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COMMUNICATIONS

MAKING A CONNECTION: PI AND PID CONTROLLER TUNING BY ANALOGY WITH THE MAXIMUM POWER TRANSFER THEOREM OF CIRCUIT THEORY

Annraoi de Paor [*] — **Brian Cogan** [**]

The well known maximum power transfer theorem of linear AC circuit theory is reformulated as a relation in a single-loop feedback system and used to motivate a new idea for the design of PI and PID controllers for a class of asymptotically stable processes. Designs are compared with those yielded by a root locus-based optimum stability approach, and with a parameter space approach due to Datta, Ho and Bhattacharyya.

K e y w o r d s: control theory, circuit theory, maximum power transfer, optimum stability, root locus, phase margin, pid controllers, Nyquist stability theory

1 INTRODUCTION

Some years ago [1] we proved that nonlinear resistive loading of a series-wound, self-excited DC generator driven by a wind turbine, in such a way as to optimise power transfer from wind to electrical load, resulted in a very well damped dynamic response to varying wind speeds. This prompted us recently to explore whether there might be some other favourable consequences for control lying unexploited in results on optimum power transfer. This note presents a resulting new idea for tuning PI and PID controllers for a class of asymptotically stable processes, discovered by viewing the Maximum Power Transfer Theorem of linear AC circuit theory as a relation in a single loop, negative feedback system. We hope that it will be of interest to the general community of electrical engineers, insofar as it brings together ideas from the cognate subjects of Circuit Theory and Control Theory, continuing an old but often overlooked tradition (Truxal, [2]). We hope that it will also be regarded as a modest contribution in its own right to the many methods available for designing PI and PID controllers. We do not review the vast literature on such methods here. The reader is referred to sources such as the comprehensive survey by O'Dwyer [3], and the book by Datta *et al* [4].

In order to set the scene, Fig. 1 shows a single-loop linear electric circuit, operating in steady state under sinusoidal excitation at angular frequency ω radians per second. The upper case boldface quantities are phasor (complex number) representatives of real sinewaves. Thus, $\mathbf{V} = |\mathbf{V}| \exp(j\angle\mathbf{V})$ is the phasor representative of the sinusoidal voltage $v = |\mathbf{V}| \sin(\omega t + \angle\mathbf{V})$. The source has sinusoidal emf and internal impedance $Z(j\omega)$ or, as we shall more conveniently characterise it below, admittance $Y(j\omega) = 1/Z(j\omega)$, where $Z(j\omega) = r + jx$ ohms. The Maximum Power Transfer Theorem states that, in order to maximise the mean power delivered to the load over any integral number of cycles, the load impedance should have the value $Z_L(j\omega) = r - jx$, *ie*, it should be the complex conjugate of the source impedance. In terms of source admittance, $Y(j\omega) = [r - jx]/\left[r^2 + x^2\right]$, this leads to magnitude and phase conditions

$$|Z_L(j\omega)) = 1/|Y(j\omega)|\,, \qquad \angle Z_L(j\omega) = \angle Y(j\omega)\,. \tag{1}$$

From Fig. 1 we write the relations

$$\mathbf{V} = Z_L(j\omega)\,\mathbf{I}\,, \qquad \mathbf{I} = Y(j\omega)\,\mathbf{E}\,, \qquad \mathbf{E} = \mathbf{U} - \mathbf{V}\,. \tag{2}$$

Equation (2) may be represented by the single-loop error-actuated feedback system shown on Fig. 2 (see, for example, the same idea in a different context, D'Azzo and Houpis [5]).

Figure 2 suggests that, in designing the controller $C(s)$ for the process $G(s)$ in Fig. 3, we might explore the counterpart of (1), *ie*, we might examine the possibility of specifying a design angular frequency ω in such a way that

$$|C(j\omega)| = 1/|G(j\omega)|\,, \qquad \angle C(j\omega) = \angle G(j\omega)\,. \tag{3}$$

Equation (3) shows that on invoking this idea, the gain of the controller would be the inverse of that of the process under control, at the chosen design frequency, and the phase shift introduced by the controller would be equal to that introduced by the process.

The initial results of an exploration of this idea, including a comparison of one of the resulting controller designs with a root-locus-inspired eigenvalue-based optimum stability approach, and with a parameter plane one due to Datta *et al* [4], are presented below.

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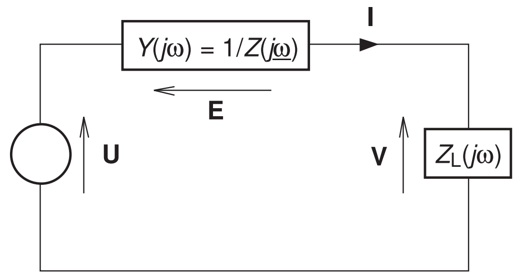

Fig. 1. Single-loop electric circuit, to illustrate maximum power transfer theorem

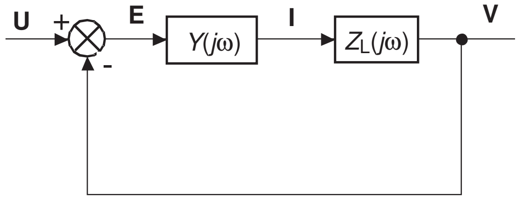

Fig. 2. Single-loop feedback control system

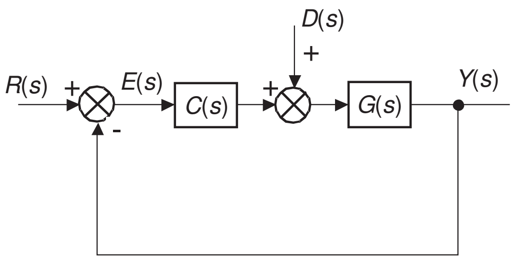

Fig. 3. Electric circuit as a feedback loop

2 PRELIMINARY DEVELOPMENT OF THE THEORY

In this initial presentation, both for analytical convenience for ease of comparison with a case study by Datta *et al* [4], attention is restricted to process transfer functions of the form

$$G(s) = k/(s+a)^n \tag{4}$$

with $k, a > 0$ and n a positive integer. This represents a cascade of n identical first order lags, each with time constant $1/a$. The method applies to any asymptotically stable process, but we have the advantage here of being able to do a comparison with other designs.

For economy in notation, controller transfer functions are taken to have the form

$$C(s) = K\,(s+b)^m/s \tag{5}$$

with $K, b > 0$ and $m = 1, 2$. The case $m = 1$ corresponds to PI control with integral action time $T_i = 1/b$, while $m = 2$ gives a restricted class of PID control with $T_i = 2/b$ and derivative action time $T_d = 1/2b$, thus giving $T_d = T_i/4$. That is actually the classical Ziegler-Nichols relation [6] between T_d and T_i.

We have found it convenient, and immediately interpretable in terms of Nyquist stability theory [7], to base our design procedure on choosing ω so that

$$\angle G(j\omega) = -\pi/2 + \phi/2 \tag{6}$$

with the angle ϕ, $\{0 < \phi < \pi\}$, to be specified by the designer. For an asymptotically stable $G(s)$ with $G(0) > 0$, such a choice is always possible. This leads, via eqns. (4) and (6) to

$$\omega = a\ \tan\big((\pi - \phi)/(2n)\big)\,. \tag{7}$$

Equation (6) and the second line of (3) give

$$m\ \tan^{-1}(\omega/b) - \pi/2 = -\pi/2 + \phi/2 \tag{8}$$

which combines with (7) to evaluate the first controller parameter as

$$b = a\ \tan\big((\pi - \phi)/(2n))/tan(\phi/(2m)\big)\,. \tag{9}$$

The first line of (3) now yields

$$K\,\left(\omega^2 + b^2\right)^{m/2}/\omega = \left(\omega^2 + b^2\right)^{m/2}/k\,. \tag{10}$$

Once ω has been evaluated from (7) and b from (9), K follows from (10).

These design equations give

$$\angle G(j\omega)C(j\omega) = -\pi + \phi\,. \tag{11}$$

Since $|G(j\omega)\,C(j\omega)| = 1$, (11) shows that the procedure works because the system has a specified phase margin ϕ, effective at the design angular frequency ω, which is extracted from the Nyquist diagram of $G(j\omega)$ at the phase angle given in (6). Specification of the phase margin — a classical robustness measure — is valuable, as it often gives sensitive control of the amount of damping in a system.

3 A FIRST EXAMPLE

Choosing $k = a = 1$, $n = 3$, we specify the phase margin as $\phi = \pi/4$. This leads to $\omega = \tan(\pi/8) = 0.4142$. The resulting controllers are:

$$\text{PI}: \qquad C(s) = 0.4853\,(s+1)/s\,, \tag{12}$$

$$\text{PID}: \qquad C(s) = 0.1165\,(s+2.0824)^2/s\,. \tag{13}$$

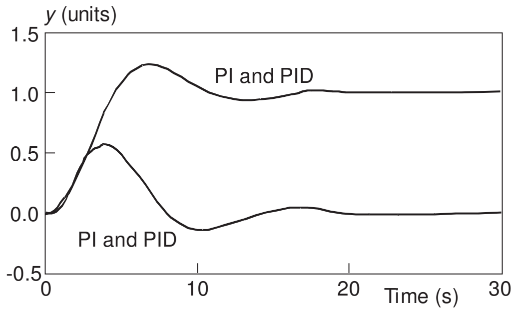

Fig. 4. Responses of third order system under PI and PID control

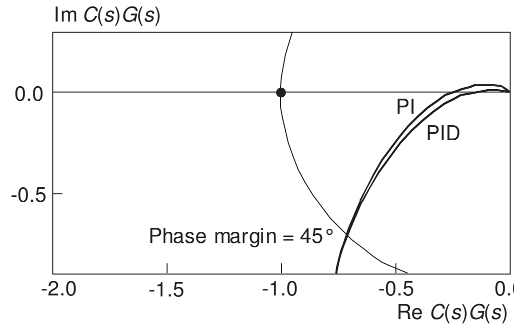

Fig. 5. Open loop frequency response loci for the systems in Fig. 4

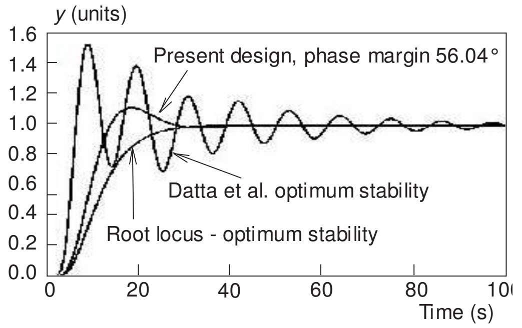

Fig. 6. Comparison of three PID designs for eighth-order process, step reference input

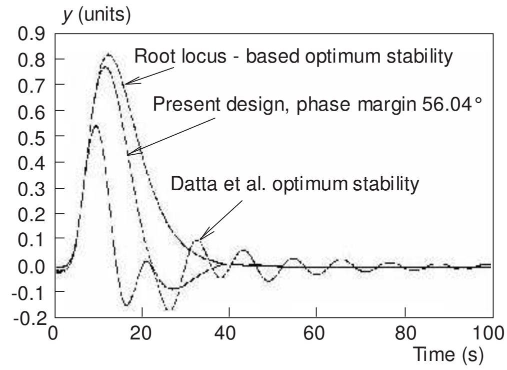

Fig. 7. Comparison of three PID designs for eighth-order process, step disturbance input

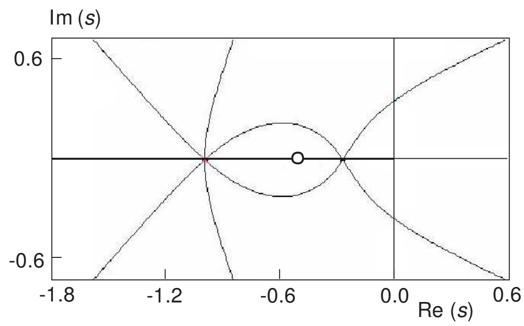

Fig. 8. Root locus to illustrate an approach to optimum stability. Design places three eigenvalues at triple breakpoint nearest origin.

Responses of the process output to unit step reference and disturbance inputs are shown on Fig. 4. PI and PID control are almost indistinguishable. This may at first sight seem surprising. However, the gain margins are so large — 6.507 for PID control (equivalent to 16.267 dB), and 4.121 (equivalent to 12.300 dB) for PI — that the dominant indicator of damping is the phase margin, and this is the same, $\pi/4$, in both cases.

The frequency responses of $C(s)G(s)$ are shown on Fig. 5. These confirm asymptotic stability using the Nyquist criterion (Franklin *et al* [7]).

4 A SECOND EXAMPLE, INCLUDING COMPARISON WITH OPTIMUM STABILITY DESIGNS

We now consider PID control of the process having $k = a = 1$, $n = 8$. An interesting study of this has been made by Datta *et al* [4] using optimum stability ideas in a parameter space. They wrote the PID controller transfer function in the form

$$C(s) = k_p + k_i/s + k_d s\,. \qquad (14)$$

For each chosen value of k_p they plotted a triangular domain of asymptotic stability in the (k_i, k_d) plane. They noted the radius of the largest circle which would just fit in this domain. They then searched for the value of k_p which maximised this radius, and chose as design parameters the values of k_i and k_d at the centre of this largest circle, along with the corresponding value of k_p. They found $k_p = 1.32759$, $k_i = 0.42563$, $k_d = 5.15291$. Corresponding process output responses to unit step reference and unit step disturbance, which are not shown by Datta *et al* [4], are on Figs. 6 and 7.

In applying the present design idea to this process, we chose the phase margin $\phi = 56.04^\circ = 0.9781$ radians. This specific value was motivated by comparison with an optimum stability design below. The resulting PID controller is

$$C(s) = 0.4636(s + 0.5453)^2/s\,. \qquad (15)$$

Responses are shown on Figs. 6 and 7.

We also compare the present design with a PID controller based on a root locus-inspired principle of optimum stability [8, 9, 10], *ie*, that the rightmost eigenvalue should, subject to structural relations between system parameters, lie as deep in the left half plane as possible. We computed that the value $b = 0.5453$ gives a triple breakpoint in the root locus of the characteristic polynomial, plotted with respect to K, and deduced that designing for the value of K which places three eigenvalues at this breakpoint confers optimum stability. In this example, but not in all that we have studied, optimum stability yields real, equal, dominant eigenvalues — three in this case — thus generalising the idea of critical damping in a second order system. The resulting controller is

$$C(s) = 0.2874(s + 0.5453)^2/s\,, \tag{16}$$

which differs only in gain from the controller in (15). The fact that the gain in (16) is less than in (15) could not readily have been predicted before its calculation.

Once optimum stability considerations had led to the value of b in (15), the corresponding phase margin, $\phi = 0.9781$, was evaluated, via a stable iterative inversion of (9). Equation (7) then gave ω and 10), K.

The root locus of the characteristic polynomial with respect to K is shown on Fig. 8. For $K = 0.2874$ (optimum stability, as in (16)) the triple rightmost eigenvalue lies at the breakpoint, $s = -0.2791$, whereas for $K = 0.4636$ (optimum power transfer analogy, (15)) the rightmost eigenvalues form the complex pair $s = -0.1420 \pm 0.2159j$.

It is interesting to note from Fig. 6 that the optimum power transfer analogy and root locus-based optimum stability give the same settling time of approximately 30 seconds, but that the latter has no overshoot. The parameter plane idea invoked by Datta et al. gives quite an underdamped response, which has not settled in 100 seconds. With regard to disturbance rejection, as portrayed on Fig. 7, the optimum power transfer analogy gives tighter control than root locus-based optimum stability, but at the expense of undershoot. The output excursion is more restricted with the Datta *et al* [4] controller, but at the expense of a much longer settling time.

5 CONCLUSIONS

A new idea for tuning PI and PID controllers has been presented, based on analogy with the maximum power transfer theorem from linear AC circuit theory. The approach has been identified as one which specifies the phase margin and the frequency at which it is effective. It has been illustrated by designs for third order and eighth order members of a restricted class of asymptotically stable processes, considered by Datta *et al* [4]. In the case $n = 3$, it is interesting to note that the performances of the PI and PID controllers are indistinguishable, both for reference input following and for disturbance rejection, despite significant differences in gain margin. In the case $n = 8$, the performance is similar in time scale to, though distinguished in overshoot (reference tracking) and undershoot (disturbance rejection) from, controllers designed by a root locus-based optimum stability approach. Controllers designed by an optimum parameter space approach due to Datta *et al* [4] give a much more oscillatory behaviour and longer settling time.

References

[1] De PAOR, A.—Cogan, B.: Windspeed-Dependent Underdamping and its Cure in the Self-Excited Series-Wound Aerogenerator, Applied Energy **42** (1992), 253–267.

[2] TRUXAL, J. G.: Automatic Feedback Control System Synthesis, McGraw-Hill, New York, 1955.

[3] O'DWYER, A.: Handbook of PI and PID Controller Tuning Rules, World Scientific Press, London, 2003.

[4] DATTA, A.—HO, M-T.—BHATTACHARYYA, S. P.: Structure and Synthesis of PID Controllers, Springer, London, 2000.

[5] D'AZZO, J. J.—HOUPIS, C. H.: Linear Control System Analysis and Design: Conventional and Modern, 4th edition, McGraw-Hill, New York, 1995.

[6] ZIEGLER, J. G.—NICHOLS, N. B.: Optimum Settings for Automatic Controllers, Trans. ASME **64** (1942), 759-768.

[7] FRANKLIN, G. F.—POWELL, J. D. EMAMI-NAEINI, A.: Feedback Control of Dynamic Systems, 4th edition, Prentice Hall, Upper Saddle River, New Jersey, 2002.

[8] COGAN, B.—de PAOR, A.: Optimum Stability and Minimum Complexity as Desiderata in Feedback Control System Design, Proc. IFAC Conference on Control Systems Design, (Bratislava, Slovakia, June 18-20, 2000) 51–53.

[9] De PAOR, A.: Concepts of Optimum Stability for Linear Feedback Systems, Int. J. Elec. Enging. Educ. **36** (1999), 46–64.

[10] De PAOR, A.: The Root Locus Method: Famous Curves, Control Designs and Non-Control Applications, Int. J. Elec. Enging. Educ. **37** (2000), 344–356.

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