Journal of ELECTRICAL ENGINEERING, VOL. 52, NO. 5-6, 2001, 154–157

COMMUNICATIONS

APPROXIMATED REPRESENTATION OF IMAGES BY SINGULAR VALUE DECOMPOSITION

Igor Mokriš [*] — Ľubomír Semančík [**]

The paper deals with approximated representation of images by Singular Value Decomposition (SVD). SVD is based on the computation of singular values of an image matrix. Since the number of singular values can be decreased, the exactness of image representation decreases too, but the error of representation of SVD is minimal in relation to other linear orthogonal transformations. Therefore SVD is a deterministic optimal transformation. As an example, the Slovak banknote in a raster of 408×805 pixels is used and exactness and error of banknote representation is evaluated.

K e y w o r d s: Singular value decomposition (SVD), approximated representation of images

1 INTRODUCTION

For representation of images linear orthogonal transformations are used. From many kinds of linear orthogonal transformations Singular Value Decomposition (SVD) is very often used because its error of image representation is minimal in relation to the other linear orthogonal transformations [1, 2, 5, 7-9, 11, 12].

SVD enables description of images by decomposition of its matrices into a vector of singular values $\mathbf{S}$ and eigenvectors $\mathbf{U}$ and $\mathbf{V}$. Eigenvectors $\mathbf{U}$, $\mathbf{V}$ and a vector $\mathbf{S}$ contain the whole information about the image. But, substantial information about the image is in vector $\mathbf{S}$ and because of it vector $\mathbf{S}$ can be used for representation of images.

Exactness and error of image representation are influenced by a number of singular values, which is used for representation of images. The number of singular values for representation of images can be decreased. This causes reduction of image representation. The quality of image representation can be evaluated by exactness and error of image representation, defined by truncation error, which is represented by the truncated singular values. When the number of singular values decreases, the exactness of image representation decreases too, but truncation error of approximated representation of images by SVD is minimal in comparison to the other linear orthogonal transformations. Therefore, from the point of truncation error, SVD is a deterministic optimal for approximated representation of images. The possibility to diminish the number of singular values of images for image representation offers the idea to use the singular values of images instead of images, which means large saving of information during processing and transmission of images.

There are many modifications of SVD. One of them is Approximated Singular Value Decomposition (ASVD), which is based on decreasing the number of singular values for representation of images [5, 7].

2 APPROXIMATED SINGULAR VALUE DECOMPOSITION

SVD for image representation uses decomposition of an image matrix into singular values [1, 4–6, 11, 12]. The decomposition of an image matrix $\mathbf{F} = [f_{i,j}]$; $i = 1, 2, \dots, M$; $j = 1, 2, \dots, N$ is given by the form

$$\mathbf{F} = \mathbf{U} \cdot \mathbf{S} \cdot \mathbf{V}^\top \tag{1}$$

where $\mathbf{U} = [u_{i,j}]$ is the matrix of orthonormal row-oriented eigenvectors $\mathbf{F} \cdot \mathbf{F}^\top$, $V = [v_{i,j}]$ is the matrix of orthonormal column-oriented eigenvectors $\mathbf{F}^\top \cdot \mathbf{F}$ and $\top$ is the matrix transpose operator.

$\mathbf{S} = [s_{i,j}]$ is a diagonal matrix of singular value decomposition (1) defined by

$$s_{ij} \geq 0\,,\ i = j\,; \quad s_{ij} = 0\,,\ i \neq j\,. \tag{2}$$

For elements $s_{i,j}$ the following relations hold

$$s_{i,j} \geq 0\,,\ i = j\,, \quad \mathbf{S} = \mathrm{diag}\,[s_1, s_2, \dots, s_N]\,. \tag{3}$$

The elements α_i are eigenvalues, computed by a determinant of the characteristic equation [3]

$$\left|\mathbf{F}^\top \cdot \mathbf{F} - \alpha \cdot \mathbf{I}\right| = \mathbf{0} \tag{4}$$

where $\mathbf{I}$ is a unit matrix with dimension $N \times N$.

[*] Department of Information Systems, Faculty of Finance, Matej Bel University, 974 01 Banská Bystrica, Slovakia, E-mail: mokris@financ.umb.sk

[**] Department of Informatics and Computer Sciences, Faculty of Communication Technology and Information Systems, Military Academy, 031 19 Liptovský Mikuláš, Slovakia, E-mail: semancik@valm.sk

ISSN 1335-3632

Fig. 1. The Slovak banknote

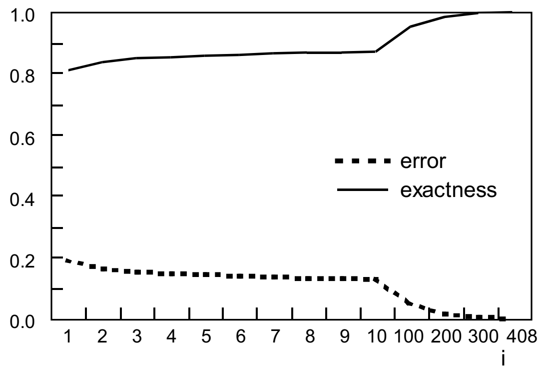

Fig. 2. Graphical representation of exactness and error of banknote approximation

Table 1. The exactness and error of banknote approximation for number $K = 1, 2, \ldots, 10, 100, 200, 300$ and 408 singular values

i	Singular Values s_i	Exactness ξ	Error ε
1	68750.078	0.811	0.189
2	12349.400	0.838	0.162
3	7664.945	0.848	0.152
4	5689.450	0.853	0.147
5	5092.428	0.858	0.142
6	4532.229	0.861	0.139
7	4495.983	0.865	0.135
8	4023.822	0.867	0.133
9	3519.830	0.870	0.130
10	3441.027	0.872	0.128
100	1724.309	0.950	0.050
200	1095.712	0.983	0.017
300	726.024	0.995	0.005
408	427.955	1.000	0.000

When matrices $\mathbf{U}$ and $\mathbf{V}$ are expressed as column vectors

$$\begin{aligned}\mathbf{U} &= [u_1, u_2, \ldots, u_N]\\ \mathbf{V} &= [v_1, v_2, \ldots, v_N]\end{aligned} \tag{5}$$

then the matrix $\mathbf{S}$ is expressed by submatrices $\mathbf{S}_i$ and decomposition (1) will be in the form

$$\mathbf{F} = \sum_{i=1}^{N} u_i \cdot s_i \cdot v_i^{\top} = \sum_{i=1}^{N} \mathbf{F}_i \,. \tag{6}$$

The expression $\mathbf{F}_i = u_i \cdot s_i \cdot v_i^{\top}$ is the outer product of the image matrix decomposition $\mathbf{F}$ into singular values and represents an elementary subimage $\mathbf{F}_i$. The elementary subimage $\mathbf{F}_i$ is a projection of image $\mathbf{F}$ in direction of eigenvector u_i in N-dimensional orthogonal space of eigenvectors U^N. The singular value s_i represents the oriented energy of subimage $\mathbf{F}_i$ [1].

The singular values are positive real numbers. If singular values are in a monotonically decreasing order and the next relation holds

$$s_1 \ge s_2 \ge \cdots \ge s_K > s_{K+1} \approx s_{K+2} \approx \cdots s_N \approx 0 \tag{7}$$

then for decomposition (1) an approximated relation holds

$$\mathbf{F} = \sum_{i=1}^{N} u_i \cdot s_i \cdot v_i^{\top} \approx \sum_{i=1}^{K} u_i \cdot s_i \cdot v_i^{\top} = \mathbf{F}_K \,. \tag{8}$$

Relation (8) expresses the approximated representation of an image by SVD. The error of approximation of an image by Euclidean metric is given as

$$\|\mathbf{F} - \mathbf{F}_K\|^2 = \sum_{i=K+1}^{N} s_i^2 \tag{9}$$

and is minimal in relation to the other linear orthogonal transformations and therefore SVD is deterministic optimal. Because the singular values s_i for $i = K+1, \ldots, N$ represent truncated singular values, the approximation error is the truncation error of an image representation.

Number K follows from the needed approximated representation of image $\mathbf{F}$. This modification of SVD is called the Approximated SVD (ASVD) [7].

3 APPROXIMATED REPRESENTATION OF IMAGES BY SINGULAR VALUE DECOMPOSITION

With aim to show the approximated representation of images by ASVD the Slovak banknote (Fig. 1) was chosen. The Slovak banknote was figured as an image in raster 408×805 pixels in 256 pseudo-colour levels obtained by a scanner.

After decomposition (1) of the Slovak banknote (Fig. 1) 408 singular values $s_i \neq 0$ at Single format by Pascal were obtained. By (7) and (8) the number of singular values was decreased and exactness of approximation was evaluated by equation

$$\xi = \frac{\sum_{i=1}^{K} s_i^2}{\sum_{i=1}^{N} s_i^2} \tag{10}$$

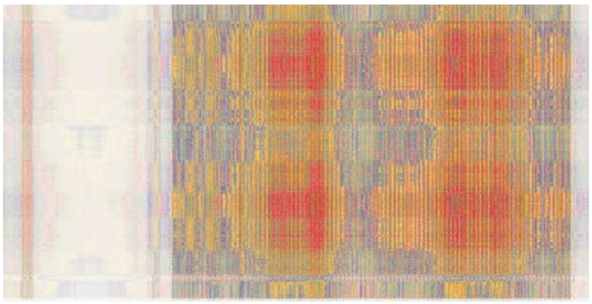

Fig. 3. Slovak banknote represented by 1 singular value

Fig. 4. Slovak banknote represented by 10 singular values

Fig. 5. Slovak banknote represented by 100 singular values

Fig. 6. Slovak banknote represented by 200 singular values

Fig. 7. Slovak banknote represented by 300 singular values

Fig. 8. Slovak banknote represented by 408 singular values

and error of approximation by equation

$$\varepsilon = \frac{\sum_{i=K+1}^{N} s_i^2}{\sum_{i=1}^{N} s_i^2}\,. \tag{11}$$

For evaluation of exactness and error of banknote approximation the computation for number $K = 1, 2, \ldots, 10$, 100, 200, 300 and 408 singular values was performed. The results of evaluation are stored in Tab. 1 and figured by graph in Fig. 2.

It follows from the results that the exactness of banknote approximation starts from 0.811 and reaches 1 and the error of banknote approximation starts from 0.189 and reaches 0. For every singular values s_i the next equation can be satisfied

$$\xi + \varepsilon = 1 \tag{12}$$

and the choice of the number K of singular values for approximate representation of images depends on real application.

The banknote (Fig. 1) consists of the $408 \times 805 =$ 328 440 pixels. This number of pixels can be represented by 408 singular values and the corresponding number of eigenvectors $\mathbf{U}$ and $\mathbf{V}$ by relation (1). If the number of singular values is decreasing, the exactness of image approximation decreases too but it can be sufficiently large. Examples of approximated representation of banknote (Fig. 1) in dependence on the number of singular values K are shown in Fig. 3 to 8.

The payment for the approximation representation of images by singular values is the computation time needed for computation of singular values and eigenvectors $\mathbf{U}$ and $\mathbf{V}$. Because the algorithm for computation of singular values is iterative, the time of singular value computation for approximated representation of images can be minimised [4, 6, 10, 12].

As it was mentioned above, the approximation representation of banknote by 1st singular value is large. It follows from the fact that 1st singular value is larger in relation to the other singular values. Due to this fact the computation of the 1st singular value and corresponding exactness and error of approximation was performed for all Slovak banknotes (face and back) 5000.- Sk, 1000.- Sk, ..., 20. -Sk (Tab. 2). The exactness and error of approximation is in "adequate regions" for all banknotes.

Table 2. Representation of Slovak banknotes by the 1st singular value

Banknote	Singular Values s_1	Exactness ξ_1	Error ε_1
5000 face	72882.305	0.815	0.185
5000 back	70218.563	0.783	0.217
1000 face	72939.227	0.835	0.165
1000 back	72070.7426	0.816	0.184
500 face	68320.063	0.820	0.180
500 back	70292.328	0.793	0.207
200 face	69030.578	0.794	0.206
200 back	70419.656	0.804	0.196
100 face	68750.078	0.811	0.189
100 back	65532.266	0.759	0.241
50 face	65022.398	0.813	0.187
50 back	63385.715	0.781	0.219
20 face	70465.719	0.827	0.173
20 back	68884.781	0.813	0.187

CONCLUSION

Decreasing the number of singular values decreases the exactness and increases the error of approximate representation of images. It follows from the analysis of approximated representation of images that by a suitably chosen number of singular values the exactness and error of approximation can be in satisfactory regions. The criteria for the choice of the number singular values follow from real application during processing and transmission of images.

References

[1] ANDREWS, H. C.—PATTERSON, C. L.: Outer Product Expansion and Their Uses in Digital Image Processing, IEEE Trans. on Computer **C-25** No. 2 (1976), 140–148.

[2] ANDREWS, H. C.—PATTERSON, C. L.: Singular Value Decomposition and Digital Image Processing, IEEE Trans. on ASSP **ASSP-24** No. 1 (1976), 26–53.

[3] ANGOT, A.: Applied Mathematics for Electrical Engineers, SNTL, Prague, 1972. (in Czech)

[4] BAI, Z.—DEMMEL, J. W.: Computing the Generalized Singular Value Decomposition, SIAM J. Comput. **14** No. 6 (1993), 1464–1486.

[5] DEPRETTERE, E. F. (Editor): SVD and Signal Processing: Algorithms, Applications and Architectures, Elsevier Science Publishers B.V. (North-Holland), The Netherlands, 1988.

[6] GOLUB, G. H.—REINSCH, C.: Singular Value Decomposition and Least Squares Solutions, Numer. Math. **14** (1970), 403–420.

[7] HANSEN, P. C.: SVD — Theory and Applications. Report No. 84-05, ISSN 0105-4988, Numerisk Institut, Lyngby, Denmark, 1984.

[8] HANSEN, P. C.—NIELSEN, H. B.: Singular Value Decomposition of Images, Third Scandinavian Conference of Image Analysis, Copenhagen, 1983, pp. 301–307.

[9] JAIN, A. K.: Advances in Mathematical Models for Image Processing, Proc. of the IEEE **69** No. 5 (1981), 502–528.

[10] MOKRIŠ, I.: Theory and Application of SVD in Image Processing, ISBN 80-8040-030-X, Military Academy Press, Liptovský Mikuláš, 1995,. (in Slovak)

[11] VACCARO, R. (Editor): SVD and Signal Processing II: Algorithms, Analysis and Applications, Elsevier Science Publishers B.V. (North-Holland), The Netherlands, 1991.

[12] WILKINSON, J. H.: Singular Value Decomposition — Basic Aspects. Numerical Software — Needs and Availability, Academic Press, London, 1978.

Received 13 October 2000

Igor Mokriš (Prof, Ing, CSc) was born in Prešov, Slovak Republic, on July 30, 1948. He received Ing (MSc) degree in technical cybernetics from the Faculty of Electrical Engineering, Technical University, Košice, in 1972 and CSc (PhD) degree from the Faculty of Electrical Engineering, Slovak Technical University, Bratislava, in 1980. Sine 1985 he has been Associate Professor, and from 1997 Professor in technical cybernetics and artificial intelligence at the Department of Informatics and Computers, Military Academy, Liptovský Mikuláš. From 1998 he works at the Department of Information Systems, Faculty of Finance, Matej Bel University, Banská Bystrica. His research interests include signal and image processing, pattern recognition and neural networks.

Ľubomír Semančík (Ing) was born in Poprad, Slovak Republic, on August 10, 1966. He received Ing (MSc) degree in electrical engineering from the Military Academy at Liptovský Mikuláš in 1989 and Dr (PhD) degree from the Military Academy at Liptovský Mikuláš in 1997. Now he works at the Department of Informatics and Computers, Military Academy, Liptovský Mikuláš. His research interests include image processing, pattern recognition and programming theory and applications.