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ASYMPTOTIC ANALYSIS OF PIECE–WISE UNIFORM PRODUCT POLAR QUANTIZERS

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In this paper, simple and complete asymptotical analysis is given for a piecewise uniform product polar quantizer (PUPPQ) with respect to mean-square error (mse). PUPPQ is based on uniform product polar quantizers. Polar quantizer optimality conditions and all main equations for the number of phase divisions and optimal number of levels for each partition are presented. PUPPQ has complexity implementation between optimal nonuniform polar quantization (NPQ) and uniform product polar quantization (UPPQ). These systems, although not optimal, may have asymptotic performance arbitrarily close to the optimum. The gain of PUPPQ over the optimum uniform product polar quantization is obtained.

K e y w o r d s: asymptotic analysis, piece-wise uniform product polar quantization, optimal mean-square error

1 INTRODUCTION

Quantization is the heart of analog-to-digital conversion. Quantizers play an important role in the theory and practice of modern-day signal processing. Extensive results have been developed on scalar quantization but more on vector quantization. The simplest vector quantization is polar quantization. Polar quantization techniques as well as their applications in areas such as computer holography, discrete Fourier transform encoding, image processing and communications have been studied extensively in the literature. The solution of polar quantization problem is determination of decision and output levels for every magnitude and phase value such that the distortion is minimized. The resulting optimal quantizer is nonuniform. A special class of NPQ is a PUPPQ. In previous works that involve polar quantization [1-3], product uniform quantization was considered ($N = M \times L$). Uniform product polar quantization was optimized numerically in [1] and analytically in [2]. However, the analysis in [2] assumed a fixed support region. In paper [3] UPPQ was optimized asymptotically by uniform scalar quantization support region.

One of the most important results in polar quantization is due to Swaszek and Ku who derived the asymptotically optimal nonuniform polar quantization [4]. However, they do not consider the problem of finding the optimal maximal amplitude, the so-called, support region. The support region for scalar quantizers has been found in [5–6] by minimization of the total distortion D, which is a combination of granular (D_g) and overload (D_o) distortions, $D = D_g + D_o$. In paper [7] only granular distortion was examined and although arrangement of points N_i in L partitions was defined, the type of cells and their arrangement within partitions was not considered. Paper [8] is an annex of paper [7], but the imperfection of this paper lies in using cubic cells for partitions and subpartitions. Due to this fact, optimal arrangement of points in a partition cannot be found. The importance of using the optimal density of points (using rectangular cells) for product quantization and Gaussian source is considered in [9, 10]. The goal of this paper is to solve quantization problem in the case of PUPPQ and to find corresponding support region. It is done by analytical optimization of the granular distortion and numerical optimization of the total distortion. We improve the cell size and use more optimal cell division in each partition. More precisely, our quantizer divides the input plane into L regions and every region is further subdivided into L_i $(1 \le i \le L)$ subregions. i-th partition in the signal plane is allowed to have M_i $(1 \le i \le L)$ cells in the phase quantizer. We perform two-steps optimization: 1) distortion optimization (D_i) in every partition under the constraint $L_i \times M_i = N_i$ and 2) optimization of the total granular distortion $D_g = \sum_{i=1}^{L} D_i$ which achieves the optimal number of points N_i on each subpartition under the constraint $\sum_{i=1}^{L} N_i = N$. We give the construction procedure and present the Gaussian source example.

2 DESCRIPTION AND OPTIMIZATION

In polar quantization, a vector $(x_1\ x_2)$ is quantized in terms of its magnitude and phase.

This is natural for circularly symmetric densities such as independent identically distributed Gaussian random variables.

We define an N-points product polar quantizer Q $(L \times M)$ as a mapping $Q\colon R^2 \to C$ where R^2 is the

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real two dimensional space and

$$C \equiv \{y_1 = (m_1, \psi_{1,1}, \ldots, y_M = (m_1, \psi_{1,M}), \ldots, y_{N-M+1} = (m_L, \psi_{L,1}, \ldots, y_N = (m_L, \psi_{L,M})\}$$

is the output set or codebook with size $|C| = N$. The output vectors $\boldsymbol{y}_i$ are sometimes referred to as output points, or reproduction vectors. Associated with every N point quantizer is a partition of the real space R^2 into N cells R_i, for $i = 1, \ldots, N$. The i-th cell is given by $R_i = \{x \in R^2 : Q(x) = y_i\}$, which is inverse image of y_i under Q. From this definition it follows that $\bigcup_i R_i = R^2$ and $R_i \cap R_j = 0$ for $i \neq j$. A cell that is unbounded is called an overload cell. Each bounded cell is called a granular cell. Together all of overload (granular) cells are called the overload region (granular region). The most important results in polar quantization are ascribed to Swaszek and Ku who derived the asymptotically optimal unrestricted polar quantization (NPQ) [4]. The nonlinear compressor characteristic is used in paper [4]. Although the smooth and differentiable compressor characteristic is convenient for mathematical manipulations, there are problems of accurately implementing analog nonlinearities [11]. Today's technology allows uniform quantizers or piecewise linear compressor characteristics implementation. PUPPQ can approximate smooth curves of the nonlinear compressor characteristics. A piecewise uniform quantizer range consists of several segments, each containing several quantization cells and output points corresponding to a uniform quantizer. Different segments, however, may have different step-sizes. In this paper, we give the simplest piecewise uniform quantization and show that it has approximately same performances as NPQ but it is much simpler for application.

The transformed probability density function for the Gaussian source is [4]

$$f(r,\phi) = \frac{1}{2\pi\sigma^2} \cdot re^{\frac{-r^2}{2\sigma^2}} = \frac{f(r)}{2\pi}\,.$$

Without losing generality, the variance is assumed to be $\sigma^2 = 1$. Let consider PUPPQ of L partitions, each partition containing L_i subpartitions. In order to minimize the total distortion we proceed as follows: magnitude partition decision levels and reconstruction subpartition levels are given as

$$\begin{aligned} r_i &= (i-1)\Delta\,,\ 1 \le i \le L\,,\ r_{L+1} = r_{\max}\,, \\ r_{i,j} &= r_i + (j-1)\Delta/L_i\,,\ 1 \le i \le L\,,\ 1 \le j \le L_i + 1\,, \\ &\qquad r_{L,L+1} = r_{L+1} + r_{\max}\,, \\ m_{i,j} &= r_i + (j-1/2)\Delta/L_i\,,\ 1 \le i \le L\,,\ 1 \le j \le L_i\,, \end{aligned}$$

where $\Delta = r_{\max}/L$ Let $\phi_{i,j,k}$ be phase decision level, and let $\psi_{i,j,k}$ be k-th phase reconstruction level for the i-th partition and j-th subpartition. Then $\phi_{i,j,k} = (k-1)2\pi/M_i$, $1 \le k \le M_i + 1$ and $\psi_{i,j,k} = (2k-1)\pi/M_i$, $1 \le k \le M_i$.

The distortion is a sum of granular and overload distortion $D = D_g + D_o$:

$$D = \sum_{i=1}^{L} D(i) + D_o = \frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{L_i}\sum_{k=1}^{M_i} \int_{\phi_{i,j,k}}^{\phi_{i,j,k+1}} \int_{r_{i,j}}^{r_{i,j+1}} \big[r^2 + m_{i,j}^2 - 2rm_{i,j}\cos(\phi - \psi_{i,j,k})\big] + \frac{f(r)}{2\pi}\,\mathrm{d}r\mathrm{d}\phi \\ \frac{1}{2}\sum_{k=1}^{M_L} \int_{\phi_{L,L_L,k}}^{\phi_{L,L_L,k+1}} \int_{r_{\max}}^{\infty} \big[r^2 + m_{L,L_L}^2 - 2rm_{L,L_L}\cos(\phi - \psi_{L,L_L,k})\big]\frac{f(r)}{2\pi}\,\mathrm{d}r\mathrm{d}\phi\,, \quad (1)$$

where $D(i) = \frac{1}{2}\big(\frac{1}{12}\frac{\Delta^2}{L_i^2}P_i + \frac{\pi^2 I_i}{3M_i^2}$, with $M_i = N_i/L_i$ and $I_i = \int_{r_i}^{r_{i+1}} r^2 f(r)\,\mathrm{d}r$. After integration over ϕ and the reordering, D becomes

$$D = \sum_{i=1}^{L} \frac{1}{2}\Big(\frac{\Delta_i^2}{12}P_i + \frac{\pi^2 I_i}{3M_i^2}\Big) + \frac{1}{2}\int_{r_{\max}}^{\infty} \big(r - m_{L,L_L}\big)^2 f(r)\,\mathrm{d}r + \frac{\pi^2}{6}\int_{r_{\max}}^{\infty} \frac{r^2 f(r)}{M_L^2}\mathrm{d}r$$

(we use $\int_{r_{i,j}}^{r_{i,j+1}} rf(r)\,\mathrm{d}r \approx m_{ij}f(m_{ij})\Delta_i$, $\Delta_i = \frac{r_{i+1}-r_i}{L_i}$ and $frac\sin(x)x \approx 1 - \frac{1}{6}x^2$). The i-th partition distortion is

$$D_i = \frac{1}{2}\Big(\frac{1}{12}\frac{(r_{i+1}-r_i)^2}{L_i^2}P_i + \frac{\pi^2 I_i L_i^2}{3N_i^2}\Big). \quad (2)$$

After solving $\frac{\partial D_i}{\partial L_i} = 0$ we obtain

$$L_{iopt} = \sqrt[4]{\frac{(r_{i+1}-r_i)^2 P_i N_i^2}{4\pi^2 I_i}} \quad (3)$$

and $M_i = N_i/L_{iopt} = \sqrt[4]{\frac{4\pi^2 I_i N_i^2}{(r_{i+1}-r_i)^2 P_i}}$, that gives

$$D_i = \frac{1}{6}\frac{(r_{i+1}-r_i)\pi\sqrt{P_i I_i}}{N_i} \quad (4)$$

minimum distortion in i-th partition D_i^{opt} because of $\frac{\partial^2 D_i}{\partial L_i^2} > 0$. Substituting (4) in equation for granular distortion we obtain

$$D_g = \frac{1}{6}(r_{i+1}-r_i)\pi\sum_{i=1}^{L}\frac{\sqrt{P_i I_i}}{N_i}\,. \quad (5)$$

The optimization of granular distortion (5) can be formulated in this way: we use equation $J = D_g + \lambda\sum_{j=1}^{L_i} N_i$ where λ represents Lagrangian multiplier and after solving $\frac{\partial J}{\partial N_i} = 0$ under constraint $\sum_{j=1}^{L} N_i = N$ we obtain

$$N_i = \frac{N\sqrt[4]{P_i I_i}(r_{i+1}-r_i)^2}{\sum_{j=1}^{L}\sqrt[4]{P_j I_j}(r_{j+1}-r_j)^2}\,. \quad (6)$$

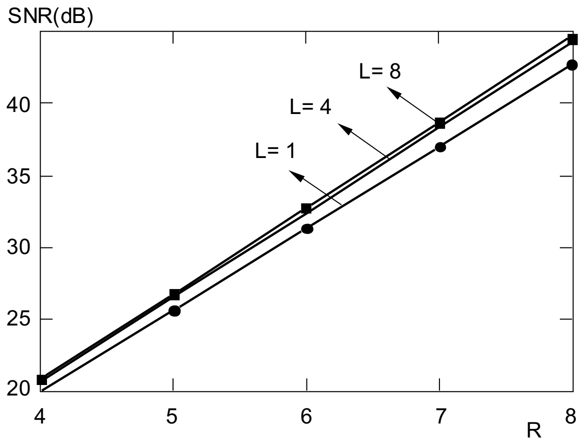

Fig. 1. PUPPQ performances (SNR) versus rate for different numbers of partitions..

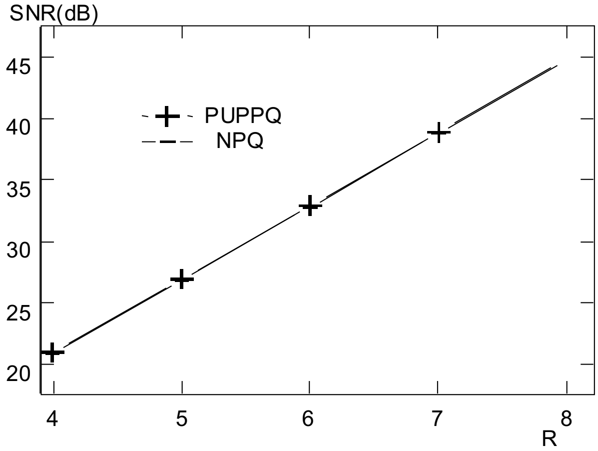

Fig. 2. PUPPQ performances (SNR) versus rate for $L = 8$ compared to theoretical NPQ bound [4].

Table 1.

$L=8$	$R=4$ $r_{\max}=3.7$		$R=6$ $r_{\max}=4.47$		$R=8$ $r_{\max}=5.05$	
i	L_i	M_i	L_i	M_i	L_i	M_i
1	2	6	9	41	40	176
2	2	15	9	91	39	383
3	2	21	8	122	33	508
4	1	47	6	141	25	529
5	1	44	5	115	17	468
6	1	36	3	105	11	342
7	1	27	2	71	6	237
8	1	18	1	54	3	145

Table 2.

$L=4$	$R=4$ $r_{\max}=3.5$		$R=6$ $r_{\max}=4.3$		$R=8$ $r_{\max}=4.95$	
i	L_i	M_i	L_i	M_i	L_i	M_i
1	4	13	46	27	84	290
2	4	23	16	112	68	427
3	3	25	10	85	37	283
4	2	18	5	42	14	119

These are optimal values for N_i because $D_g(N_1, \ldots, N_L)$ is convex function and constraint $\sum_{j=1}^{L} N_i = N$ forms convex set (Hessian matrix are positive semidefinite). Finally, granular distortion becomes

$$D_g = \frac{1}{6}\frac{\pi}{N}\Big(\sum_{i=1}^{L} \sqrt[4]{P_i I_i (r_{i+1} - r_i)^2}\Big)^2 . \qquad (7)$$

The exact optimal value for $r_{\max}$ is obtained repeating our optimization method for different $r_{\max}$ and choosing the values for which $D = D_g + D_o$ is minimal.

3 NUMERICAL ANALYSIS AND RESULTS

As an illustration of the PUPPQ performance, we show the signal-to-quantization noise ratio $SNR = 10\log(1/D)$ as a function of the number of bits per sample R. We will make analysis for $r_{i+1} - r_i = const$. In order to see advantages of PUPPQ we performed numerical calculations of total distortion for $L = 1, 4, 8$ and rates $R = (4\text{–}8)$ bits/sample. For $L = 8$, PUPPQ performances is up to 2 dB (for $R = 8$) in regard to optimal product polar quantization. (Fig. 1).

Application of NPQ requires nonlinear compressor, expander and lookup table of number phase divisions and decision levels and reconstruction levels (r_i, m_i). For $R = 8$ lookup table has $((r_i, m_i),\ M_i,\ 1 \le i \le L;\ L = \sqrt{N/2} = 181)$ 543 elements, while PUPPQ for $L = 8$ requires: lookup table of 18 memory elements $((L_i, M_i),\ L,\ r_{\max})$ and no nonlinear compressor.

For $L = 8$ and rates $R = (4, 6, 8)$, optimal integer values of (L_i, M_i) are given in Table 1.

For $L = 4$ and rates $R = (4, 6, 8)$, optimal integer values of (L_i, M_i) are given in Table 2. For $R = 8$ lookup table has $((r_i, m_i),\ M_i,\ 1 \le i \le L;\ L = \sqrt{N/2} = 181)$ 543 elements, while PUPPQ for $L = 4$ requires: lookup table of 10 memory elements $((L_i, M_i),\ L,\ r_{\max})$ and no nonlinear compressor.

By exceeding L better performances can be achieved but complexity becomes greater. Theoretical bound for the optimal NPQ is $D = 2\pi/(3N) = 2.09N^{-1}$ which is for only 0.17 dB worse than for the case of optimal two-dimensional vector quantization $D = 10\pi/(9\sqrt{3}N) = 2.016N^{-1}$ [4, 9]. The comparison of PUPPQ $(L = 8)$ with theoretical distortion is shown on Fig. 2. If we compare our results for $R = 4$ $(L = 8)$ to those from paper [8] gain is $0.375,$dB. Greater gain can be obtained for greater L and R.

4 CONCLUSION

In this paper the simple asymptotical analysis is given for piece-wise uniform polar quantizer construction for Gaussian source. For the fixed number of partitions, we give equations for optimal number of levels L_i and optimal number of phase divisions M_i. We also give optimal number of points N_i for each partition, while optimal support region is obtained numerically. Application of NPQ requires nonlinear compressor, expander and lookup table of number phase divisions and decision levels and reconstruction levels (r_i, m_i). The PUPPQ method achieves nearly theoretical NPQ performances with less complexity (lookup table contains much less memory elements $((L_i, M_i),\ L,\ r_{\max})$ and hasn't nonlinear compressor).

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