ON THE CONTROL OF INTEGRATING AND UNSTABLE PROCESSES WITH TIME DELAY

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A two stage procedure — stabilization followed by disturbance rejection — is developed for the control of integrating and unstable processes with time delay. A fundamental limit on stabilizability by PID control is overcome. Example responses are characterized by superior damping, along with settling times comparable to those obtained by minimization of integral performance measures. Optimum stability arguments are invoked, based on root locus and Nyquist approaches.

Keywords: time delay, optimum stability, two-stage design, Padé Approximations, Nyquist Analysis

1 INTRODUCTION

In a recent paper, Visioli [1] presented PID controller designs for integrating and unstable processes characterized by the transfer functions

\[ G(s) = \frac{k_1}{s} \exp(-Ls) \]  
(1)

and

\[ G(s) = \frac{k_1}{(s - \lambda)} \exp(-Ls) \]  
(2)

Designs were computed to minimize selected integral performance measures separately for responses to step reference inputs and step disturbances applied at the process input. Very useful formulae, obtained by curve-fitting, encapsulated the derived parameter values.

Reference [1] makes no mention of the difficulties associated with the parameter

\[ \varepsilon = \lambda L \]  
(3)

This was highlighted some years ago in papers by de Paor [2], de Paor and O’Malley [3], and de Paor and Egan [4,5]. Basically, following an observation in reference [2], de Paor and O’Malley [3] produced PID controller designs for eqn. (2), using concepts of optimum stability in a parameter plane and the idea of phase margin, associated with frequency response, but showed that asymptotic stability could be obtained only for \( \varepsilon < 1 \). This very important restriction must also limit the applicability of the designs derived by Visioli [1]. It has also been pointed out by later workers, eg references [6] and [7]. de Paor and Egan [5] showed how to remove this restriction, using a two-stage design in which an observer-based eigenvalue-assigning controller was first used for stabilization, followed by a PI or PID controller for disturbance rejection. However, the parameter \( \varepsilon \) remained an index of difficulty of control, with output excursions becoming wider as \( \varepsilon \) increased, and with very fragile designs.

Continuing interest in this problem has prompted the author to revisit it, in the hope of simplifying and improving the approach used in reference [5]. The idea of designing stabilizing and disturbance-rejecting controllers in sequence has been retained here, but a new approach has been used to the former, involving the Padé approximation to time delay [8] — but only as a design aid — in conjunction with a cascade eigenvalue-assigning controller whose tuning invokes a root locus-based principle of optimum stability, ie that the rightmost eigenvalue should lie as deep as possible in the left half plane [2,9]. At both stages of the design, stability is confirmed by the Nyquist criterion, using a true time delay representation in the frequency response of the process. Responses have been obtained which we judge to be considerably superior to those using PID controllers based on integral performance measures, with no greater control effort. It must be admitted at the outset, however, that the comparison is unfair, since the eigenvalue-assigning controllers are of considerably higher order than PID controllers. Given the vastly increased computing power available nowadays, as compared to that when Ziegler and Nichols [10] produced their great paper on P, PI and PID controllers, the increase in controller order is not seen as an obstacle to practical implementation. Attention is drawn to the fact that one of the examples considered below has \( \varepsilon \) considerably greater than 1, to illustrate that the present approach removes both the stability restriction and the control difficulty, although as \( \varepsilon \) increases, admissible ranges of controller parameters have been noted to contract, emphasizing the seeming inevitable fragility alluded to above.

2 DEVELOPMENT OF THE THEORY

The complete control scheme considered here is shown in Fig. 1. Although not a necessary restriction, the study
The degree of the transfer function relating the upper loop in Fig. 1, has transfer function

$$M(s) = \frac{s^4 + 20 s^3 + 180 s^2 + 840 s + 1680}{L^2 s^3 + L s^2 + L^2 s + L^3} \tag{7}$$

The roots of $M(s)$ are all in the left half plane, and those of $N(s)$ are their reflections in the imaginary axis. With the replacement in eqn. (6), the characteristic polynomial of the upper loop in Fig. 1 is

$$P(s) = M(s)\{A(s)H(s) + k_1 k_2 B(s)F(s)N(s)\} = M(s)P_a(s). \tag{8}$$

We refer to $P_a(s)$ as the “apparent characteristic polynomial” of the upper loop, since $M(s)$ is masked in the transfer function relating $Y(s)$ to $U_2(s)$, subject to eqn. (6). We propose to choose the monic polynomials $H(s)$, $F(s)$, and $k_2$, to satisfy

$$P_a(s) = A(s)H(s) + k_1 k_2 B(s)F(s)N(s) = (s + \alpha)^m \tag{9}$$

with $\alpha > 0$. Although complete design freedom is not exploited in this paper, the degree $m$ is chosen such that the coefficients of $H(s)$ along with those of $F(s)$ and $k_2$ constitute just the right number of free parameters to permit $P_a(s)$ to be assigned as any real Coefficient $m$th degree monic polynomial.

$F(s)$ is specified as a monic polynomial of degree $c$ — thus containing $c$ free parameters — and $H(s)$ as monic of degree $c + n$. This gives a balanced controller, with numerator and denominator degrees both being $c + n$. The degree of $P_a(s)$ is now

$$m = p + c + n. \tag{10}$$

The number of free parameters is $c + n + c + 1$ ($c + n$ in $H(s)$, $c$ in $F(s)$ and $1$ as $k_2$). Thus, to have the possibility of assigning $P_a(s)$ arbitrarily, we require

$$p + c + n = c + n + c + 1, \text{ i.e., } c = p - 1 \tag{11}$$

thus giving

$$m = 2p + n - 1 \tag{12}$$

$H(s)$, $F(s)$ and $k_2$ are now to be found by solving the polynomial equation (9), given $A(s)$, $B(s)$, $N(s)$, $k_1$ and $\alpha$. A simple, general way to effect this solution is to rearrange eqn. (9) as in reference [11]:

$$\frac{(s + \alpha)^m}{A(s)B(s)N(s)} = \frac{H(s)}{B(s)N(s)} + \frac{k_1 k_2 F(s)}{A(s)}. \tag{13}$$

The left hand side of eqn. (13) is a monic polynomial of degree $r - 1$, plus a partial fraction expansion in the roots of $A(s)$, $B(s)$ and $N(s)$. We expand it conveniently into this form using the “pfc” command in the well-known Program CC. Recombining the $(r - 1)^{th}$ degree monomial with the portion of the partial fraction expansion involving the roots of $B(s)$ and $N(s)$ gives, as numerator, the $(n + c)^{th}$ degree monic polynomial $H(s)$. Reorganizing the terms in the roots of $A(s)$ gives as the numerator the $c^{th}$ degree monomial $k_1 k_2 F(s)$, with $k_1$ known. The controller transfer function $C(s)$ in eqn. (5) is thus determined.

The preceding paragraph summarizes a general solution procedure for eqn. (13). However, simpler, special solutions are used in the design examples presented below.

At this stage the stability of the upper loop in Fig. 1 is confirmed by applying the Nyquist stability criterion to $C_1(s)G(s)$. Using Program CC, this can be readily done with an exact time delay representation, $\exp(-j\omega L)$, incorporated in the frequency response. We find in design examples that Nyquist analysis sometimes suggests an optimum gain margin setting for $\alpha$, and this is chosen where applicable.

The upper loop in Fig. 1 has now been stabilized, optimally if possible, and it remains to design the disturbance-rejecting controller, $C_2(s)$. In the case that $F(s)$ is Hurwitz, we propose the structure

$$C_2(s) = k_3(s + \alpha)/[sF(s)]. \tag{14}$$

If $F(s)$ is not Hurwitz, its non-Hurwitz part may be omitted from eqn. (14). If $F(s)$ is omitted altogether, $C_2(s)$ is a PI controller. The parameter $k_3$ is tuned by a root locus-based optimum stability criterion, namely that, subject to the approximation in eqn. (6), the rightmost root of the overall “apparent” characteristic polynomial should lie as deep in the left half plane as possible. Before exploring responses, the stability of the overall system is checked by applying the Nyquist criterion to the transfer function $[1 + C_2(s)]C_1(s)G(s)$, again with an exact time delay representation in $G(s)$.

The scheme is now illustrated by four design examples. Two are chosen from Visioli [1]. The third illustrates how the procedure copes with $\varepsilon > 1$. The fourth is the well-known double integrator test case [12, 13], subject to a time delay.
Subject to eqns. (11) and (12), Wang and Cluett [14], and has This is considered by Visioli [1], following Example 1.

Fig. 1. The two stage control scheme considered

Fig. 2. Frequency response locus, stage one, Example 1

Fig. 3. Root locus to design $C_2(s)$ in Example 1

Fig. 4. Frequency response locus, stage two, Example 1

Fig. 5. $y(t)$ for unit step reference and disturbance, Example 1

Fig. 6. $u_1(t)$ for unit step reference, Example 1

Fig. 7. $u_1(t)$ for unit step disturbance, Example 1

Fig. 8. Frequency response, stage one, Example 2

Fig. 9. Frequency response, stage two, Example 2

Fig. 10. $y(t)$ for unit step reference and disturbance, Example 2

Fig. 11. $u_1(t)$ for unit step reference, Example 2

3 DESIGN EXAMPLES

EXAMPLE 1. This is considered by Visioli [1], following Wang and Cluett [14], and has

$$G(s) = \frac{0.0506}{s} \exp(-6s).$$

Subject to eqns. (11) and (12), $c = 0$, giving

$$F(s) = 1.$$  \hspace{1cm} (16)  

Eqn. (9) becomes

$$\begin{align*}
(s + \alpha)^5 &= sH(s) + 0.0506k_2N(s). \hspace{1cm} (17) \\
\alpha^5 &= 0.0506k_21680/6^4. \hspace{1cm} (18)
\end{align*}$$

Setting $s = 0$, eqn. (17) gives

$$(s + \alpha)^5 = 0.0506k_21680/6^4.$$
For a reason to be explained below, we assign

$$\alpha = 1.165 \implies k_2 = 32.717117.$$  \hfill (19)

Eqn. (17) now yields,

$$H(s) = \left[(s + 1.165)^5 - 1.6554861N(s)\right]/s = s^4 + 4.169514s^3 + 19.09054s^2 + 7.534241s + 15.6483$$ \hfill (20)

thus giving the stabilising controller as

$$C_1(s) = 32.717117\left[s^4 + 3.333333s^3 + 5s^2 + 3.888889s + 1.296293\right]/H(s).$$  \hfill (21)

It is readily confirmed — most easily by factorization — that $H(s)$ is Hurwitz, and so the Nyquist criterion for
asymptotic stability of the upper loop [15] is that the complete Nyquist diagram of $C_1(s)G(s)$ should not encircle the point $(-1,0)$. This is readily confirmed from Fig. 2, which shows the frequency response locus of $C_1(s)G(s)$. The complete Nyquist diagram is this augmented by its conjugate, along with one clockwise half revolution out at infinity, going from the point corresponding to $\omega = 0-$ (on the conjugate locus) to that corresponding to $\omega = 0+$ (on the frequency response locus itself).

In exploring the choice of $\alpha$, we noticed that, as $\alpha$ was increased, the lowest frequency crossing of the negative real axis in Fig. 2 moved to the right — of itself increasing the gain margin — but the spiral developed local resonances at higher frequency, so that a higher frequency crossing of the negative real axis in Fig. 2 moved to the right. This pair of complex conjugate eigenvalues will, giving instability.

At this stage, if eqn. (6) were effective, we would have the transfer function of the upper loop on Fig. 1, with $D(s) = 0$, as

$$Y(s)/U_2(s) = k_1 k_2 N(s)/(s + \alpha)^3$$

and with

$$C_2(s) = k_3(s + \alpha)/s$$

the apparent characteristic polynomial of the overall double-loop system is

$$P_3(s) = s(s + \alpha)^4 + k_1 k_2 k_3 N(s).$$

The positive-parameter root locus of eqn. (24) is plotted in Fig. 3, with $k_3$ as parameter. It is noted that choosing $k_3$ to give a root at the rightmost breakpoint gives an optimally stable system, in the sense that, under this condition, the rightmost eigenvalue of the closed loop system is as deep in the left half plane as possible. (Any smaller value of $k_3$ gives a real eigenvalue lying between this breakpoint and the origin, and any larger value of $k_3$ gives a pair of complex conjugate eigenvalues lying to the right. This pair of complex conjugate eigenvalues will, for $k_3$ sufficiently large, migrate into the right half plane, giving instability.)

The value of $k_3$ at the relevant breakpoint is readily computed, or found using the cursor facility of Program CC, as

$$k_3 = 0.04799.$$

The design is now complete. However, because the last stage involves the Padé approximation again, it is essential to confirm overall stability using the Nyquist criterion. If the loop in Fig. 1 is considered broken just below the label $Y(s)$, it is appreciated that the stability is the same as that of a unity negative feedback loop with forward path transfer function $[1 + C_2(s)]C_1(s)G(s)$. The frequency response locus of this transfer function, which has two poles at the origin and none in the right half plane, is plotted in Fig. 4. It is readily deduced that this locus, along with its conjugate and a two half-revolution clockwise closure, leaves the critical point $(-1,0)$ unencircled, thus guaranteeing asymptotic stability of the closed loop.

Figure 5 shows the responses of $y(t)$ to a unit step reference input (the curve which settles at $y = 1$) and to a unit step disturbance input (the curve which settles at $y = 0$). These were obtained using the simulation package SIMNON, by representing process and controllers in state variable form and using the exact time delay facility provided. (There is no Padé approximation involved here, as there would be were we to do the simulation using Program CC.) The settling times are almost the same as those shown by Visioli [1], but the underdamped behaviour yielded by minimization of all the integral performance measures has been eliminated.

Figure 6 shows the control signal $u_1(t)$ corresponding to the unit step reference, while Fig. 7 shows $u_1(t)$ in response to a unit step disturbance. Excursions and settling times are close to those shown in [1]. As suggested earlier, however, these comparisons are unfair to [1], since $C_1(s)$ is of fourth order and $C_2(s)$ of first order, thus giving an overall control scheme of fifth order, much higher than PID.

**Example 2.** This is also considered by Visioli [1], following Ho and Xu [16]:

$$G(s) = [1/(s - 1)]\exp(-0.2s).$$

This is of the class considered in eqn. (2), but has $\varepsilon = 0.2$, and so is, in our terms, fairly benign.

Using exactly the same approach as in Example 1, we select an optimum value

$$\alpha = 33.5$$

and compute

$$C_1(s) = 51.480881s^4 + 100s^3 + 4500s^2 + 105000s + 1050000/H(s),$$

$$H(s) = s^4 + 117.0191s^3 + 16487.61s^2 + 160777.4s + 11863490.$$  

Since $H(s)$ is Hurwitz, $G(s)$ has one pole in the right half plane, and $G(s)C(s)$ has no pole at the origin, the Nyquist criterion requires the frequency response locus of $G(s)C(s)$ plus its conjugate to encircle the critical point $(-1,0)$ once in the anticlockwise direction, as is confirmed in Fig. 8. As in Example 1, the value of $\alpha$ selected is that which puts as far to the right as possible the first crossing of the negative real axis which lies to the right of $(-1,0)$.

Using a root locus approach to tune $k_3$ as before, $C_2(s)$ is now obtained as

$$C_2(s) = 0.03812(s + 33.5)/s.$$  

The stability of the overall scheme is confirmed by the frequency response locus in Fig. 9.
Figure 10 shows responses of $y(t)$ to unit step reference and unit step disturbance inputs, with corresponding waveforms of $u_1(t)$ shown on Figs. 11 and 12. As in Example 1, these are far better damped than the corresponding records in [1], while having roughly the same settling times.

**Example 3.** Here we consider

$$G(s) = [1/(s - 1.5)] \exp(-s).$$  \hspace{1cm} (30)

This has $\varepsilon = 1.5$ and so cannot be stabilized by a PID controller.

Following the same procedure as before, we select

$$\alpha = 4.65$$ \hspace{1cm} (31)

and compute

$$C_1(s) = 11.537192(s^4 + 20s^3 + 180s^2 + 840s + 1680)/H(s),$$

$$H(s) = s^4 + 13.37481s^3 + 469.0629s^2 - 297.4482s + 11415.77.$$ \hspace{1cm} (32)

$H(s)$ has two roots in the right half plane and $G(s)$ one. Thus, the Nyquist criterion requires that the complete Nyquist diagram of $C_1(s)G(s)$ encircle $(-1, 0)$ three times in the anticlockwise sense, as confirmed by Fig. 13. Figure 13 also illustrates the basis for choosing $\alpha$.

As before, root locus-based tuning of $k_3$ gives the disturbance-rejecting controller as

$$C_2(s) = 0.006255(s + 4.65)/s.$$ \hspace{1cm} (33)

Stability of the overall scheme is confirmed by the frequency response locus shown in Fig. 14.

Figure 15 shows responses of $y(t)$ to unit step reference and unit step disturbance inputs, with the corresponding waveforms of $u_1(t)$ shown on Figs. 16 and 17.

**Example 4.** The double integrator process runs as a test case right through the famous textbook by Åström and Wittenmark [13] and has also been used by de Paor [12] to highlight difficulties in maintaining asymptotic stability when discretising continuous-time controllers. It seemed instructive, therefore, to conclude the examples with a time-delayed version of it:

$$G(s) = [1/s^2] \exp(-s).$$  \hspace{1cm} (34)

Here we have $m = 7$, $p = 2$, so that $F(s)$ makes its first appearance as a monic first degree polynomial. $H(s)$ is of fifth degree.

Frequency response analysis indicated no optimum value for $\alpha$. The value

$$\alpha = 2$$ \hspace{1cm} (35)

was chosen to give a closed loop settling time of about 30 seconds. The stabilizing controller was computed as

$$C_1(s) = 0.3047619(s + .25)(s^4 + 20s^3 + 180s^2 + 840s + 1680)/H(s),$$

$$H(s) = s^5 + 14s^4 + 83.68524s^3 + 286.019s^2 + 506.6679 + 914.2857.$$ \hspace{1cm} (36)

$G(s)C_1(s)$ has no poles in the right half plane and two at the origin, and so the complete Nyquist diagram, involving a closure of two half-revolutions clockwise at infinity, must not encircle the point $(-1, 0)$. This is confirmed by the frequency response locus in Fig. 18.

The root locus used to tune $k_3$ is shown in Fig. 19: it corresponds to

$$P_b(s) = s(s + 2)^6 + k_1k_2k_3N(s).$$ \hspace{1cm} (37)

The disturbance-rejecting controller is obtained as

$$C_2(s) = 0.01239(s + 2)/[s(s + .25)].$$ \hspace{1cm} (38)

Stability of the overall system is confirmed by Fig. 20, since the single clockwise encirclement of $(-1, 0)$ by the three half-revolution closure at infinity offsets the anticlockwise encirclement by the frequency response locus of $G(s)C_1(s)[1 + C_2(s)]$ and its conjugate.

Responses of $y(t)$ to a unit step reference and a unit step disturbance input are shown in Fig. 21, with corresponding waveforms of $u_1(t)$ in Figs. 22 and 23.

### 4 Conclusions

Inspired by the recent study by Visioli [1] and by other works, the problem of the control of integrating and unstable processes with time delay has been revisited. The fundamental idea invoked by de Paor and Egan [5], of a two stage approach — stabilization followed by disturbance-rejection — has been adopted. The fairly complicated observer-based stabilization stage used earlier has been replaced, however, by a cascade eigenvalue-assigning controller, designed by imagining the time delay replaced by a Padé approximation and solving a polynomial equation. Although a generalized partial fraction expansion approach to solving the polynomial equation is described, a simpler, problem-specific idea is used in three of the four design examples. Two of the design examples show superior dynamics as compared with those obtained with PID controllers tuned by optimizing integral performance measures [1]. It is acknowledged, however, that the comparison is unfair, since the controllers involved here are of higher order than PID and are, at the stabilization stage, specifically tailored to the process being controlled. Nonetheless, it is perhaps time to reflect that, with the great computing power available nowadays, the reign of the classical PID controller, in this situation, may be coming to an end.
For an unstable process having a single pole \( \lambda \) in the right half plane, and latency \( L \), it was pointed out in reference [3] that PID control could not give stabilization in the case \( \varepsilon = \lambda L > 1 \). One of the design examples given here shows that this limitation does not apply to the present scheme, and it even seems to overcome — although in a fragile and therefore system-restricted manner — the difficulty in control associated with \( \varepsilon > 1 \), which was highlighted in reference [5]. A final design example considers control of the famous double-integrator process — which runs right through the famous textbook by Åström and Wittenmark [13] — augmented by a time delay. Some years ago, de Paor [12] used this process to illustrate difficulties encountered in guaranteeing stability when discretizing continuous-time control schemes using popular algorithms. Much of the difficulty arose there by using the common form of the PID controller transfer function, which is improper, i.e., when combined into a rational function it has a numerator of higher degree than denominator. It was shown in [12] that the difficulties disappeared when the differentiator was fitted with a first order lowpass filter, which gives a “balanced” controller, one having equal numerator and denominator degrees. One of the motivations for specifying a balanced controller at the stabilization stage in this paper is to avoid stabilization difficulties when the current scheme, as it would inevitably be if applied in practice, is discretized. The final disturbance-rejecting controller used here is either balanced (PI) or has a proper rational transfer function, and no stabilization problems would arise in discretizing it.

References


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