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COMMUNICATIONS

ROBUST CONTROLLER DESIGN FOR LINEAR SYSTEMS WITH PARAMETRIC AND DYNAMIC UNCERTAINTIES

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The paper deals with the synthesis of a robust PID controller for uncertain linear continuous-time SISO systems with specified parametric and dynamic uncertainty. First, the robust controller design for parametric uncertainty is performed in the frequency domain, based upon necessary and sufficient closed-loop robust stability conditions. Then, applying the small gain, the stability of the closed-loop with dynamic uncertainty is verified. The design procedure is demonstrated on an example.

K e y w o r d s: parametric uncertainty, dynamic uncertainty, robust stability

1 INTRODUCTION

A time-invariant mathematical model never describes the properties of a real dynamic plant sufficiently exactly. The reason for it is a perpetual time change of its parameters (due to aging, influence of environment, working point changes *etc*), as well as its unmodelled dynamics. The former uncertainty type is denoted as the parametric uncertainty and the latter one the dynamic uncertainty. A controller ensuring closed- loop stability under both of these uncertainty types is called a robust controller. A lot of robust controller design methods are known from the literature [1, 4], in the time- as well as in the frequency domain. In this paper a method is proposed the main aim of which is to reject as much as possible the unfavourable effect of dynamic uncertainty on the closed-loop stability. The design philosophy is based upon the general stability theory and application of the vector Lyapunov function.

2 PROBLEM FORMULATION

Let the real dynamic plant be described by differential equations

$$\dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1) + b_1(\mathbf{x}_1, \boldsymbol{u}_1) + h_1(\mathbf{x}_2)\,, \tag{1a}$$

$$\dot{\mathbf{x}}_2 = f_2(\mathbf{x}_2) + h_2(\mathbf{x}_1)\,. \tag{1b}$$

Here $\mathbf{x}_1$ and $\mathbf{x}_2$ are state vectors of the plant, whereby $\mathbf{x}_1$ and $f_1(\mathbf{x}_1)$ characterize the modelled, and $\mathbf{x}_2$ and $f_2(\mathbf{x}_2)$ the unmodelled dynamics of the plant. The functions $h_1(\mathbf{x}_2)$ and $h_2(\mathbf{x}_1)$ express mutual interaction between both uncertainty types. It is assumed that the functions f_i and h_i, $i = 1, 2$ are continuous, bounded and differentiable, whereby f_1 is a known function in difference to unknown functions f_2, h_1, and h_2. The task is to design such a control $\boldsymbol{u}_1 = \boldsymbol{u}_1(\mathbf{x}_1)$, which guarantees the closed-loop stability of the real plant described by (1).

3 ROBUST STABILITY PROBLEM

It is assumed that there exists a unique solution to differential Eqs. (1)

$$\mathbf{x}^\top(\mathbf{x}_0, t_0) = [\mathbf{x}_1(\mathbf{x}_{10}, t_0), \mathbf{x}_2(\mathbf{x}_{20}, t_0)] \tag{2}$$

bounded as follows
$\|\mathbf{x}\| \le d$, $\|\mathbf{x}_1\| \le d_1$, $\|\mathbf{x}_2\| \le d_2$, whereby $\mathbf{x} \in R^n$, $\mathbf{x}_1 \in R^{n1}$, $\mathbf{x}_2 \in R^{n2}$, where d, d_1, d_2 are real positive numbers and $\mathbf{x}_0$ and t_0 are initial conditions. For (1) a Lyapunov function is to be found $V(\mathbf{x})$ defined over $R_d = \{\mathbf{x} \in R^n \colon \|x\| \le d\}$. Using the Lyapunov function we can express the general stability conditions of the plant (1) as stated in the following theorem [2, 5]:

THEOREM 3.1. *Suppose that function f_1 satisfies the condition*

$$\|f_1(\mathbf{x}_1, \mathbf{c}^*)\| \le c_1\,, \quad c_1 > 0\,, \tag{3}$$

*$t \ge t_0$, and $f_1(\mathbf{0}, \mathbf{c}^*) = 0$,
where $\mathbf{c}^*$ denotes parameters of the nominal model of the plant. If there exists a Lyapunov function $V(\mathbf{x})$ which satisfies*

$$a\big(\|\mathbf{x}\|\big) \le V(\mathbf{x})\,, \tag{4a}$$

$$V(\mathbf{x}) \le -b\big(\|\mathbf{x}_1\|\big)\,, \tag{4b}$$

where $a\big(\|\mathbf{x}\|\big)$ and $b\big(\|\mathbf{x}_1\|\big)$ are continuous, scalar monotonously growing functions, then the equilibrium $\mathbf{x} = 0$ is stable, and asymptotically stable with respect to vector $\mathbf{x}_1$.

For (1), consider a closed-loop system under the control law

$$\boldsymbol{u}_1 = \boldsymbol{u}_1(\mathbf{x}_1, \boldsymbol{r}_1) \tag{5}$$

where $\boldsymbol{r}_1$ is the vector of controller parameters. Generate a Lyapunov function for the subsystem (1a) $V_1(\mathbf{x}_1)$

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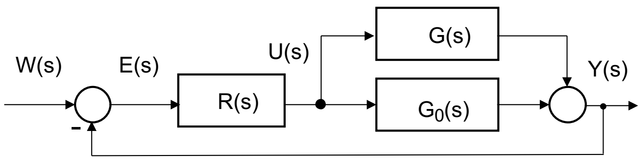

Fig. 1. Closed-loop block scheme

defined over $R_{d1} = \{\mathbf{x}_1 \in R^{n1}\colon \|\mathbf{x}_1\| \leq d_1\}$ and for the subsystem (1b) a Lyapunov function $V_2(\mathbf{x}_2)$ over $R_{d2} = \{\mathbf{x}_2 \in R^{n2}\colon \|\mathbf{x}_2\| \leq d_2\}$. The resulting Lyapunov function is the sum of $V_1(\mathbf{x}_1)$ and $V_2(\mathbf{x}_2)$. Its time derivative is

$$\dot{V} = \sum_{i=1}^{2} \left[(\operatorname{grad} V_i)^\top (f_i + h_i) + (\operatorname{grad} V_1)^\top b_1 \right] . \tag{6}$$

Splitting it into three parts we can bound each of them by defining three positive real coefficients $w_{11} = w_{11}(\boldsymbol{r}_1)$, w_{22} and w_{12}. Thus, a system of three inequalities is obtained

$$(\operatorname{grad} V_1)^\top [f_1(\mathbf{x}_1) + b_1(\mathbf{x}_1, \boldsymbol{r}_1)] \leq -w_{11}(\boldsymbol{r}_1)\|\mathbf{x}_1\|^2, \tag{7a}$$

$$(\operatorname{grad} V_2)^\top [f_2(\mathbf{x}_2)] \leq -w_{22}\|\mathbf{x}_2\|^2 , \tag{7b}$$

$$0.5 \sum_i \left\|(\operatorname{grad} V_i)^\top \boldsymbol{h}_i\right\| \leq w_{12}\|\mathbf{x}_1\|\|\mathbf{x}_2\| , \tag{7c}$$

where $w_{11} = w_{11}(\boldsymbol{r}_1)$, w_{22} characterizes the exponential stability of the modelled and unmodelled dynamics and w_{12} specifies the size ("intensity") of their interactions. Joining inequalities (7), we can directly bound $\dot{V}$ as follows

$$\dot{V} \leq \|\mathbf{x}_1\|\|\mathbf{x}_2\| \begin{bmatrix} -w_{11}(\boldsymbol{r}_1) & w_{12} \\ w_{12} & -w_{22} \end{bmatrix} \begin{bmatrix} \|\mathbf{x}_1\| \\ \|\mathbf{x}_2\| \end{bmatrix} . \tag{8}$$

Based upon inequalities (7), (8), the following theorem can be formulated:

THEOREM 3.2. *The equilibrium* $\mathbf{x} = 0$ *of* (1) *is stable, and asymptotically stable with respect to vector* $\mathbf{x}_1$, *if inequalities* (7), *or* (8) *hold.*

a) From inequality (7a) the stabilizability of subsystem (1a) results by means of parameters $\boldsymbol{r}_1$.
b) Inequality (7b) expresses the asymptotic stability of the unmodelled dynamics, *ie* of subsystem (1b).
c) Stability of (1) requires fulfillment of the inequality

$$w_{11}w_{22} \geq w_{12}^2 \tag{9}$$

which results from (8).

Analysing (9) we can come to the conclusion that the stability of the overall plant (1) can be guaranteed either by increasing the exponential stability of (1a), *ie* by increasing the coefficient $w_1(\boldsymbol{r}_1)$, or by decreasing the interactions between the modelled and unmodelled dynamics of (1) (*ie* the coefficient w_{12}). Decreasing w_{12} can be realized by the controller design due to which the amplitude frequency spectrum of the controlled plant (1a) is shifted to lower frequencies. Thus, the effect of the amplitude frequency spectrum of the unmodelled dynamics of (1b) (which, as a rule, is situated in the range of higher frequencies) is simultaneously decreased thus slowing down the control dynamics.

4 APPLICATION FOR CONTINUOUS LINEAR SYSTEMS

Let the plant (1) be a linear uncertain system described by a transfer function $G(s)$ with additive dynamic uncertainty $\delta G(s)$, *ie*

$$G(s) = G_0(s) + \delta G(s) , \tag{10}$$

where $G_0(s)$ represents the linear interval transfer function and $\delta G(s)$ the unmodelled dynamics. The corresponding closed-loop block scheme is in Fig. 1, where $R(s)$ is the controller transfer function.

According to the necessary and sufficient condition for the closed-loop stability, the zeros of

$$1 + R(s)G(s) = 0 \tag{11}$$

have to be situated in the left half complex plane. After substituting (9) to (10) and some manipulations we obtain

$$\begin{aligned} &1 + R(s)\left[G_0(s) + \delta G(s)\right] = \\ &= \left[1 + R(s)G_0(s)\right]\Big[1 + \frac{R(s)G_0(s)}{1 + R(s)G_0(s)} \frac{\delta G(s)}{G_0(s)}\Big] . \end{aligned} \tag{12}$$

According to the Small Gain theory [4], the closed-loop is stable if the characteristic equation without considering dynamic uncertainty is stable

$$1 + R(s)G_0(s) = 0 \tag{13}$$

as well as the characteristic equation

$$1 + G_u(s)\frac{\delta G(s)}{G_0(s)} = 0 , \tag{14}$$

where

$$G_u(s) = \frac{R(s)G_0(s)}{1 + R(s)G_0(s)} \tag{15}$$

denotes the closed-loop without considering uncertainty.

The stability of the characteristic equation (13) is tested using the criterion

$$\left\|G_u(j\omega)\frac{\delta G(j\omega)}{G_0(j\omega)}\right\| < 1 , \tag{16}$$

which results from the Small Gain theory and further yields the inequality

$$\|G_u(j\omega)\| < \left\|\frac{G_0(j\omega)}{\delta G(j\omega)}\right\| = M_0(\omega)\,, \qquad (17)$$

where $M_0(\omega)$ is the modulus of the ratio of the modelled- and unmodelled transfer function magnitudes of the considered system (10).

According to (17), the modulus of the closed-loop without considering the dynamic uncertainty has to lie below the function $M_0(\omega)$. The likelihood of fulfillment of this requirement increases with the decrease of the dynamic uncertainty modulus $\|\delta G(j\omega)\|$ (especially in the range of low frequencies), or with the location of its amplitude frequency spectrum in the range of high frequencies. This fully complies with the conclusions drawn from inequality (9).

Substituting (15) into inequality (17) we obtain

$$\frac{1}{\left\|\frac{R(j\omega)}{1+R(j\omega)G_0(j\omega)}\right\|} > \|\delta G(j\omega)\| \qquad (18)$$

where $G_0(s=\omega)$ represents an infinite number of transfer functions generated by means of interval coefficients. In [4] it is shown that instead of the infinite number of $G_0(s)$ in (18) it is sufficient to consider a finite number of the so-called boundary transfer functions $G_i(s)$, $i = 1,2,\dots,p$, generated from the Kharitonov polynomials and segments. Using the minimum- and maximum values of interval coefficients of $G_0(s)$, 4 Kharitonov polynomials of the numerator $G_0(s)$ are generated, denoted $P_{c1}(s)$ through $P_{c4}(s)$, and 4 Kharitonov polynomials $P_{m1}(s)$ through $P_{m4}(s)$ for the denominator of $G_0(s)$. Further, 4 segments $S_{c1}(s)$ through $S_{c4}(s)$ are generated for the numerator and 4 segments $S_{m1}(s)$ through $S_{m4}(s)$ for the denominator of $G_0(s)$. The numerator (denominator) segments are actually convex combinations of two neighbouring numerator (denominator) Kharitonov polynomials. For the numerator, the 4 couples ordered as $P_{c1}(s) - P_{c2}(s)$, $P_{c2}(s) - P_{c3}(s)$, $P_{c3}(s) - P_{c4}(s)$ and $P_{c4}(s) - P_{c1}(s)$ have been considered. For the numerator $G_0(s)$, the first segment is $S_{c1}(s) = \beta P_{c1}(s) + (1-\beta)P_{c2}(s)$, where $\beta \in \langle 0,1\rangle$ is the coefficient of the convex combination. The 4 segments of the denominator are generated by analogy, *eg* $S_{m1}(s) = \beta P_{m1}(s) + (1-\beta)P_{m2}(s)$. The first 4 boundary transfer functions $G_1(s)$ through $G_4(s)$ have been defined as follows: $G_1(s) = P_{c1}(s)/S_{m1}(s)$ through $G_4(s) = P_{c1}(s)/S_{m4}(s)$. The other 12 transfer functions $G_i(s)$ have been derived from them by successively exchanging $P_{c1}(s)$ for $P_{c2}(s)$, $P_{c3}(s)$ for $P_{c4}(s)$ *etc.* In the denominators of the further 16 transfer functions $G_i(s)$, there will be the polynomials $P_{mi}(s)$ and in their numerators the segments $S_{ci}(s)$. The total number of the boundary transfer functions is $p = 32k$, where k denotes the number of selected discrete values of the coefficient β. All the p transfer functions have been gathered in the set G_E.

The closed-loop system comprising the controlled system with parametric uncertainties $G_0(s)$ and the dynamic uncertainty $\delta G(s)$ is robustly stable if all the p closed-loops under the controlled systems $G_i(s)$ and the controller $R(s)$ are stable, ie if the following inequality holds

$$\alpha(\omega) = \frac{1}{\sup\limits_{G_i\in G_E} r\left|\frac{R(j\omega)}{1+R(j\omega)G_i(j\omega)}\right|} > |\delta G(j\omega)|\,. \qquad (19)$$

In case we do not consider the dynamic uncertainty, on the r.h.s. of (19) is zero. The supremum of the norm on the set G_E is evaluated for discrete values ω_k from the chosen interval $\omega_k \in \langle\omega_{\min},\omega_{\max}\rangle$ with a chosen step. For the known structure and parameters of the dynamic uncertainty it is possible to plot the magnitude of $\delta G(j\omega)$. For the robustly stable closed-loop this plot has to lie under the plotted $\alpha(\omega)$, *ie* the following inequality has to be satisfied

$$\alpha(\omega > |\delta G(j\omega)|\,. \qquad (20)$$

The robust stability conditions (18), (19) a (20) are used for the analysis of the closed-loop robust stability under the known controller transfer function $R(s)$. These conditions imply that all closed-loops under the controlled systems $G_i(s)$, $i = 1,2,\dots,p$ and the controller $R(s)$ are to be stable. Therefore, the controller design has been performed by means of D-curves of the p closed-loop characteristic equations under the controlled systems $G_i(s)$ and the controller $R(s)$, plotted in the plane of controller parameters. The controller parameters have been chosen from the intersection of stable regions determined by all D-curves.

5 EXAMPLE

In a power plant, a step change of the synchronous generator voltage setpoint brings about transition change in its real power according to the transfer function

$$\frac{\Delta P_G(s)}{\Delta U_{GZ}(s)} = \frac{b_2s^2+b_1s+b_0}{a_3s^3+a_2s^2+a_1s+a_0}\,, \qquad (21)$$

where the parameters vary with the operating point within the following intervals

$a_0 = 1188.8 \div 1464.9$ $\quad b_0 = 0.7773 \div 3.4916$

$a_1 = 83.2 \div 89.2$ $\quad b_1 = 323.4378 \div 383.9109$

$a_2 = 25.0 \div 30.5$ $\quad b_2 = -6.1073 \div -3.6038$

$a_3 = 1.0 \div 1.0$

A stabilizing feedback from the real power is to be designed with the following structure

$$R(s) = \frac{c_1s+c_0}{Ts+1}\,. \qquad (22)$$

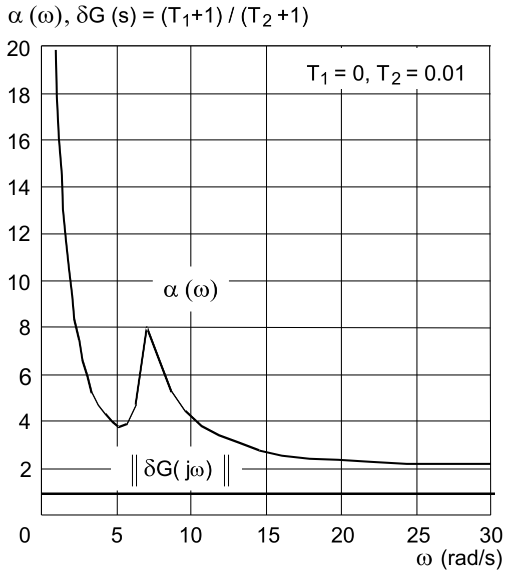

Fig. 2. Illustraion to above given example

The time constant $T = 0.12$ s has been determined with respect to the required noise filtering and $c_0 = 0$ has been chosen in order to ensure a derivative character of the stabilizing element. Thus, only the coefficient c_1 is to be determined to ensure robust stability with respect to specified interval changes of coefficients in (21) and to the estimated dynamic uncertainty.

$$\delta G(s) = \frac{1}{(0.01 \div 1)s + 1} \tag{23}$$

The coefficient c_1 has been found using the D-partition method for the required closed-loop stability degree. The final value of c_1 to be chosen is the one with the highest stability degree, in our case $c_1 = 0.05$. For the identified stabilizing element (22) the function $\alpha(\omega)$ is plotted, as well as $\|\delta G(j\omega)\|$ in (Fig.2). If $\|\delta G(j\omega)\|$ lies under $\alpha(\omega)$ within the specified frequency range, then, according to (18), the closed-loop robust stability for the specified parametric as well as dynamic uncertainty is ensured. By changing the transfer function coefficients of the dynamic uncertainty (23) we can find the borderline transfer functions (23) for which the closed-loop is on the stability limit.

6 CONCLUSION

In the proposed paper, the general stability conditions of dynamic systems (7) have been closer specified for linear dynamic SISO systems with parametric and dynamic uncertainty using the inequality (18).

The main contribution of the paper is the knowledge that the closed-loop stability of a plant with parametric and dynamic uncertainty can be ensured either by increasing the exponential stability of the controlled plant via increasing the coefficient $w_{11}(\boldsymbol{r}_1)$, or by decreasing the interactions (w_{12}) between the modelled- and the unmodelled plant dynamics which can be realized by designing a controller which shifts the amplitude frequency spectrum of the modelled dynamics towards lower frequencies.

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