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OPTIMAL PIECEWISE UNIFORM VECTOR QUANTIZATION OF THE MEMORYLESS LAPLACIAN SOURCE

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In this paper we will present a design procedure of an optimal piecewise uniform vector quantizer of a memoryless Laplacian source. We will derive expressions for the granular distortion and the optimum number of the output points in each subregion. We will also compare the obtained results with the results presented in reference [7] and on the basis of that we will derive some conclusions about the proposed models.

K e y w o r d s: piecewise uniform vector quantizer, Laplacian source, optimal asymptotic analysis

1 INTRODUCTION

Quantizers play an important role in the theory and practice of modern day signal processing. The asymptotic optimal quantization problem, even for the simplest case uniform scalar quantization, is very topical nowadays [1, 2]. It was shown in the literature that vector or multidimensional quantization can yield a smaller average mean squared error per dimension than scalar quantization for the case of fine quantization. Designing an optimal vector quantizer is equivalent to finding a partition of the vector space and assigning a representative point to each partition such that a predefined distortion measure between the input and output is minimized. Unfortunately, the optimal partitions in higher dimensional spaces are unknown for even the simplest source distributions and the most common distortion measures. Extensive results have been developed on finding the optimal output points distribution in multidimensional space for a specific probability distribution. The first approximation to the long-time-averaged probability density function (pdf) of amplitudes is provided by Laplacian model [3]. In a number of papers the vector quantization of the memoryless Laplacian source was analysed since the pdf of the difference signal for an image waveform follows the Laplacian function [3]. Also, the Laplace source is a model for the speech [4].

The analysis of a vector quantizer for an arbitrary distribution of the source signal was given in reference [5]. The authors derived an expression for the optimum granular distortion and optimum number of output points. However, they did not prove the optimality of the proposed solutions. Also, they did not define the partition of the multidimensional space into subregions. In reference [6] they derived expressions for the optimum number of output points, however, the proposed partitioning of the multidimensional space for memoryless Laplacian source does not take into consideration the geometry of the multidimensional source. In reference [7], vector quantizers of Laplacian and Gaussian sources were analysed. The proposed solution for the quantization of memoryless Laplacian source, unlike in [6], takes into consideration the geometry of the source, however, the proposed vector quantizer design procedure is complicated and gives results that are quite different from optimal for low bit rates. We presented an improved variant of the method in [7] and also the optimal solution for piecewise uniform vector quantization.

In this paper we will give an exact analysis of the proposed quantizer and present our systematic analysis of a piecewise uniform vector quantizer of the Laplacian memoryless source. We will give a general and simple way for the design of the piecewise uniform vector quantizer. We will derive the optimum number of output points and prove the optimality of the proposed solutions. Also we will give a short discussion of the vector quantization solution proposed in reference [7] by Jeong and Gibson.

2 OPTIMAL ASYMPTOTIC ANALYSIS

Let us consider a multidimensional piecewise uniform quantizer, where the input space is partitioned into subregions. Each subregion is divided using cubic cells with different sizes. The n-dimensional vector of Laplacian random variables is the input to the n-dimensional quantizer. For n-dimensional vector $\mathbf{x} = [x_1\ x_2\ \dots\ x_n]^\top$ consisting of independent and identically distributed (i.i.d.) Laplacian variables x_i with zero mean and unit variance, the joint pdf of $\mathbf{x}$ is

$$f(x) = \prod_{i=1}^{n} f(x_i) = 2^{-\frac{n}{2}} \exp\Big(-\sqrt{2}\sum_{i=1}^{n}|x_i|\Big). \tag{1}$$

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The contour of constant probability density function (pdf) is given by

$$\sum_{i=1}^{n} |x_i| = -\frac{1}{2\sqrt{2}} \ln\left(2^n f_c^2\right) \equiv r_0\,,\ r_0 > 0\,, \tag{2}$$

where f_c is the value of pdf. This is an expression for the n-dimensional hyperpyramid with radius r_0, where we define the radius as

$$r = \sum_{i=1}^{n} |x_i|\,. \tag{3}$$

Since each $|x_i|$ has an exponential distribution with mean $1/\sqrt{2}$ and variance $1/2$, the random variable r has a gamma pdf given as

$$f_n(r) = K_n r^{n-1} e^{-\sqrt{2}r}\,,\ r \geq 0\,, \tag{4}$$

where

$$K_n = \frac{2^{\frac{n}{2}}}{\Gamma(n)}\,. \tag{5}$$

The probability that the vector $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^\top$ is in the kth cell denoted by P_k is equal to

$$P_k = \int_{R_k} f(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{r_k}^{r_{k+1}} f_n(r)\mathrm{d}r\,, \tag{6}$$

where r_k and r_{k+1} denote the radius of the kth and $(k+1)$th region R_k and R_{k+1}, respectively.

Granular distortion for piecewise vector quantization of the signal generated by the Laplacian source can be written as follows

$$D_g = \frac{1}{12}\sum_{k=0}^{N_q-1} \Delta_k^2 P_k = \frac{1}{12}\sum_{k=0}^{N_q-1} \left(\frac{V_k}{N_k}\right)^{\frac{2}{n}} P_k\,, \tag{7}$$

where the volume V_k of the kth subregion is given by the following equation

$$V_k = \frac{2^n r_{k+1}^n}{\Gamma(n+1)} - \frac{2^n r_k^n}{\Gamma(n+1)}\,. \tag{8}$$

Now, we will derive the optimum solution for the piecewise vector quantization of the memoryless Laplacian source. We can obtain the optimum number of cells in one subregion N_k by using the Lagrangian multipliers on equation (7). In order to do that we consider the expanded function J

$$J = D_g + \lambda \sum_{k=0}^{N_q-1} N_k\,. \tag{9}$$

After differentiating J with respect to N_k, $k = 0, N_q - 1$, and equalizing with zero, with some mathematical manipulation, we obtain the number of points in the kth region as

$$N_k = N\frac{V_k^{\frac{2}{n+2}} P_k^{\frac{n}{n+2}}}{\sum_{i=0}^{N_q-1} V_i^{\frac{2}{n+2}} P_i^{\frac{n}{n+2}}}\,, \tag{10}$$

with the respect to the condition that the total number of output points is equal to

$$N = \sum_{k=0}^{N_q-1} N_k\,. \tag{11}$$

If we substitute N_k from equation (10) in equation (7), we have

$$D_g = \frac{1}{12N^{2/n}}\left(\sum_{k=0}^{N_q-1} P_k^{\frac{n}{n+2}} V_k^{\frac{2}{n+2}}\right)^{\frac{n+2}{n}}\,. \tag{12}$$

In the literature [8] we found that this form of equation (equation (12)) is Zador Gersh formula for the piecewise uniform vector quantizer.

The total distortion can be calculated using the following expression

$$D = \frac{1}{12N^{2/n}}\left(\sum_{k=0}^{N_q-1} P_k^{\frac{n}{n+2}} V_k^{\frac{2}{n+2}}\right)^{\frac{n+2}{n}} + \frac{1}{n}\int_{r_{N_q}}^{\infty}\left[\left(r - r_{N_q}\right)^2 + \frac{(n-1)\Delta_{N_q}^2}{12}\right] f_n(r)\mathrm{d}r\,, \tag{13}$$

where r_{N_q} denotes the radius of the input space.

From the equation (13) we can see that the total distortion depends only on the total number of the output points, dimension of the quantizer, the number of the subregions and the radius of the input space. If we assume that the total number of the output points and the dimension of the quantizer are known, minimizing the total distortion given by the equation (13) we can obtain the optimum radius of the input space and the optimum number of the subregions. It is important to emphasize that this solution is valid for all kinds of space partition into subregions. One example of partition is presented with the following equation for radius of subregion

$$r_k = k\cdot\frac{r_{N_q}}{N_q}\,,\ \ k = 0, \ldots N_q\,, \tag{14}$$

where r_{N_q} and N_q denote optimum radius of the input space and the number of the subregions, respectively.

Now, we perform asymptotic analysis, *ie* we assume the infinite number of output points. For this special case the constant pdf $f(r)$ over the volume of the kth subregion V_k can be supposed. Than we can substitute P_k with $P_k = V_k f(r_k)$ and after that the optimum number of the output points is given with

$$N_k = N\frac{V_k f^{\frac{n}{n+2}}(r_k)}{\sum_{i=0}^{N_q-1} V_i f^{\frac{n}{n+2}}(r_i)}\,, \tag{15}$$

while the granular distortion is

$$D_g \approx \frac{1}{12N^{2/n}}\Bigg(\sum_{k=0}^{N_q-1} f^{\frac{n}{n+2}}(r_k)V_k\Bigg)^{\frac{n+2}{n}} . \qquad (16)$$

The assumption that $P_k = V_k f(r_k)$ has to be very carefully applied. The problem is that the former assumption is not valid for the small bit rates per dimension, *ie* for small codebook. The equations (15) and (16) are suitable for high-resolution case, *ie* in practice for design of the high bit rate quantizer or the large dimension quantizer.

3 JEONG AND GIBSON ASYMPTOTIC ANALYSIS

In [7] Jeong and Gibson proposed a codebook dilution where a codebook is constructed by a series of successive subregions filled with different sets of cubic cells of length c_k. It was shown that for the minimum distortion the following equation must be satisfied

$$c_B = \exp\Big[\frac{\sqrt{2}(r_B - r_A)}{n+2}\Big]\cdot c_A , \qquad (17)$$

where c_A, c_B, r_A and r_B represent the lengths of cells and radii of two successive subregions. Also, they supposed the constant pdf $f(r)$ no matter which bit rate it is and from this they derived that the lengths of cells in two successive subregions satisfy the following constraint

$$c_{k+1} = s\cdot c_k , \qquad (18)$$

where s represents the scale factor ratio. After some mathematical manipulation with equations (17) and (18) they obtain

$$\exp\Big(\frac{\sqrt{2}r_k}{n+2}\Big) = s^k . \qquad (19)$$

From equation (19), they concluded that the radial parameters have to be equidistant

$$r_k = \frac{(n+2)\ln s}{\sqrt{2}}\cdot k , \quad k = 0, N_q - 1 . \qquad (20)$$

The number of the output points in each subregion was calculated in [7] as

$$N_k = \frac{V_k}{(s^k c_0)^n} , \qquad (21)$$

where c_0 is the length of cells in the first subregion. Than the total number of output points in [7] is equal to the infinite summation of the output points in each subregion, which is a defectiveness of this model. From equations (21), (8) and (20) the total number of output points can be calculated as

$$N = \sum_{k=0}^{\infty} N_k =$$

$$\frac{c_0^{-n}}{n!}\big[\sqrt{2}(n+2)\ln s\big]^n \sum_{k=0}^{\infty} s^{-kn}\big[(k+1)^n - k^n\big] . \qquad (22)$$

Since the sum in the above equation converges to some constant S_n when $s > 1$ and $N = 2^{nR}$, the length of the cells in the first subregion can be calculated from (22) as

$$c_0 = 2^{-R+0.5}(n+2)\ln s\Big[\frac{(s^n-1)S_n}{n!}\Big]^{\frac{1}{n}} . \qquad (23)$$

The way to determine the desired value of the scale factor ratio s is to calculate the minimum value of the distortion. In order to do that, we need to assume a value for scale factor ratio, and calculate the length of the cells in the first subregion using (23). After that we calculate radial parameter from equation (20), the volume of each subregion using (8) and the number of output points using equation (21). With this parameters we can than calculate the total distortion. We have to repeat this procedure in order to obtain the minimum distortion and the optimum value for the scale factor ratio s.

4 PERFORMANCE COMPARISON

In this section we will compare performance of the quantizers proposed in [7] and in this paper. We will do that on the example of the two dimensional uniform piecewise quantizer described in [7, Fig. 5]. Total number of output points is $N = 165$ which reflect on a bit rate per dimension ($R = 3.68316$ bits/dimension) and the number of subregions ($N_q = 3$). Scale factor ratio is $s = 1.5$, *ie* the radius of input space $r_{N_q} = 3.44049$.

First we will derive one improvement of the model proposed in [7]. We will start from the equations (21) and (22), and replace the infinite summations with the summations over N_q subregions

$$N = \sum_{k=0}^{N_q-1} N_k = \frac{1}{c_0^n}\sum_{k=0}^{N_q-1}\frac{V_k}{s^{nk}} . \qquad (24)$$

The length of the first cell c_0 can than be obtained from the previous equation as

$$c_0 = \frac{1}{N^{1/n}}\Bigg(\sum_{k=0}^{N_q-1}\frac{V_k}{s^{nk}}\Bigg)^{1/n} . \qquad (25)$$

Using equation (1) and equation (19) we can express s^{-k} as

$$s^{-k} = 2^{\frac{n}{2(n+2)}}\cdot f^{\frac{1}{n+2}}(r_k) . \qquad (26)$$

If we substitute s^{-k} from equation (26) in equation (25) we can write

$$c_0 = \frac{2^{\frac{n}{2(n+2)}}}{N^{1/n}}\Bigg(\sum_{k=0}^{N_q-1} V_k f^{\frac{n}{n+2}}(r_k)\Bigg)^{1/n} . \qquad (27)$$

Then, if we substitute (26) and (27) in (21) we obtain the elegant way of calculating the number of output points in each subregion

$$N_k = N \frac{V_k f^{\frac{n}{n+2}}(r_k)}{\sum_{i=0}^{N_q-1} V_i f^{\frac{n}{n+2}}(r_i)} . \tag{28}$$

The number of output points in subregions obtained on this way (equation (28)) is equivalent to our optimal solution for the high-resolution case (equation (15)). The distortions in these cases are calculated by equation (16).

Now, we will compare the numbers of output points in subregions as they are given in [7, Fig. 5], equation (21), with our results obtained using equations (10) and (15), *ie* (28) (Table 1.). If the number of the output points in subregion is calculated by means of equation (28), than the minimizing of the granular distortion (equation (16), $N = 165$ and $N_q = 3$) gives the scale factor ratio optimum value $s = 1.62958$.

Table 1.

k \ N_k	[7, Fig. 5]	eq. (10)	eq. (28) $s = 1.62958$
1	61	70	58
2	60	66	66
3	44	29	41

The values of the total distortion calculated using the values given in Table 1 are given in Table 2.

Table 2.

	[7, Fig. 5]	eq. (10)	eq. (28) $s = 1.62958$
Distortion	0.0393872	0.0171547	0.0238895
r_{opt}	3.44049	5.69727	4.14355

It is obvious from Table 2 that distortion calculated as in [7] is more than two times larger than our optimal distortion, *ie* our distortion is about 3.6 dB lesser than distortion in [7]. This means that the uniform piecewise vector quantization with our space partition into subregions has better performance than the quantization proposed in [7]. Also for given bit rate in this example, *ie* for given codebook dimension, the assumption that the pdf $f(x)$ is constant inside subregion is not suitable since the distortion in this case (Table 2, column 3) is very different from optimal (Table 2, column 2).

In order to show ones more that the assumption of the constant pdf function over a subregion is not suitable for the low bit rate, in the following Table 3 we compare the values for the probability that the input vector is in the kth subregion and the product of the pdf function and the volume of that subregion. The difference between these two is obvious.

Next, in Table 4 we will show the values for the optimum radii of the two-dimensional quantizer, number of subregions and distortion for various values of the bit rate per dimension R.

Distortion as a function of the input load $r_{\max}$ and the dimension of the vector quantizer n is plotted in Fig. 1. The bit rate per dimension is $R = 4$ bit/dimension, number of subregions $N_q = 5$. As we can see the distortion decreases and the optimum input load increases as the dimension of the vector quantizer increases.

Table 3.

k	$R = 1$		$R = 2$	
	P_k	$f(r_k)V_k$	P_k	$f(r_k)V_k$
1	5.32482E-1	1.5919E0	3.6412E-1	8.12188E-1
2	3.38709E-1	8.01893E-1	3.58495E-1	6.81185E-1
3	9.87331E-2	2.24411E-1	1.71988E-1	3.17396E-1
4	2.36071E-2	5.27535E-2	6.81455E-2	1.24227E-1
5	5.14411E-3	1.13887E-2	2.46603E-2	4.46527E-2
6			8.46231E-3	1.52576E-2
7			2.80418E-3	5.04107E-3
8			9.06518E-4	1.62614E-3
9			2.87697E-4	5.15232E-4
10			9.00098E-5	1.60988E-4

Table 4.

R	r_{opt}	$N_{q,opt}$	D	SNRQ (dB)
4	6.30853	5	1.06009E-2	19.75
6	9.01215	10	6.79786E-4	31.68
8	11.9251	18	4.17316E-5	43.795

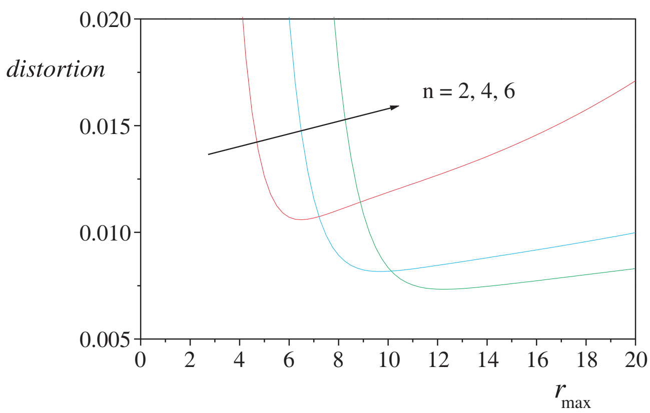

Fig. 1. Influence of the dimension of the vector quantizer on the distortion

5 CONCLUSION

In this paper we have presented a design procedure of an optimal piecewise uniform vector quantizer of memoryless Laplacian source. The presented design procedure of piecewise uniform quantizer is based on minimum distortion criterion. By optimising the total distortion, that includes granular and overload distortion, we can exactly determine all the parameters of the quantizer that are needed. We derived the expressions for the granular distortion and the optimum number of the output points in each subregion. We use closed forms to calculate all parameters of the quantizer. We proved the optimality of the proposed solutions. Also, the presented method is simpler and more practical for application than the method in reference [7].

References

[1] HUI, D.—NEUHOFF, D. L.: Asymptotic Analysis of Optimal Fixed-Rate Uniform Scalar Quantization, IEEE Trans. **47** No. 3 (2001), 957977.

[2] NA, S.—NEUHOFF, D. L.: On the Support of MSE-Optimal, Fixed-Rate Scalar Quantizers, IEEE Trans. on Information Theory **47** No. 6 (2001), 29722982.

[3] JAYANT, N. S.—NOLL, P.: Digital Coding of Waveforms Principles and Applications to Speech and Video, Prentice-Hall, New Jersey, 1984.

[4] GERSHO, A.—GRAY, R. M.: Vector Quantization and Signal Compression, Kluwer Academ. Pub, 1992.

[5] KUHLMANN, F.—BUCKLEW, J. A.: Piecewise Uniform Vector Quantizers, IEEE Trans. on Information Theory **34** No. 5 (1988), 12591263.

[6] SWASZEK, P. F.: Unrestricted Multistage Vector Quantizers, IEEE Trans. on Information Theory **38** No. 3 (1992), 11691174.

[7] JEONG, D. G.—GIBSON, J. D.: Uniform and Piecewise Uniform Lattice Vector Quantization for Memoryless Gaussian and Laplacian Sources, IEEE Trans. on Information Theory **39** No. 3 (1993), 786–804.

[8] GRAY, R. M.—NEUHOFF, D. L.: Quantization, IEEE Trans. on Information Theory **44** No. 6 (1998), 2325–2384.

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