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ROBUST OUTPUT FEEDBACK QUADRATIC CONTROLLER DESIGN

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The paper provides a survey of some recent quadratic stability methods for static output feedback robust controller design for linear continuous-time invariant systems with convex polytopic uncertainty and their mutual comparison. Robust controller design is based on linear matrix inequalities (LMIs) conditions and single Lyapunov functions. The presented quadratic stability methods are compared on numerical examples and randomly generated ones and it is shown which of them provides the least conservative results.

K e y w o r d s: quadratic stability, linear matrix inequality, robust controller, output feedback

1 INTRODUCTION

One of the most challenging problems in control theory remains to find numerically tractable necessary and sufficient conditions for the stabilizability of linear time-invariant (LTI) systems via static output feedback. The output feedback problem is one of the most important open questions of control engineering. In a simple way, the problem can be formulated as follows: for a given complex linear system a robust controller with a static output feedback is to be found which would provide some desirable characteristics to the closed-loop systems, or determine that such a feedback does not exist.

Lyapunov functions have been used in the study of stability of dynamic systems since many years ago. Concerning linear systems with uncertain parameters, the use of Lyapunov functions has allowed important developments that are mainly related to the concept of quadratic stability and convex optimization applied to robust control problems during the last two decades. Thanks to quadratic stability, the stability of a polytope of matrices can be attested by means of a convex feasibility test performed only at the vertices of the uncertainty domain, which means a feasibility test of a set of linear matrix inequalities (LMIs). A drawback of quadratic stability is that it guards against arbitrary fast parameter variations and thus it uses a single Lyapunov function for testing over the whole uncertainty box [3].

The aim of this paper is to provide a numerical comparison of four LMI based quadratic methods for the robust stability design of uncertain linear systems in polytopic domains: the V-F iteration method [6], the two-step method [16] and the linearization method [5, 7, 8, 12, 13]. The V-F iteration algorithm is based on an alternative solution of two convex LMI optimization problems obtained by fixing the Lyapunov matrix or the gain controller matrix. The two-step method does not require iteration of LMI problems, the linearization and Henrion's quadratic method uses an iterative LMI algorithm. The aspect of conservatism is investigated on numerical examples and randomly generated ones. The proposed LMI based algorithms are computationally simple and tightly connected with the Lyapunov function, quadratic stability, guaranteed cost and LQ optimal state feedback design.

2 PROBLEM FORMULATION AND PRELIMINARIES

Consider the following linear continuous time-invariant uncertain system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A}+\delta\mathbf{A})\mathbf{x}(t)+(\mathbf{B}+\delta\mathbf{B})\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \mathbf{C}\mathbf{x}(t)\,, \quad \mathbf{x}(0)=\mathbf{x}_0\end{aligned} \tag{1}$$

where $\mathbf{x}\in R^n$, $\boldsymbol{u}\in R^m$ and $\boldsymbol{y}\in R^l$ are state, control and output vectors, respectively; $\mathbf{A}\in R^{n\times n}$, $\mathbf{B}\in R^{n\times m}$ and $\mathbf{C}\in R^{l\times n}$ are known matrices of appropriate dimensions; $\delta\mathbf{A}$, $\delta\mathbf{B}$ are unknown but norm bounded uncertainties. In the next development the matrix affine type uncertain structure will be used

$$\begin{aligned}&\delta\mathbf{A}=\sum_{j=1}^{p}\varepsilon_j\mathbf{A}_j\,, \quad \delta\mathbf{B}=\sum_{j=1}^{p}\varepsilon_j\mathbf{B}_j\,,\\ &\underline{\varepsilon}_j\le\varepsilon_j\le\overline{\varepsilon}_j\,, \qquad j=1,2,\dots,p\end{aligned} \tag{2}$$

where $\mathbf{A}_j$, $\mathbf{B}_j$ are known matrices; $\varepsilon_j\in\langle\underline{\varepsilon}_j,\overline{\varepsilon}_j\rangle$ are uncertain parameters with known lower and upper uncertainty bounds. In general, the polytope characterization of uncertainties results in less conservative controller designs than using other characterizations of uncertainty [3].

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The problem studied in this paper can be formulated as follows. For linear continuous time-invariant system described by (1) a robust static output feedback controller is to be designed for the control algorithm

$$\boldsymbol{u} = \mathbf{FC}\boldsymbol{x}\,, \quad \mathbf{F} \in R^{m\times l} \tag{3}$$

such that the closed loop system

$$\dot{\boldsymbol{x}} = (\mathbf{A} + \mathbf{BFC})\boldsymbol{x} + (\delta\mathbf{A} + \delta\mathbf{BFC})\boldsymbol{x} \tag{4}$$

is stable for all admissible uncertainties described by (2) and simultaneously guaranteeing the suboptimal solution to the performance index

$$J = \int_0^\infty \big(\boldsymbol{x}^\top \mathbf{Q}\boldsymbol{x} + \boldsymbol{u}^\top \mathbf{R}\boldsymbol{u}\big)\mathrm{d}t \tag{5}$$

where $\mathbf{Q} = \mathbf{Q}^\top > 0$ and $\mathbf{R} = \mathbf{R}^\top > 0$ are matrices of compatible dimensions, $\mathbf{Q} \in R^{n\times n}$, $\mathbf{R} \in R^{m\times m}$.

The nominal model of the system (1) is

$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}\,,\\ \boldsymbol{y} &= \mathbf{C}\boldsymbol{x}\,. \end{aligned} \tag{6}$$

The following lemma is well known [11].

LEMMA 1. *Let* $\mathbf{Q} = \mathbf{Q}^\top > 0$. *Matrix* $\mathbf{A}$ *is quadratically stable if and only if there exists matrix* $\mathbf{P} = \mathbf{P}^\top > 0$ *such that the following Lyapunov matrix equation is satisfied*

$$\mathbf{A}^\top\mathbf{P} + \mathbf{PA} + \mathbf{Q} = 0\,. \tag{7}$$

A linear time invariant system is stable if and only if it is quadratically stable. It is possible, however, that *eg* linear polytopic systems can be stable without being quadratically stable [3].

The closed loop polytopic system with output feedback algorithm (3) can be described by the list of its vertices ($N = 2^p$)

$$\dot{\boldsymbol{x}} = \big(\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC}\big)\boldsymbol{x} = \mathbf{A}_{ci}\boldsymbol{x}\,, \quad i = 1,2,\dots,N\,. \tag{8}$$

The linear uncertain system (4) belongs to a convex polytopic set defined as

$$\dot{\boldsymbol{x}} = \mathbf{A}(\alpha)\boldsymbol{x} \tag{9}$$

whereby

$$S := \Big\{\mathbf{A}(\alpha)\colon \mathbf{A}(\alpha) = \sum_{i=1}^N \alpha_i\mathbf{A}_{ci}\,,\ \sum_{i=1}^N \alpha_i = 1\,,\ \alpha_i \ge 0\Big\}. \tag{10}$$

LEMMA 2. *The system represented by (9) is quadratically stable if and only if there is a common Lyapunov matrix* $\mathbf{P} > 0$ *such that*

$$\mathbf{A}_{ci}^\top\mathbf{P} + \mathbf{PA}_{ci} < 0\,, \quad i = 1,2,\dots,N\,. \tag{11}$$

3 OUTPUT FEEDBACK CONTROLLER DESIGN

In this section we present four quadratic stability methods (the V-F iteration method, the two-step method, the linearization method and the Henrion method) to design a static output feedback controller for linear continuous time-invariant systems (1) that ensures the guaranteed cost (5) of the closed loop system.

3.1 V-F iteration method [6]

Inequalities (11) can be extended and modified to the form

$$\begin{aligned} \big(\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC}\big)^\top\mathbf{P} + \mathbf{P}\big(\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC}\big)&\\ + \mathbf{Q} + \mathbf{C}^\top\mathbf{F}^\top\mathbf{RFC} < 0\,, \quad i = 1,2,\dots,N\,.& \end{aligned} \tag{12}$$

In the system of inequalities (12) the positive definite matrix $\mathbf{P}$ and the feedback gain $\mathbf{F}$ are unknown.

If such matrices exist then the polytopic system is quadratically stable and simultaneously matrix $\mathbf{F}$ ensures a minimum value of the quadratic performance criterion (5).

For the functional (5) the following inequality holds

$$\int_0^\infty \big(\boldsymbol{x}^\top\mathbf{Q}\boldsymbol{x} + \boldsymbol{u}^\top\mathbf{R}\boldsymbol{u}\big)\mathrm{d}t < \boldsymbol{x}_0^\top\mathbf{P}\boldsymbol{x}_0 \tag{13}$$

where matrix $\mathbf{P}$ is the solution to inequalities (12) and $\boldsymbol{x}_0$ is the initial condition. With regard to matrices $\mathbf{P}$ and $\mathbf{F}$ a solution to (12) belongs to a class of bilinear matrix inequalities (BMI) and in the case of a convex problem (*eg* when matrix $\mathbf{F}$ is known) to the class of linear matrix inequalities (LMI). In general a modification of non-linear (convex) inequalities to the LMI form employs the Schur complement [3] *ie* for any matrices $\mathbf{D}_{11}$, $\mathbf{D}_{22}$ and $\mathbf{D}_{12}$, where $\mathbf{D}_{11}$ and $\mathbf{D}_{22}$ are symmetric, the following statements are equivalent:

$$\text{a)}\quad \begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix} > 0\,, \tag{14}$$

$$\text{b)}\quad \text{if } \mathbf{D}_{22} > 0\colon \mathbf{D}_{11} - \mathbf{D}_{12}\mathbf{D}_{22}^{-1}\mathbf{D}_{12}^\top > 0\,, \tag{15}$$

$$\text{c)}\quad \text{if } \mathbf{D}_{11} > 0\colon \mathbf{D}_{22} - \mathbf{D}_{12}^\top\mathbf{D}_{11}^{-1}\mathbf{D}_{12} > 0\,. \tag{16}$$

Next we present the solution to (12) by the V-F iteration. Its principle consists in alternately solving two convex LMI optimization problems, where in the first problem we compute the matrix $\mathbf{P}$ for a fixed $\mathbf{F}$ and in the second problem vice versa.

The first task is to find matrix $\mathbf{F}$ for which systems $\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC}$, $i = 1,2,\dots,N$ are all stable. This matrix determines the initial condition for the problem solution.

Algorithm 1

1. $j=1$, $\mathbf{Q}=\mathbf{Q}^\top>0$, $\mathbf{R}=\mathbf{R}^\top>0$ and $\mathbf{F}=\mathbf{F}_0$ (For stable matrices $\mathbf{A}_{vi}$, $\mathbf{F}_0=0$)
2. Using the LMI algorithm compute the matrix $\mathbf{P}_j$ from the following inequalities
$$\big(\mathbf{A}_{vi}+\mathbf{B}_{vi}\mathbf{F}_{j-1}\mathbf{C}\big)^\top\mathbf{P}_j+\mathbf{P}_j\big(\mathbf{A}_{vi}+\mathbf{B}_{vi}\mathbf{F}_{j-1}\mathbf{C}\big)+\mathbf{Q}+\mathbf{C}^\top\mathbf{F}_{j-1}^\top\mathbf{R}\mathbf{F}_{j-1}\mathbf{C}<0\,,\quad i=1,\dots,N\,,\qquad 0<\mathbf{P}_j<\rho\mathbf{I}\,,\ \rho>0 \tag{17}$$
where ρ is a given positive upper bound for the maximal eigenvalue of $\mathbf{P}_j$.
3. For the known matrix $\mathbf{P}_j>0$ compute $\mathbf{F}_j$ using LMI
$$\begin{bmatrix}\big(\mathbf{A}_{vi}+\mathbf{B}_{vi}\mathbf{F}_j\mathbf{C}\big)^\top\mathbf{P}_j+\mathbf{P}_j\big(\mathbf{A}_{vi}+\mathbf{B}_{vi}\mathbf{F}_j\mathbf{C}\big)+\mathbf{Q} & \mathbf{C}^\top\mathbf{F}_j^\top\mathbf{R}\\ \mathbf{R}\mathbf{F}_j\mathbf{C} & -\mathbf{R}\end{bmatrix}<0\,,\ i=1,\dots,N\,. \tag{18}$$
4. Compute $er=\|\mathbf{F}_j-\mathbf{F}_{j-1}\|$.
If $er\le tolerance$ stop, else $j=j+1$ and go to Step 2.

The V-F iteration algorithm is guaranteed to converge, but not necessarily to the global optimum of the problem depending on the starting conditions.

3.2 Two-step method [16]

Consider the following algebraic Riccati inequalities
$$\mathbf{A}_{vi}^\top\mathbf{P}+\mathbf{P}\mathbf{A}_{vi}-\mathbf{P}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}+\mathbf{Q}<0\,. \tag{19}$$
Define $\mathbf{P}=\mathbf{S}^{-1}$. Using the Schur complement formula (14) and (15), inequality (19) is equivalent to the following linear matrix inequality
$$\begin{bmatrix}\mathbf{S}\mathbf{A}_{vi}^\top+\mathbf{A}_{vi}\mathbf{S}-\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top & \mathbf{S}\\ \mathbf{S} & -\mathbf{Q}\end{bmatrix}<0\,,\qquad \gamma\mathbf{I}<\mathbf{S}\,,\ i=1,2,\dots,N \tag{20}$$
where $\gamma\ge 0$ is some non-negative constant.

Inequalities (12) can be extended and modified to the form
$$\mathbf{A}_{vi}^\top\mathbf{P}+\mathbf{P}\mathbf{A}_{vi}+\mathbf{Q}+\mathbf{C}^\top\mathbf{F}^\top\mathbf{B}_{vi}^\top\mathbf{P}+\mathbf{P}\mathbf{B}_{vi}\mathbf{F}\mathbf{C}+\mathbf{C}^\top\mathbf{F}^\top\mathbf{R}\mathbf{F}\mathbf{C}<0\,,\quad i=1,2,\dots,N \tag{21}$$
and after some manipulation
$$-\mathbf{R}+\big(\mathbf{R}\mathbf{F}\mathbf{C}+\mathbf{B}_{vi}^\top\mathbf{P}\big)\mathbf{G}_i^{-1}\big(\mathbf{R}\mathbf{F}\mathbf{C}+\mathbf{B}_{vi}^\top\mathbf{P}\big)^\top<0\,,\quad i=1,2,\dots,N \tag{22}$$
where $\mathbf{G}_i=-\big(\mathbf{A}_{vi}^\top\mathbf{P}+\mathbf{P}\mathbf{A}_{vi}-\mathbf{P}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}+\mathbf{Q}\big)$.

With $\mathbf{P}=\mathbf{S}^{-1}$, inequality (22) can be rewritten using Schur complement as follows
$$\begin{bmatrix}-\mathbf{R} & \mathbf{R}\mathbf{F}\mathbf{C}+\mathbf{B}_{vi}^\top\mathbf{P}\\ \big(\mathbf{R}\mathbf{F}\mathbf{C}+\mathbf{B}_{vi}^\top\mathbf{P}\big)^\top & -\mathbf{G}_i\end{bmatrix}<0\,,\ i=1,2,\dots,N. \tag{23}$$
The algorithm for static output feedback simultaneous stabilization for system (9) with a guaranteed cost (13) using the non-iterative LMI approach is given as follows.

Algorithm 2

1. Select $\mathbf{Q}=\mathbf{Q}^\top>0$ and $\mathbf{R}=\mathbf{R}^\top>0$ and using the LMI based algorithm calculate $\mathbf{S}$ from the inequality (20). $\mathbf{P}=\mathbf{S}^{-1}$.
2. For the known matrix $\mathbf{P}$ compute $\mathbf{F}$ from the inequality (23).

If the solution (20) is not feasible, the polytope system (8) is not simultaneously stabilizable and if (23) is not feasible (the closed loop system (8) is not stable) change $\mathbf{Q}$ and $\mathbf{R}$ or decrease $|\varepsilon_j|$, $j=1,2,\dots,p$.

If solutions (20) and (23) are feasible with respect to $\mathbf{S}$ and $\mathbf{F}$ then the uncertain system (1) is quadratically stable with a guaranteed cost control algorithm $\boldsymbol{u}=\mathbf{F}\boldsymbol{y}$ and $\mathbf{J}^*=\mathbf{x}_0^\top\mathbf{P}\mathbf{x}_0$ is the guaranteed cost for the uncertain closed loop system.

3.3 Linearization method [4, 7, 12, 13].

Inequalities (21) can be modified to the following quadratic matrix inequalities (QMIs)
$$\mathbf{A}_{vi}^\top\mathbf{P}+\mathbf{P}\mathbf{A}_{vi}+\mathbf{Q}-\mathbf{P}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}+\big(\mathbf{F}\mathbf{C}+\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}\big)^\top\mathbf{R}\big(\mathbf{F}\mathbf{C}+\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}\big)<0\,,\quad i=1,2,\dots,N\,. \tag{24}$$
If it is possible to find $\mathbf{P}>0$ and $\mathbf{F}$ satisfying the QMI in equation (24), then a stabilizing static output feedback gain exists. An advantage of this approach to obtain a stabilizing feedback gain $\mathbf{F}$ is that $\mathbf{F}$ is no longer assumed to be a function of the solution $\mathbf{P}$ of a special equation or inequality.

Due to the negative sign in the $-\mathbf{P}\mathbf{B}_{iv}\mathbf{R}^{-1}\mathbf{B}_{iv}^\top\mathbf{P}$ term, equations (24) cannot be simplified to LMI. To accommodate the $-\mathbf{P}\mathbf{B}_{iv}\mathbf{R}^{-1}\mathbf{B}_{iv}^\top\mathbf{P}$ term, we introduce an additional design variable $\mathbf{X}$. By linearization using inequality $\big(\mathbf{X}-\mathbf{P}\big)^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\big(\mathbf{X}-\mathbf{P}\big)\ge 0$ for any $\mathbf{X}$ and $\mathbf{P}$ of the same dimension, we obtain
$$\mathbf{X}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}+\mathbf{P}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X}-\mathbf{X}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X}\le\mathbf{P}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}\,,\quad i=1,2,\dots,N \tag{25}$$
with equality holding for $\mathbf{X}=\mathbf{P}$. By combining inequalities (25) and (24), we obtain a sufficient condition for the existence of static output feedback matrix $\mathbf{F}$ given by
$$\mathbf{A}_{vi}^\top\mathbf{P}+\mathbf{P}\mathbf{A}_{vi}+\mathbf{Q}-\mathbf{X}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}-\mathbf{P}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X}+\mathbf{X}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X}+\big(\mathbf{F}\mathbf{C}+\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}\big)^\top\mathbf{R}\big(\mathbf{F}\mathbf{C}+\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}\big)<0\,,\quad i=1,2,\dots,N\,. \tag{26}$$
Using the Schur complement (14) and (15), inequalities (26) for fixed matrix $\mathbf{X}$ are equivalent to the following LMIs
$$\begin{bmatrix}\mathbf{A}_{vi}^\top\mathbf{P}+\mathbf{P}\mathbf{A}_{vi}+\mathbf{Q}-\mathbf{X}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}-\mathbf{P}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X} & 0\\ \mathbf{F}\mathbf{C}+\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P} & 0\end{bmatrix}+\begin{bmatrix}\mathbf{X}\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X} & \big(\mathbf{F}\mathbf{C}+\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{P}\big)^\top\\ 0 & -\mathbf{R}^{-1}\end{bmatrix}<0\,. \tag{27}$$

These LMIs can be solved by an iterative approach. The LMI problem is convex and can be solved efficiently if a feasible solution exists.

Algorithm 3

1. Select $\mathbf{Q} = \mathbf{Q}^\top > 0$ and $\mathbf{R} = \mathbf{R}^\top > 0$ and choose the initial value of $\mathbf{X}$ for example from the following algebraic Riccati equation
$$\mathbf{A}^\top\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^\top\mathbf{P} + \mathbf{Q} = 0\,. \qquad (28)$$
Set $j = 1$ and $\mathbf{X} = \mathbf{P}$.
2. For the known matrix $\mathbf{X}$ compute $\mathbf{F}$ and $\mathbf{P} = \mathbf{P}^\top > 0$ using matrix inequalities (27).
3. Compute $er = \|\mathbf{X} - \mathbf{P}\|$. If $er \le tolerance$ stop, else $j = j + 1$, $\mathbf{X} = \mathbf{P}$ and go to Step 2.

If the algorithm fails to arrive at a stabilizing solution, we may select another $\mathbf{Q}$ and run the LMI algorithm again. Our numerical experience indicates that the initial choice with $\mathbf{Q} = \mathbf{I}$ always leads to a convergent result.

3.4 Henrion's quadratic method [8]

Consider the following Henrion's lemma for continuous-time systems.

LEMMA 3. *The matrix* $\mathbf{A}_{ci} = \mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC}$ *is robustly stable if there exists a matrix* $\mathbf{E}$ *and a matrix* $\mathbf{P} = \mathbf{P}^\top$ *satisfying the LMI*

$$\begin{bmatrix} \mathbf{E}^\top\mathbf{A}_{ci} + \mathbf{A}_{ci}^\top\mathbf{E} & -\mathbf{E}^\top - \mathbf{A}_{ci}^\top - \mathbf{P} \\ -\mathbf{E} - \mathbf{A}_{ci} - \mathbf{P} & 2\mathbf{I} \end{bmatrix} > 0\,,\ i = 1,2,\dots,N\,. \qquad (29)$$

The linear matrix inequalities (29) can be extended and modified to the form

$$\begin{bmatrix} \mathbf{E}^\top(\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC}) + (\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC})^\top\mathbf{E} - \mathbf{Q} & 0 \\ -(\mathbf{E} + \mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC} + \mathbf{P}) & 0 \end{bmatrix} + \begin{bmatrix} -\mathbf{C}^\top\mathbf{F}^\top\mathbf{RFC} & -(\mathbf{E} + \mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC} + \mathbf{P})^\top \\ 0 & 2\mathbf{I} \end{bmatrix} > 0\,. \qquad (30)$$

Term (1,1) of BMIs (30) can be modified to the following quadratic matrix inequalities

$$\mathbf{A}_{vi}^\top + \mathbf{E}^\top\mathbf{A}_{vi} - \mathbf{Q} + \mathbf{E}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E} + (\mathbf{FC} - \mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E})^\top(-\mathbf{R})(\mathbf{FC} - \mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E}) > 0\,. \qquad (31)$$

By linearization using inequality $(\mathbf{X}-\mathbf{E})^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top(\mathbf{X}-\mathbf{E}) \ge 0$ for any matrix $\mathbf{X}$ and $\mathbf{E}$ of the same dimension, we obtain

$$\mathbf{X}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E} + \mathbf{E}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X} - \mathbf{X}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X} \le \mathbf{E}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E}\,, \quad i = 1,2,\dots,N \qquad (32)$$

with equality holding for $\mathbf{X} = \mathbf{E}$. By combining inequalities (31) and (32), we obtain a sufficient condition for the existence of static output feedback matrix $\mathbf{F}$ and matrix $\mathbf{E}$ given by

$$\mathbf{A}_{vi}^\top\mathbf{E} + \mathbf{EA}_{vi} - \mathbf{Q} + \mathbf{X}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E} + \mathbf{E}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X} - \mathbf{X}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X} + (\mathbf{FC} - \mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E})^\top(-\mathbf{R})(\mathbf{FC} - \mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E}) > 0\,, \quad i = 1,2,\dots,N\,. \qquad (33)$$

Using the Schur complement inequalities (33) for fixed $\mathbf{X}$ are equivalent to the following LMIs

$$\begin{bmatrix} \mathbf{U}_i & (\mathbf{FC} - \mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E})^\top \\ \mathbf{FC} - \mathbf{R}^{-1}\mathbf{B}_{iv}^\top\mathbf{E} & \mathbf{R}^{-1} \end{bmatrix} > 0 \qquad (34)$$

where $\mathbf{U}_i = \mathbf{A}_{vi}^\top\mathbf{E} + \mathbf{E}^\top\mathbf{A}_{vi} - \mathbf{Q} + \mathbf{X}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E} + \mathbf{E}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X} - \mathbf{X}^\top\mathbf{B}_{vi}\mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{X}$ for $i = 1,2,\dots,N$. Applying LMIs (34) to matrix inequalities (30) we obtain the following LMIs

$$\begin{bmatrix} \mathbf{U}_i & (\mathbf{FC} - \mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E})^\top & 0 \\ \mathbf{FC} - \mathbf{R}^{-1}\mathbf{B}_{vi}^\top\mathbf{E} & \mathbf{R}^{-1} & 0 \\ -(\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC} + \mathbf{E} + \mathbf{P}) & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -(\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{FC} + \mathbf{E} + \mathbf{P})^\top \\ 0 & 0 & 0 \\ 0 & 0 & 2\mathbf{I} \end{bmatrix} > 0\,. \qquad (35)$$

These LMIs can be solved by an iterative approach. The LMI problem is convex and can be solved efficiently if a feasible solution exists.

Algorithm 4

1. Select $\mathbf{Q} = \mathbf{Q}^\top > 0$ and $\mathbf{R} = \mathbf{R}^\top > 0$ and choose the initial value of $\mathbf{X}$ for example $\mathbf{X} = \mathbf{A}$. Set $j = 1$.
2. For the known matrix $\mathbf{X}$ compute $\mathbf{F}$ and matrix $\mathbf{E}$ using matrix inequalities (35)
3. Compute $er = \|\mathbf{X} - \mathbf{E}\|$. If $er \le$ tolerance stop, else $j = j + 1$, $\mathbf{X} = \mathbf{E}$ and go to Step 2.

If the algorithm fails to arrive at a stabilizing solution, we may select another $\mathbf{Q}$ and run the LMI algorithm again.

4 EXAMPLES

4.1 Method of evaluation

In this section the properties and power of individual methods presented in Sections 3 have been tested on several examples taken from references and laboratory plants at our department as well as on 50 randomly generated ones. To be able to evaluate the conservatism of each particular method, the term "stability region size" has been adopted. In each tested example, it has been measured in terms of the parameter ε corresponding to

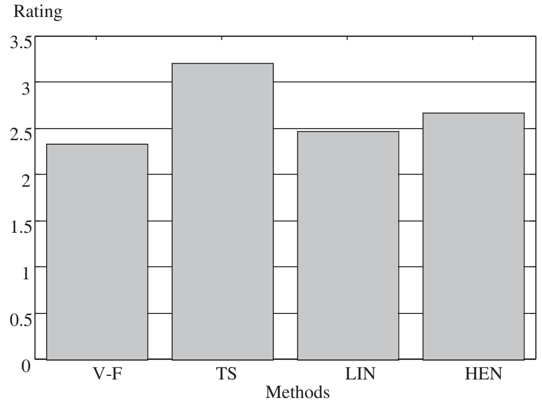

Fig. 1. Result of robust stability evaluation in terms of rating

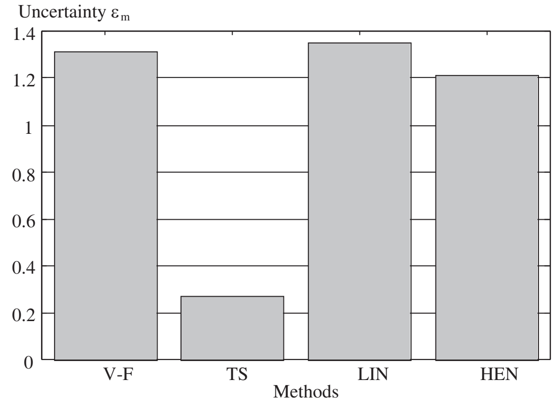

Fig. 2. Result of robust stability evaluation in terms of uncertainty ε_m

Table 1.

Methods	V–F	TS	LIN	HEN
Rating	2.3333	3.2000	2.4667	2.6667
ε_m	1.3117	0.2693	1.3484	1.2108

where the above acronyms have the following meaning:
V-F — V-F iteration method [6], Algorithm 1
TS — two-step method [16], Algorithm 2
LIN — linearization method [4], Algorithm 3
HEN — Henrion's quadratic method [8], Algorithm 4

where rating and maximum uncertainty parameter ε_m have been calculated using the mean values obtained from the 15 examples.

the maximum uncertainty polytope for which the closed loop affine uncertain system with the gain matrix $\mathbf{F}$ still remains stable.

Consider the following closed loop polytopic system with output feedback algorithm (3) described by the list of its vertices ($N = 2^p$)

$$\dot{\mathbf{x}} = \big(\mathbf{A}_{vi} + \mathbf{B}_{vi}\mathbf{F}\mathbf{C}\big)\mathbf{x} = \mathbf{A}_{ci}\mathbf{x}\,, \quad i = 1, 2, \dots, N\,. \tag{36}$$

The closed loop robust stability problem can be extended, as a question may arise about "how robust" the closed loop under the considered controller is:

What is the maximum range for uncertainty parameters such that the closed loop affine uncertain system with the gain matrix (36) remains stable?

$$\varepsilon = \max|\varepsilon_j|\,, \quad j = 1, 2, \dots, p\,. \tag{37}$$

The following test has been applied:

- first, all considered methods were tested on ten continuous-time models taken from references and five laboratory plants at our department,
- then, 50 matrix $\mathbf{A}_{vi}$, $\mathbf{B}_{vi}$ for closed loop polytopic system (36) were generated considering $\varepsilon_j \in \langle -1, 1 \rangle$, $j = 1, 2, \dots, p$ for the pairs ($n = 4$, $p = 2$) and ($n = 5$, $p = 2$); (the matrix $\mathbf{C}$ was constant: $\mathbf{C} = [1\,0\,1\,0\,;\,0\,1\,0\,1]$ and $\mathbf{C} = [1\,0\,1\,0\,0\,;\,0\,1\,0\,1\,0]$),
- finally, in each example, the maximum value of the uncertainty parameter ε was evaluated for each considered robust stability condition.

The obtained results have been evaluated as follows:

- For each example, all methods were arranged according to the maximum value of the uncertainty parameter ε and number of points assigned with respect to their priority (the highest value of ε-best rating — 1 point, ... *etc*), *ie* the fewer points, the better rating of the respective method.
- For each method, the mean value ε_m of all uncertainty parameters obtained in the considered examples was computed, and the methods were arranged according to decreasing values of ε_m. Hence, in this case, the higher value of ε_m, the better rating of the respective method.

These two proposed criteria have been chosen due to their obvious interpretation. While the first criterion evaluates the method's priority ("the fewer points — the better method"), the second one estimates the "size" of the stability region ("the higher ε_m — the better method").

4.2 Results for real plants

All considered methods were tested on ten continuous-time models taken from references:

- [2], ($p = 1$, $n = 4$, $m = 1$, $l = 2$), ($p = 2$, $n = 4$, $m = 2$, $l = 1$),
- [1], ($p = 2$, $n = 3$, $m = 2$, $l = 2$),
- [14], ($p = 1$, $n = 3$, $m = 1$, $l = 3$),
- [5], ($p = 2$, $n = 3$, $m = 1$, $l = 2$), ($p = 1$, $n = 3$, $m = 1$, $l = 2$),
- [10], ($p = 1$, $n = 5$, $m = 2$, $l = 3$),
- [15], ($p = 1$, $n = 3$, $m = 1$, $l = 2$),
- [17], ($p = 2$, $n = 3$, $m = 1$, $l = 2$),
- [16], ($p = 2$, $n = 10$, $m = 2$, $l = 4$), and five laboratory plants at our department.

Results obtained for the above-considered pairs are summarized in Table 1 and the corresponding charts are in Fig. 1 and Fig. 2. For all four quadratic methods (V-F,

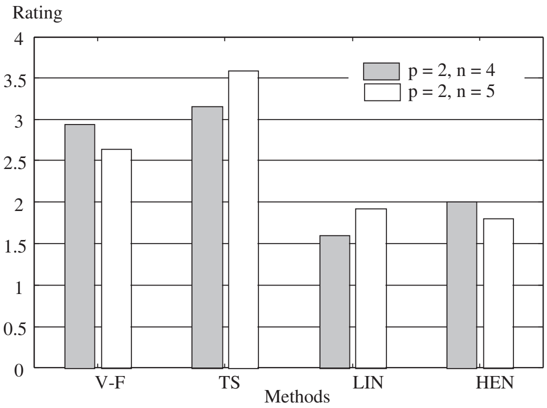

Fig. 3. Result of robust stability evaluation in terms of rating

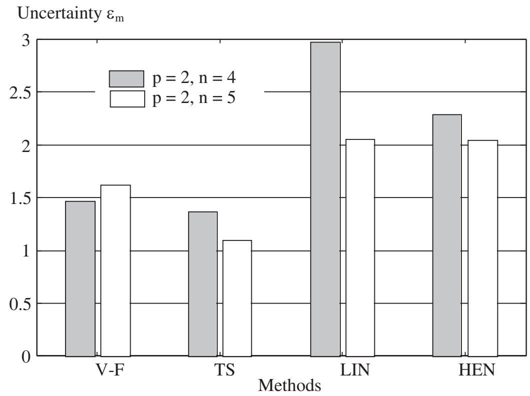

Fig. 4. Result of robust stability evaluation in terms of uncertainty ε_m

Table 2.

p	n		V–F	TS	LIN	HEN
2	4	Rating	2.94	3.16	1.60	2.00
		ε_m	1.47	1.36	2.97	2.29
		σ_s	0.80	1.14	2.05	1.07
	5	Rating	2.64	3.58	1.92	1.80
		ε_m	1.62	1.10	2.05	2.04
		σ_s	0.79	0.63	1.27	0.92

where rating, maximum uncertainty parameter ε_m and standard deviation σ_s have been calculated using the mean values obtained from the 50 closed loop polytopic systems.

TS, LIN and HEN) the V–F iteration method is the less conservative: the method has obtained a minimum of points. The second place goes to the method LIN but the mean value ε_m is larger than in other methods. The third place belongs to the method HEN. On the average, for fifteen examples the two-step method is the most conservative with smallest mean value of uncertainty parameter ε_m . Note that all systems were been stable.

4.3 Results for generated examples

For 50 matrix $\mathbf{A}_{vi}$, $\mathbf{B}_{vi}$ for closed loop polytopic system (36) were generated considering $\varepsilon_j \in \langle -1, 1 \rangle$, $j = 1, 2, \dots, p$ for the pairs ($n = 4$, $p = 2$) and ($n = 5$, $p = 2$). The results have been evaluated as explained in Section 4.1. Results obtained for the above-considered pairs are summarized in Table 2 and the corresponding charts are in Fig. 3 and Fig. 4. From all four quadratic stability methods (V–F, TS, LIN and HEN) the linearization method is the least conservative for the first pair ($n = 4$, $p = 2$) and Henrion's quadratic method is the least conservative for the second pair ($n = 5$, $p = 2$): these methods have obtained a minimum of points. The third place goes to the V–F method and the most conservative is the two-step method. When the affine system has become more complex (increased n), the LIN and TS methods have become more conservative on the contrary to the V–F method and HEN method.

5 CONCLUSIONS

The paper provides a numerical comparison of four quadratic stability methods based on LMI conditions and a single Lyapunov function for continuous-time linear uncertain system with polytopic uncertainties. According to the proposed comparative test the linearization method and Henrion's quadratic method provides less conservative results, generally. The presented algorithms are heuristic and may fail to determine the feedback gain, even if it exists. One of possible reasons could be too high requirements on performance. The advantage of static output feedback robust controller design by quadratic stability methods is that they are computationally simple.

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