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COMMUNICATIONS

CONTROL OF FRACTIONAL–ORDER CHUA'S SYSTEM

Ivo Petráš *

This paper deals with feedback control of a fractional-order Chua's system. The fractional-order Chua's system with the total order less than three which exhibits chaos as well as other nonlinear behavior, and a theory for control of chaotic systems using sampled data are presented. A numerical experimental example is shown to verify the theoretical results.

K e y w o r d s: fractional calculus, fractional-order Chua's system, control of chaos.

1 INTRODUCTION

It is well-known that chaos cannot occur in continuous systems of total order less than three. This assertion is based on the usual concepts of order, such as the number of states in a system or the total number of separate differentiations or integrations in the system. The model of system can be rearranged to three single differential equations, where one of the equations contains the non-integer (fractional) order derivative. The total order of the system is changed from 3 to $2+q$, where $0<q\leq 1$. To put this fact into context, we can consider the fractional-order dynamical model of the system. Hartley *et al.* [4] considered the fractional-order Chua's system and demonstrated that chaos is possible in systems where the order is less than three. In their work, the limits on the mathematical order of the system having a chaotic response, which was measured from bifurcation diagrams, are approximately from 2.5 to 3.8. In work [1], chaos was discovered in a fractional-order two-cell cellular neural network and also in work [6] chaos was exhibited in a system with total order less than three.

The control of chaos has been studied and observed in experiments (*eg*, works [3], [5], [8], [12]). Especially, the control of well-known Chua's system [9] by sampled data has been studied [13]. The main motivation for the control of chaos via sampled data are well-developed digital control techniques.

In this brief study, practical results from sampled-data feed-back control of a fractional-order chaotic dynamical system are presented. The system was modeled by the state equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, where $\boldsymbol{x}\in\Re^n$ is state variable, $\boldsymbol{f}\colon \Re^n \to \Re^n$ is nonlinear function and $\boldsymbol{f}(0)=0$.

The approach used in this paper is concentrating on the feed-back control of the chaotic fractional-order Chua's system, where the total order of the system is 2.9.

2 FRACTIONAL CALCULUS

2.1 Definitions of Fractional Derivatives

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibnitz and L'Hospital in 1695.

Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^r$, where a and t are the limits of the operation. The continuous integro-differential operator is defined as

$$ {}_aD_t^r = \begin{cases} \dfrac{\mathrm{d}^r}{\mathrm{d}t^r} & \Re(r)>0\,, \\ 1 & \Re(r)=0\,, \\ \int_a^t (\mathrm{d}\tau)^{-r} & \Re(r)<0. \end{cases} $$

The two definitions used for the general fractional differintegral are the Grünwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition [7], [11]. The GL is given here

$$ {}_aD_t^r f(t) = \lim_{h\to 0} h^{-r} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{r}{j} f(t-jh)\,, \qquad (1) $$

where $[\cdot]$ means the integer part. The RL definition is given as

$$ {}_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{r-n+1}} \mathrm{d}\tau\,, \qquad (2) $$

for $(n-1<r<n)$ and where $\Gamma(\cdot)$ is the *Gamma* function.

2.2 Numerical Methods for Calculation of Fractional Derivatives

For numerical calculation of fractional-order derivative we can use the relation (3) derived from the Grünwald-Letnikov definition (1). This approach is based on the

* Department of Informatics and Process Control, BERG Faculty, Technical University of Košice, B. Němcovej 3, 042 00 Košice, Slovak Republic, e-mail: ivo.petras@tuke.sk

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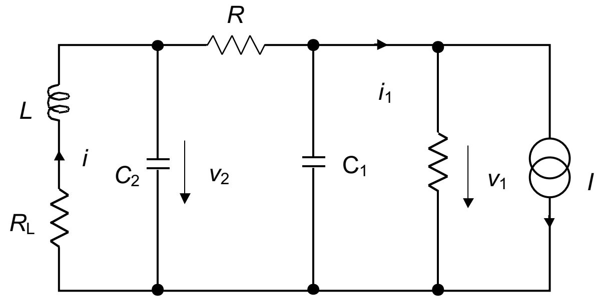

Fig. 1. Chua's circuit.

fact that for a wide class of functions, two definitions — GL (1) and RL (2) — are equivalent. The relation for the explicit numerical approximation of r-th derivative at the points kT, $(k = 1, 2, \dots)$ has the following form [2], [10], [11]:

$$ {}_{(k-L/T)}D^r_{kT} f(t) \approx T^{-r} \sum_{j=0}^{k} (-1)^j \binom{r}{j} f_{k-j} \,, \tag{3}$$

where L is the "memory length", T is the step size of the calculation (sampling period) and $(-1)^j \binom{r}{j}$ are binomial coefficients $c_j^{(r)}$, $(j = 0, 1, \dots)$. For its calculation we can use the following expression:

$$ c_0^{(r)} = 1 \,, \qquad c_j^{(r)} = \left(1 - \frac{1+r}{j}\right) c_{j-1}^{(r)} \,. \tag{4}$$

2.3 Some Properties of Fractional Derivatives

Two general properties of the fractional-order derivative will be used. The first is a composition of the fractional with an integer-order derivative, and the second is the property of linearity.

The fractional-order derivative commutes with integer-order derivative [11],

$$ \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_aD^r_t f(t)\right) = {}_aD^r_t \left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right) = {}_aD^{r+n}_t f(t) \,, \tag{5}$$

under the condition $t = a$ we have $f^{(k)}(a) = 0$, $(k = 0, 1, 2, \dots, n-1)$. Relationship (5) says that operators $\frac{\mathrm{d}^n}{\mathrm{d}t^n}$ and ${}_aD^r_t$ commute.

Similar to integer-order differentiation, fractional differentiation is a linear operation [11]:

$$ {}_aD^r_t \left(\lambda f(t) + \mu g(t)\right) = \lambda \, {}_aD^r_t f(t) + \mu \, {}_aD^r_t g(t) \,. \tag{6}$$

3 FRACTIONAL–ORDER CHUA'S SYSTEM

The classical Chua's oscillator, which is shown in Fig. 1, is a simple electronic circuit that exhibits nonlinear dynamical phenomena such as bifurcation and chaos. This circuit can be described by equations:

$$\begin{aligned} \frac{\mathrm{d}v_1}{\mathrm{d}t} &= \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] \,, \\ \frac{\mathrm{d}v_2}{\mathrm{d}t} &= \frac{1}{C_2}[G(v_1 - v_2) + i] \,, \\ \frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{L}[-v_2(t) - R_L i] \,, \end{aligned} \tag{7}$$

where $G = 1/R$, i is the current through inductor L, v_1 and v_2 are the voltages across capacitors C_1 and C_2, respectively, and $f(v_1)$ is the piecewise linear $v - i$ characteristic of the nonlinear Chua's diode, which can be described as

$$ f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)\big(|v_1 + E| - |v_1 - E|\big)$$

with E being the breakpoint voltage of the diode, and $G_a < 0$ and $G_b < 0$ are some appropriate constants (slope of the piecewise linear resistance).

Given the techniques of fractional calculus, there are still a number of ways in which the order of the system could be amended. One approach would be to change the order of any or all of the three constitutive equations (7) so that the total order gives the desired value.

In our case, in equation one, we replace the first differentiation by fractional differentiation of order q, $q \in \Re$. The final dimensionless equations of the system for $R_L = 0$ are $(x_1 = v_1/E,\ x_2 = v_2/E,\ x_3 = i/(EG))$:

$$\begin{aligned} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= \alpha \, {}_0D^{1-q}_t (x_2(t) - x_1(t) - f(x_1)) \,, \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= x_1(t) - x_2(t) + x_3(t) \,, \\ \frac{\mathrm{d}x_3(t)}{\mathrm{d}t} &= -\beta x_2(t) \,, \end{aligned} \tag{8}$$

where $\alpha = C_2/C_1$, $\beta = C_2R^2/L$ and

$$ f(x_1) = m_2 x_1 + \frac{1}{2}(m_1 - m_2)(|x_1 + 1| - |x_1 - 1|),$$

in which $m_1 = RG_a$, $m_2 = RG_b$ and with $E = 1$ therein.

4 FEEDBACK CONTROL OF CHAOS

The structure of the control system with sampled data [13] is shown in Fig. 2. The state variables of the chaotic system are measured and the result is used to construct the output signal $\boldsymbol{y}(t) = D\mathbf{x}(t)$, where D is a constant matrix. The output $\boldsymbol{y}(t)$ is then sampled by a sampling block to obtain $\boldsymbol{y}(k) = D\mathbf{x}(k)$ at discrete moments kT, where $k = 0, 1, 2, \dots$, and T is the sampling period. Then $D\mathbf{x}(k)$ is used by the controller to calculate the control signal $\boldsymbol{u}(k)$, which is fed back into the chaotic system.

The controlled chaotic system is defined by relations [13]:

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} &= \boldsymbol{f}(\mathbf{x}(t)) + B\boldsymbol{u}(k) \,, \quad t \in [kT, (k+1)T) \\ \boldsymbol{u}(k+1) &= C\boldsymbol{u}(k) + D\mathbf{x}(k) \,, \quad k = 0, 1, 2, \dots \end{aligned} \tag{9}$$

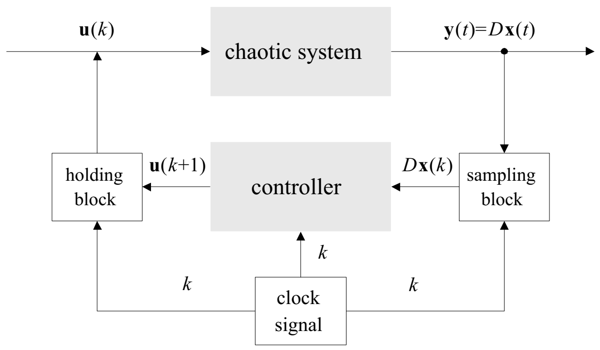

Fig. 2. Structure of the control system.

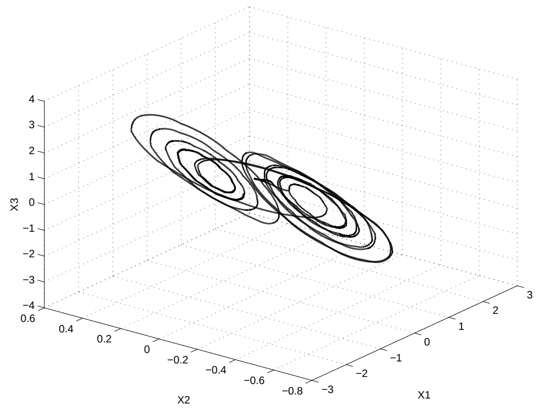

Fig. 3. Strange attractor of the fractional-order Chua's system without control

where $\boldsymbol{u} \in \Re^m$, $B \in \Re^n \times \Re^m$, $C \in \Re^m \times \Re^m$, $D \in \Re^m \times \Re^n$ and $t \in R_+$; $\mathbf{x}(k)$ is the sampled value of $\mathbf{x}(t)$ at $t = kT$. Observe that since $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ is an equilibrium point of the system (9).

The controlled fractional-order Chua's system is defined by

$$\begin{aligned}
\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= \alpha\, {}_0D_t^{1-q}(x_2(t) - x_1(t) - f(x_1)) + u_1(t)\,,\\
\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= x_1(t) - x_2(t) + x_3(t) + u_2(t)\,,\\
\frac{\mathrm{d}x_3(t)}{\mathrm{d}t} &= -\beta x_2(t) + u_3(t)\,.
\end{aligned} \tag{10}$$

5 ILLUSTRATIVE EXAMPLE

For numerical simulations the following parameters of the fractional Chua's system (8) were chosen:

$$\alpha = 10,\ \beta = \frac{100}{7},\ q = 0.9,\ m_1 = -1.27,\ m_2 = -0.68,$$

and the following parameters (experimentally found) of controller:

$$B = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0.8 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -3.3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Using the above parameters (11) the digital controller in state space form is defined as

$$u_1(k+1) = 0.8u_1(k) - 3.3x_1(k), \tag{12}$$

for $k = 0, 1, 2, \dots$, where $u_2(t) = 0$ and $u_3(t) = 0$. The initial conditions for Chua's circuit were $(x_1(0), x_2(0), x_3(0)) = (0.2, -0.1, -0.01)$ and the initial condition for the controller (12) was $u_1(0) = (0)$. The sampling period was $T = 0.01$ sec.

For computation of the fractional-order derivative in equations (10), relations (3), (4) and properties (5), (6) were used. The length of memory was $L = 10$ (1000 values and coefficients from history for $T = 0.01$ sec).

Figure 3 shows the attractor of Chua's circuit (8) without control. A similar behaviour was shown in work [4], where piecewise linear nonlinearity was replaced by cubic nonlinearity, which yields very similar properties.

In Fig. 4, axonometric view of the fractional-order Chua's strange attractor to the $x_2 - x_3$ plane is shown.

In Fig. 5 the controlled trajectory is shown of state variables of the fractional-order Chua's system (10), which tends to origin asymptotically.

Figure 6 shows the control signal from the digital controller (12).

6 CONCLUSION

We have considered an example of control of chaotic fractional-order Chua's circuit, which exhibits chaotic behaviour with the total order less than three. As it has been demonstrated, the idea of fractional calculus requires one to reconsider dynamic system concepts that are often taken for granted. So by decreasing the order of a system from 3 to $2 + q$ in this way, we also move from a three-dimensional system to one of infinite dimension. This system can be controlled by sampled data. The sampled data of output are sufficient for constructing the control signals in the digital controller. This advantage is important because the digital controllers had been widely used in industry.

The conclusion of this work confirms the conclusions of the works [4], [6], [11] that there is a need to refine the notion of the order of the system which can not be considered only by the total number of differentiations. For fractional-order differential equations the number of terms is more important than the order of differentiation.

The fractional-order model for chaotic Chua's system can be also directly derived because electrical elements used in circuits are not ideal. As it was mentioned in

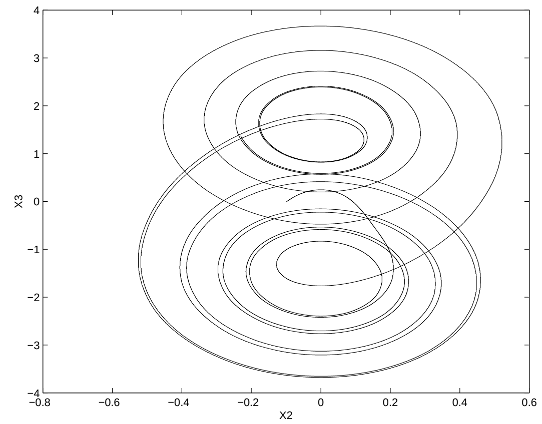

Fig. 4. The chaotic trajectory of the fractional-order Chua's system, depicted in Fig. 3, projected onto the $x_2 - x_3$ plane.

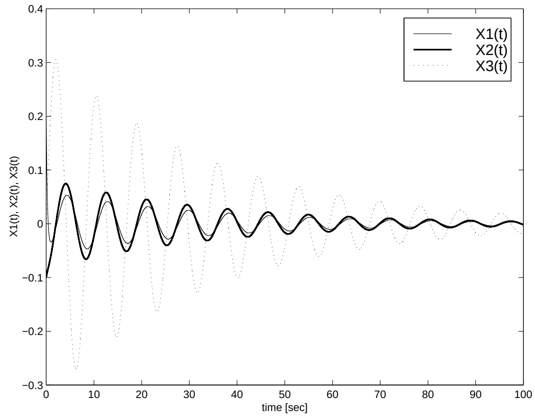

Fig. 5. Controlled state variables $x_1(t), x_2(t), x_3(t)$.

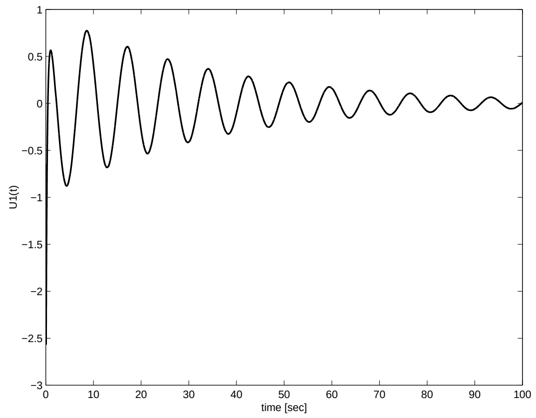

Fig. 6. Control signal $u_1(t)$.

works [10] and [11], the real electrical elements (*eg*, capacitors, inductors) have fractional orders and should be described by fractional-order models.

The results presented in this contribution give basis for controlling chaotic fractional-order systems. An alternative approximation of fractional-order derivative, stability investigation, and also other chaotic fractional-order system will be used in further work.

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Ivo Petráš (Ing, PhD) was born in Košice, Slovak Republic, in 1973. He received the Ing (1997) and PhD (2000) degrees in process control of raw materials extraction and processing from the Department of Informatics & Process Control, Faculty B.E.R.G. of the Technical University of Košice. His main research interests include fractional calculus in robust control and process control. He is a member of the Slovak Society for Applied Cybernetics and Informatics (SSAKI) and of the organizing committee of the International Carpathian Control Conference (ICCC).