\mathrm{tg}\,\varphi}{\frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\mathrm{tg}\,\varphi + r} \tag{8}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{r\big(\mathrm{tg}\,\varphi - \frac{\mathrm{d}y}{\mathrm{d}x}\big)}{\mathrm{tg}\,\varphi\frac{\mathrm{d}y}{\mathrm{d}x} + 1}\,. \tag{9}$$

Now we substitute (6) to (9) by which the Cartesian coordinates x and y will be eliminated.

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = r\frac{\mathrm{tg}\,\varphi + \mathrm{cotg}\,\frac{\varphi\pm\alpha}{2}}{-\mathrm{tg}\,\varphi\,\mathrm{cotg}\,\frac{\varphi\pm\alpha}{2} + 1} = r\frac{1 + \mathrm{tg}\,\varphi\,\mathrm{tg}\,\frac{\varphi\pm\alpha}{2}}{\mathrm{tg}\,\frac{\varphi\pm\alpha}{2} - \mathrm{tg}\,\varphi}$$
$$= r\frac{1}{\mathrm{tg}\big(\frac{\varphi\pm\alpha}{2} - \varphi\big)} = r\cdot\mathrm{cotg}\,\frac{\varphi\pm\alpha - 2\varphi}{2}\,.$$

Finally we obtain the differential equation of reflection in its pure form, suitable for analysis of reflectors of luminaries with LIDCs as input and output data:

$$\mathrm{d}r\varphi = -r\,\mathrm{cotg}\,\frac{\varphi\mp\alpha}{2}\,. \tag{10}$$

3 DISCUSSION

Comparison of Eqs. (3) and (10) shows a small difference caused by reversal orientation of angle φ. Equation (10) is reciprocal to (Eq. 3) in terms of signs and fraction. It can be shown (see [2]) that if all the conditions (*ie* orientation of angles) taken for the classical approach remain conserved, the use of the direct derivation of the differential equation of reflection will necessarily lead to Eq. 3 and this is the evidence that the chosen methodology is correct. Moreover, the evidence was in [2] laid down in two manners.

4. CONCLUSION

The result of the work presented within this article is basically equation (10). The novel approach based on expressing the situation in the plane and an the use of conformal orientation of angles is intended to contribute to the theory of lighting technology and to make education and theory adoption more friendly and comprehensive.

The new equation is valuable for computer analysis of reflectors and consequent simulation with ray-tracing procedures, where LIDCs as input and output data are required. Thus, for creation of future software products the new formula (10) instead of the previously used is recommended. The same is valid for further works in the field of theory where the differential equation is applied (see [3], [4]).

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