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A NOVEL BINARY PROGRAMMING APPROACH FOR OPTIMAL DESIGN OF CELLULAR MOBILE COMMUNICATION SYSTEM

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The present paper deals with the problem of determining the minimum number of base stations required to cover a given radio network area, together with the associated cell size and the best topographical configurations. Existing solutions to this NP-Complete problem are based on the heuristic approaches but these do not guarantee convergence to a true globally optimal solution. In this paper, a strong binary programming formulation is developed for the problem and a special branch and cut algorithm is applied for obtaining guaranteed optimal solutions. The presented examples show that using the proposed techniques, it is possible to solve optimally fairly large-size network design problems for the GSM as well as CDMA based mobile radio cellular systems.

K e y w o r d s: network design, independent set, branch and cut algorithm, binary linear programming

1 INTRODUCTION

As a result of the increasing demand for an ever-widening range of services, the optimal design of mobile radio networks is becoming critically important. An important aspect of cellular network design concerns determining the minimum number of cells required to cover a given design area together with the associated cell size and best topographical configurations, whilst complying with bandwidth and other constraints. This design problem is referred to as the *cellular layout design problem* (CLDP) and is considered in the present paper. In general, one base station (BS) is established in each cell for providing radio coverage to its associated cell area. This implies that the solution to the CLDP provides the minimum number of BS's required to cover a given network area. In the engineering of any cellular radio network, the costs incurred on the deployment of BS's represent a significant investment. The number of BS's required to cover a given cellular network design area not only affects the initial infrastructure costs, but also has a significant effect on the subsequent operational and administrative costs. An effective engineering approach aimed at minimizing the number of BS's required to cover a given network area can reduce significantly the overall installation and operational costs of any cellular network. The development of such a technique is the focus of this paper.

It is shown [1] that the CLDP belongs to the family of non-polynomial (NP) complete problems. The computational complexity of exact solution algorithms associated with such NP-complete problems grows exponentially with the problem size. This fact alone is a major restraining factor for attaining true optimal solutions for large network design problems. Consequently, the existing solutions to the CLDP are mainly based on heuristic techniques. In [1], a three-stage design procedure is proposed for solving this combinatorial optimization problem. In order to solve the formulated problem, a simulated annealing algorithm is used. The proposed formulation however fails to guarantee non-formation of enclaves, and fails to ensure compliance of the compactness and the contiguity considerations. In [2], a fuzzy expert system is proposed in order to solve the CLDP. In [3], a genetic algorithm is presented in order to find the cell sites for medium size networks. Tutschku [4] has dealt with the CLDP as the minimal coverage location problem, and used a greedy heuristic algorithm in order to a find a good solution. In [5], the BS minimization problem has been formulated as a graph optimization problem. The common limitation of these techniques lies in the fact that they do not guarantee convergence to a true globally optimal solution. These heuristic techniques are usually trapped in the local optima depending on the starting conditions. The use of a sub-optimal solution can lead to significant unwarranted costs in the design of the network.

The aim of the work reported in this paper is to investigate the viability of using exact solution techniques for finding the truly optimal solution for the CLDP. Comparison in terms of computational efficiency, such as CPU time, between the already reported heuristics and the analytical approach presented in this chapter, is not the main aim of this chapter. In contrast to previously reported results, the present work is of fundamental importance since it provides the first known general analytical solution approach to the CLDP, based on a mathematical programming formulation and an associated exact solution technique.

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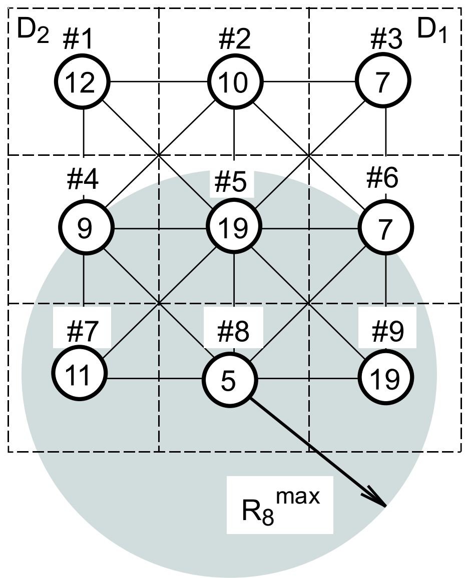

Fig. 1. The representation of a given network design area in terms of a grid and an overlaid graph. Numbers inside circles denote bandwidth demands. The maximum coverage for node No. 8 is also shown. Node indices are preceded with #.

This paper is organized into five more Sections. In Section 2, a description of the problem is presented using vectors and matrices. The mathematical formulation is presented in Section 3. The solution methodology is described in Section 4. The computational results are given in Section 5. Finally, some conclusions are presented in Section 6.

2 PROBLEM DESCRIPTION

2.1 The Topology Graph

Initially, the network area to be covered in the design procedure is partitioned into a large grid. Each grid element is of equal size and there is a bandwidth demand associated with every grid-element. The centre of each grid is represented by a 'demand node'. The Euclidean distance between two adjacent demand nodes is represented by the weight of an arc connecting the two nodes. In this way, the CLDP is transformed into a problem involving a non-directed 'topology graph' $G = (V, E)$, where $V = \{1, 2, 3, \dots, M\}$ is the set of nodes, E is the set of arcs and M is the number of grid elements. There is an arc $e_{(i,j)}$, between node v_i and v_j if and only if these are adjacent nodes.

2.2 The Demand Vector

The number of channels required in each grid element serves as the weight of the associated demand node. The weights of demand nodes by an M-element Demand Vector $\boldsymbol{D}$ are

$$\boldsymbol{D} = [\, d_1 \quad d_2 \quad \dots \quad d_j \quad \dots \quad d_M \,]^\top . \tag{1}$$

The vector $\boldsymbol{D}$ is developed a priori. It is assumed that the allocated radio spectrum consists of b discrete channels. Therefore, in the case of the CDMA based systems, the sum of channel demands of all nodes in a cell is bounded by b. For the GSM network design problem, this bandwidth constraint is estimated by dividing the available channels by a factor called cluster size. In this paper, the transmission loss is estimated using the Hata radio propagation model [6].

Considering that a BS can be located only at the demand nodes of the topology graph, and given the minimum acceptable C/I ratio at the receiver, the maximum coverage distance of a BS is determined. For a node 'i', $i \in V$, its maximum coverage distance is represented as $R_i^{\max}$. For radio networks with uniform propagation conditions it follows that

$$R_i^{\max} = R \,, \ \forall\, i \in V \,. \tag{2}$$

In the case of networks involving non-uniform conditions, the value of $R_i^{\max}$ is computed at each node of the topology graph.

2.3 The Primary Solution Matrix

For a CLDP involving M nodes, its associated *primary solution vector* $\boldsymbol{X}$ is defined as a binary N-element vector. Here $N \le M$ represents an upper bound on the number of BS's required to cover the given network design area. The elements of $\boldsymbol{X}$ are decision variables and are defined according to the rule:

$$x_j = \begin{cases} 1 & \text{if } j^{\text{th}} \text{ BS is needed to provide} \\ & \text{optimal radio coverage} \\ & \text{in the given network,} \\ 0 & \text{otherwise.} \end{cases} \qquad 1 \le j \le N \tag{3}$$

As an illustrative example, in Fig. 1, a 9-node topology graph for a CLDP is presented. For the network shown thus, if the upper bound 'N' on the number of required BS's is set to M (that is $N = M = 9$), the unknown vector X is given as

$$\boldsymbol{X} = [\, x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \,]_{1\times 9} \,. \tag{4}$$

Assume that the optimal solution yields $x_1 = x_3 = x_8 = x_9 = 1$. In this case, the optimal $\boldsymbol{X}$ is given as follows:

$$\boldsymbol{X} = [\, 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \,] \,. \tag{5}$$

Therefore, for the network shown, out of 9 possible BS's only 4 BS's (BS No. 1, 3, 8 and 9) are required in order to provide adequate radio coverage for the entire network.

Table 1. Statement of the Cellular Layout Design Problem (CLDP)

Given: 1. the topology graph G, 2. the limitation on maximum permissible cell radius R, 3. the Demand vector $\boldsymbol{D}$, 4. the Contiguity and Compactness constraints, 5. the allocated bandwidth (number of channels).
Determine: a. the optimal primary solution matrix $\mathbf{X}$, b. the Secondary Solution Matrix $\mathbf{Y}$, so as to satisfy conditions 2–5.

2.4 The Secondary Solution Matrix

For a CLDP including M-nodes and with 'N' as an upper bound on the number of BS's required, its Secondary Solution Matrix $\mathbf{Y}$ is a binary matrix of size $M \times N$. The elements of $\mathbf{Y}$, y_i^j, $1 \le i \le M$, $1 \le j \le N$, are *Secondary Decision Variables* defined according to the following rule:

$$y_i^j = \begin{cases} 1 & \text{if mode '}i\text{' is included} \\ & \text{in the cell associated with BS '}j\text{'} \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

In general, the computation of $\boldsymbol{X}$ and $\mathbf{Y}$ involves solving two distinct problems. In usual network planning techniques, these two problems are solved independently in a hierarchical order. In this paper, an integrated approach is presented for finding solutions to these two problems simultaneously. A summary of the problem statement is given in Table 1.

3 MATHEMATICAL FORMULATION

• The integration between $\mathbf{X}$ and $\mathbf{Y}$ is made possible by a special coupling, and is denoted as follows:

$$x_i^j \Rightarrow x_j \,. \quad (7)$$

This coupling implies that a node 'i' can be allocated to a BS 'j' if and only if BS 'j' is included in the design solution. The Boolean relationship implied by (7) can be formulated as the following constraint:

$$y_i^j \le x_j \,, \quad 1 \le i \le M \,, \, 1 \le j \le N \,. \quad (8)$$

• Since in any optimal solution, only one unique assignment is made for each node of the topology graph, the following constraint can be written:

$$\sum_{j=1}^{N} y_i^j = 1 \,, \quad i = 1,2,3,\dots,|M| \,. \quad (9)$$

• The sum of weights of demand nodes in a cell must be bounded by the number of allocated radio channels, therefore:

$$\sum_{i=1}^{M} y_i^j \cdot d_i \le b \,, \quad j = 1,2,3,\dots N \,. \quad (10)$$

• The number of BS's required to cover the given network is bounded by the following constraint:

$$\sum x_j \le |M| \,, \quad j \in V \mid x_j = 1 \,. \quad (11)$$

Here M is an upper bound on the number of BS's.

• In any real-world cellular layout design, each cell must be made of geographically contiguous grids. For a transmitter located at node v_i, other nodes lying within its radio coverage area are referred to as its '*compatible*' nodes. The set of such compatible nodes is represented as $\boldsymbol{P}_i$, $i \in V$. All nodes lying *outside* this coverage area are referred to as '*incompatible*' nodes, and the associated set is represented as T_i, $i \in V$.

Using the $M \times N$ binary variables y_i^j defined earlier, the contiguity constraint can be formulated as:

$$y_i^j + y_{i'}^j \le 1 \,, \quad \forall j \in \boldsymbol{X} \,, \, i \ne i' \,, \, i' \in \boldsymbol{T}_i \,, \, i \in V \,. \quad (12)$$

• As the aim of the optimization is to solve the problem using a minimum number of BS's, the following objective function can be considered:

$$Z(\boldsymbol{X}) = Min\Big(\sum_{j=1}^{N} x_j\Big) \,. \quad (13)$$

The minimum value of $Z(\boldsymbol{X})$ expresses the minimum number of BS's required to cover a given network design area. Since all constraints are linear and decision variables are binary, the resulting formulation essentially represents a BILP problem.

3.1 Generation of Valid Inequalities

The mathematical programming problem defined by (7)–(13) is an unfavorable starting point for the branch and bound algorithm based optimization techniques. The use of constraints (10) and (12) in their original form will create a linear programming relaxation with a large feasible region. Therefore, from a mathematical programming point of view, constraints (10) and (12) are particularly weak constraints. In the following, a technique is presented which yields a significant speed-up of computation time for the CLDP related BILP. First, some graph theoretic definitions are introduced.

An independent vertex set (also known as internally stable set) is a set of nodes of G such that no nodes of this set are adjacent. That is, no two nodes included in an independent vertex set are joined by an arc. Generally,

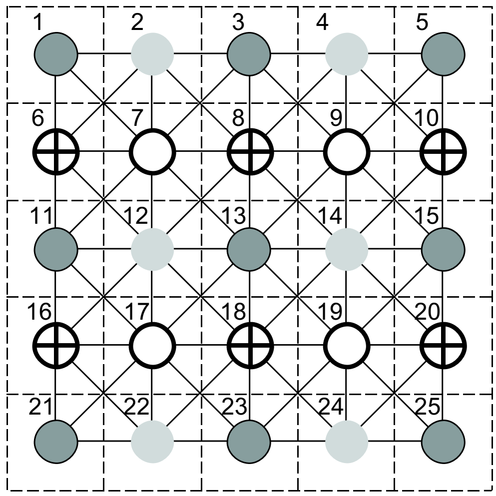

Fig. 2. A 25-node Topology graph together with the associated grid. The nodes of each MIS can be identified by having similar filling pattern.

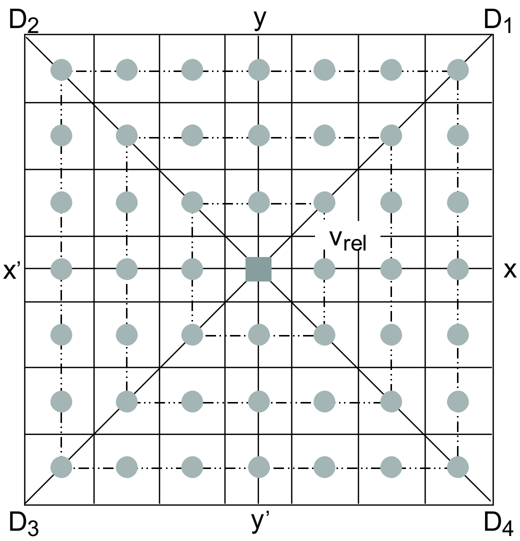

Fig. 3. The development of contiguity and compactness constraints.

these sets are referred to as *independent sets*. As an example, in Fig. 3 the sets of nodes $\{2,4\}$ and $\{6,8\}$ define two independent sets.

An independent set is called *maximal* when there is no other independent set that contains it. For the topology graph shown in Fig. 2, the sets $\{2,4\}$ and $\{6,8\}$ are not maximal but the sets $U_1 = \{2,4,12,14,22,24\}$ and $U_2 = \{6,8,10,16,18,20\}$ are maximal independent sets (MIS). Note that the number of elements (nodes) in various MIS is not the same for all the sets. For a given topology graph G, the MIS with the largest cardinality is called its *maximum independent set*.

The independent sets associated with the topology graph G are of significant importance as the elements of these sets represent the sets of mutually incompatible nodes for a given CLDP. No two nodes of such a set can be allocated together to a given BS. If $U \subset V$ is an independent set associated with G, then $\sum_{u \in U} u \geq 1$ can be used to develop a stronger valid inequality. If $|U| \geq 3$, then substituting this inequality in the constraint set of the CLDP cuts off non-integral, extremal points of the polyhedron of the relaxed linear problem. Obviously, this method is more efficient if U is an MIS.

It has been shown [7] by the application of *complexity theory* that the *exhaustive* generation of MIS for an arbitrary graph is an NP-complete problem, if an arbitrary graph is to be considered. However, this problem is made easy for the CLDP by the fact that the exhaustive generation of the MIS's is not needed. Only a selected number of MIS's of larger cardinalities are generated in order to develop necessary valid inequalities. The set of MIS's generated is governed by the following condition:

$$\bigcup_{i=1}^{p} U_i = V \quad \text{and} \quad \bigcap_{i=1}^{p} U_i = 0\,. \tag{14}$$

Such MIS are extracted heuristically for a given graph. Assuming uniform radio propagation conditions, a set of four MIS has been shown in Fig. 3 using four different filling patterns inside the nodes.

In order to explain the underlying advantage obtained, consider the node No. 1 of a topology graph shown in Fig. 2. Its associated set of compatible nodes is given as follows:

$$P_1 = \{2,6,7\}$$

whereas the set of incompatible nodes is given as follows:

$$T_1 = \{3,4,5,8,9,\ldots 25\}\,.$$

The application of (12) to node No. 1 results in the following inequalities:

$$y_1^j + y_3^j \leq 1 \qquad \forall j \in \mathbf{X}\,,\ v_3 \in T_1 \tag{15}$$

$$y_1^j + y_4^j \leq 1 \qquad \forall j \in \mathbf{X}\,,\ v_3 \in T_1 \tag{16}$$

$$\vdots$$

$$y_1^j + y_{25}^j \leq 1 \qquad \forall j \in \mathbf{X}\,,\ v_3 \in T_1 \tag{17}$$

However, using the four MIS U_1, U_2, U_3, U_4 for the graph shown in Fig. 2, (15)–(17) can be expressed by four sets of stronger valid inequalities as

$$y_1^j + \sum_{i' \in U_1, i' \in P_1} y_{i'}^j \leq 1 \tag{18}$$

$$\vdots$$

$$y_1^j + \sum_{i' \in U_4, i' \in P_1} y_{i'}^j \leq 1\,. \tag{19}$$

In the case of the CLDP's where the coverage area of a transmitter is limited to its adjacent nodes, the application of (14) limits the number of required MIS to four. In this case, a stronger constraint can be developed as follows:

$$y_i^j + \sum_{i' \in U_r, i' \in P_i} y_{i'}^j \le 1 \quad i, i' \in V,\ i \ne i',\ r = 1,2,3,4\,. \tag{20}$$

Note that for the CLDP's developed and solved during the start-up phase of the network, the observance of (20) implies the contiguity of nodes in a cell. In the case of CLDP's where the compatibility set of a node can cover the nodes beyond its adjacent nodes, similar MIS's are extracted heuristically. The extraction of these MIS is made easy by the fact that these sets must satisfy (14). An upper bound on the number of independent sets can be determined as a function of the maximum number of nodes that can be included in a cell. As an example, consider the graph shown in Fig. 2 assuming uniform radio propagation conditions and that for a BS, its sets of adjacent nodes and the set of next to adjacent nodes constitutes the full set of '*compatible*' nodes, $R_i^{\max} = 3$ nodes is attained. In this case, 9 MIS's are required. As an example, two independent sets are presented in the following:

$$U_1 = \{1, 4, 16, 19\}\,,$$
$$\vdots$$
$$U_9 = \{2, 5, 17, 20\}\,.$$

However, when $R_i^{\max} \ge 3$ is involved, additional constraints are needed in order to ensure the contiguity of the compatible nodes in a cell.

3.2 Additional Contiguity Constraints

In the case the of macro-level design where the radio coverage range of a transmitter is limited to adjacent nodes, the contiguity of nodes in a cell is guaranteed by (20). However, for the micro- level CLDP, special constraints are needed to ensure that compatible nodes are combined so that the resulting cell is free from enclaves, and is of acceptable geometry.

In order to ensure the contiguity of nodes in a cell, a '*ladder*' mechanism is proposed for developing the contiguity constraints. Starting with a reference node, the procedure combines the surrounding nodes with this reference node in a step-by-step fashion. As an example, consider the graph shown in Fig. 2. Node No. 1 is selected as reference node. In order to illustrate the underlying principle, it is assumed that the maximum coverage of every BS is limited to its three different sets of surrounding nodes. The three sets of compatible nodes for node No. 1 are given by $P_1 = \{2, 6, 7\}$, $P_2 = \{3, 8, 11, 12, 13\}$ and $P_3 = \{4, 9, 14, 16, 17, 18, 19\}$. A set of contiguity constraints required to ensure the contiguity of compatible nodes in a cell is given in Tab. 2.

Table 2. Development of contiguity constraints for node No. 1 of the graph shown in the Fig. 3

1st set (P_1) of nodes	2nd set (P_2) of nodes	3rd set (P_3) of nodes
$y_1^j + y_7^j \le 1 + y_2^j + y_6^j$	$y_1^j + y_3^j \le 1 + y_2^j$	$y_1^j + y_4^j \le 1 + y_3^j$
$y_2^j + y_6^j \le 1 + y_1^j + y_7^j$	$y_1^j + y_{11}^j \le 1 + y_6^j$	$y_1^j + y_9^j \le 1 + y_4^j$
	$y_1^j + y_{12}^j \le 1 + y_{11}^j$	$y_1^j + y_{14}^j \le 1 + y_9^j$
	$y_1^j + y_{13}^j \le 1 + y_8^j$	$y_1^j + y_{16}^j \le 1 + y_{11}^j$
		$y_1^j + y_{17}^j \le 1 + y_{16}^j$
		$y_1^j + y_{18}^j \le 1 + y_{17}^j$
		$y_1^j + y_{19}^j \le 1 + y_{14}^j$

The set of inequalities presented in Tab. 2 enforces the following condition:

'A node v_i can be combined with a given node $v_{reference}$ in order to form a cell j
if and only if at least one of surrounding nodes of v_i is also included in cell j'.

Note that Tab. 2 does not represent a unique set of inequalities. Depending on the topographical constraints, the blackout areas, existing cells, and administrative conditions, alternative sets of inequalities can be developed.

In order to develop generalized expressions for the contiguity constraints, consider the graph shown in Fig. 3. As a typical case, consider the development of generalized contiguity inequalities for a node 'v_{ref}'. All nodes of the graph shown in Fig. 3 are assumed to be *compatible* for reference node v_{ref}. The compatible nodes are identified by three groups. The first group comprises nodes lying on the perpendicular axis (XX', YY'), the second group comprises nodes lying on the diagonal axis, D_1, D_2, D_3 and D_4. The third group consists of nodes lying between the perpendicular and the diagonal axis.

All the nodes located on the perpendicular axis are related to v_{ref} by the following contiguity constraint:

$$y_{ref}^j + y_n^j \le 1 + y_{n-1}^j \ \forall v_{ref} \in V\,, v_n, v_{n-1} \in P_{ref} \quad \text{located on the perpendicular axis.} \tag{21}$$

In order to develop the contiguity constraints of a node located on the diagonal reference axis, the following inequality can be written:

$$y_{ref}^j + y_m^j \le 1 + y_{m'}^j + y_{m''}^j \tag{22}$$

where v_m is located on any diagonal axis with respect to v_{ref}, $v_{m'}$ and $v_{m''}$ are two adjacent nodes of v_m that are closest to v_{ref}, and are not located on the diagonal axis.

For the nodes that are not located on the diagonal or perpendicular axis, the following inequality can be written:

$$y_{ref}^j + y_k^j \le 1 + y_{k'}^j \quad ref, j, j'k, k' \in V \tag{23}$$

where $v_k, v_{k'} \in P_{ref}$, and $v_{k'}$ is one of two adjacent nodes of v_k located closest to v_{ref}.

Depending on the special geometrical requirements of a CLDP, additional contiguity constraints can be integrated in the formulation.

The proposed mathematical formulation for the CLDP is summarized below:

Objective Function: $$Z(\boldsymbol{X}) = Min\Big(\sum_{j=1}^{N} x_j\Big) \tag{13}$$

Set of Constraints:

$$y_i^j \le x_j\,, \qquad 1 \le i \le M\,,\ 1 \le j \le N \tag{8}$$

$$\sum_{j=1}^{N} y_i^j = 1\,, \qquad i = 1, 2, 3, \ldots, |M| \tag{9}$$

$$\sum_{i=1}^{M} y_i^j \cdot d_i \le b\,, \qquad j = 1, 2, 3, \ldots N \tag{10}$$

$$\sum x_j \le |M|\,, \qquad j \in V \mid x_j = 1 \tag{11}$$

$$y_i^j + \sum_{i' \in U_r, i' \in P_i} y_{i'}^j \le 1,\ i, i' \in V,\ i \ne i',\ r = 1, 2, 3, \ldots R \tag{20}$$

$$y_{ref}^j + y_n^j \le 1 + y_{n-1}^j\ \forall v_{ref} \in V\,, v_n, v_{n-1} \in P_{ref} \text{ located on the perpendicular axis} \tag{21}$$

$$y_{ref}^j + y_m^j \le 1 + y_{m'}^j + y_{m''}^j \tag{22}$$

$$y_{ref}^j + y_k^j \le 1 + y_{k'}^j \quad ref, j, j'k, k' \in V \tag{23}$$

$$x_j \in \{1, 0\}\,, \quad 1 \le j \le N \tag{3}$$

$$y_i^j \in \{1, 0\} \quad 1 \le i \le M\,,\ 1 \le j \le N \tag{6}$$

4 SOLUTION STRATEGY

4.1 Computational Complexity of CLDP's

In the case of CLDP's, the solution space comprises 2^N combinations for X and $2^{N \times M}$ combinations of Y. Each time the number of binary variables in increased by one, the number of possible combinations is doubled. This property is known as the exponential growth of the solution difficulty with the size of the problem. The time needed to solve this class of problems grows exponentially with the size of the problem to be solved. Nevertheless, this level of computational complexity is associated with only the worst case scenario since many of these combinations are discarded as their associated solutions violate the constraints of a given CLDP.

4.2 Reduction in Problem Complexity

A realistic upper bound N on the number of BS reduces the number of binary variables from $M\ (1+M)$ to $N\ (1+N)$. In addition, a lower bound is needed for an optimal prefixing scheme. In order to determine a lower bound 's' on the number of BS, initially (10) is relaxed and the number of BS's, s^1 required is found by applying Hata's propagation model. Finally, by relaxing the limits on the maximum power of the transmitter, the average number of BS required to cover the design area are determined as follows:

$$s^2 = \frac{\sum_{j=1}^{j=M} d_j}{b}\,. \tag{24}$$

where s^2 is an integer.The lower bound is selected as:

$$s = Min(s^1, s^2)\,. \tag{25}$$

The upper bound N on the number of BS's is determined by adding a margin Δ to the lower bound s. i.e.

$$N \equiv \Delta + s\,. \tag{26}$$

A specially developed heurstic is used to determine the best value of Δ. Note that an unrealistically smaller upper bound will make the problem solution infeasible.

4.3 A Heuristic Pre-Solving Technique

Using (25), the following constraints are added to the formulation:

$$x_j = 1\,, \quad 1 \le j \le s\,. \tag{27}$$

Subsequently, a set L of nodes is determined. All nodes of L are sufficiently mutually apart that no two nodes can be included in the same cell. This is equivalent to developing a special independent set on G. Each node of L is then allocated to a unique BS fixed using (27). Hence a one-to-one mapping $\Im$ is developed between a BS and its allocated node.

The following constraints are then added to the BILP formulation of a given CLDP:

$$y_i^j = 1\,, \qquad (i, j) \in \Im\,, i \in \boldsymbol{L}\,,\ j \in s \tag{28}$$

$$y_i^{j'} = 0\,, \qquad j \ne j'\,, i \in \boldsymbol{L}\,,\ j \in s \tag{29}$$

$$y_{i'}^j = 0\,, \qquad i \ne i'\,, i' \in \boldsymbol{L}\,,\ j \in s \tag{30}$$

In order to strengthen (10), a technique developed from the concept of minimal covers [8] is used to generate cover cuts. Note that in some cases, (10) behaves as a non-binding constraint and can be left out of the model as it has no effect on the optimal solution.

4.4 A Branch and Cut Algorithm

For the solution of BILP developed, a B&C algorithm [Garfinkel, 1972], [Namhauser, 1988] is used. The B&C algorithm is an optimization technique widely used for solving combinatorial problems. Instead of attempting to solve the problem directly over the set of all feasible solutions S, this set is successively divided into smaller and smaller sets which have the property that any optimal solution must be in at least one of the sets. This process is called 'separation' and is often illustrated by a so-called *enumeration tree* [8]. The other three fundamental elements of B&C algorithm are: relaxation (lower bounding); fathoming of sub-problems (upper bounding); and selection of sub-problems (branching). The B&C algorithm involves calculating upper bounds and lower bounds on the objective function in order to accelerate the fathoming process and to curtail the enumeration.

The essence of the *branch and cut* algorithm used to solve the BILP problem is linear programming (LP) [8]. The convex set defined by the constraints of an LP is called *polyhedron* and a bounded polyhedron is called *polytope*. The convexity property of the LP plays an important role in discarding the uninteresting sections of the enumeration tree.

If the whole all solution space of the proposed BILP model is denoted as sol^1 and that of the associated integer solutions as sol^2, then two problem sets can be defined as follows:

$$\text{P(1):}\;\; Z(\boldsymbol{X}) = Min\Big(\sum_{j=1}^{N} x_j\Big) \quad x_j \in sol^1,\, 0 \le x_j \le 1 \tag{31}$$

is said to be a relaxation of

$$\text{P(2):}\;\; Z(\boldsymbol{X}) = Min\Big(\sum_{j=1}^{N} x_j\Big) \quad x_j \in sol^2,\, x_j \in \{1,0\} \tag{32}$$

if and only if $sol^1 \supseteq sol^2$.

The theory of polyhedron states that if $\boldsymbol{X}^{Opt\_LP}$ is an optimal solution to P(1) and $\boldsymbol{X}^{Opt\_IP}$ is an optimal solution to P(2), then for the presented minimization problem, it follows that:

$$\boldsymbol{X}^{Opt\_LP} \le \boldsymbol{X}^{Opt\_IP} \tag{33}$$

Furthermore if $\boldsymbol{X}^{Opt\_LP} \in sol^2$ then $\boldsymbol{X}^{Opt\_LP}$ is an optimal binary solution.

The *simplex based* B&C algorithm begins by solving the original problem as a linear programming problem by relaxing integrality conditions (42) in the hope that the solution will be an integer feasible, and thus solves the original BILP problem. If an integer solution is not obtained, the next step is to pick one of the variables to create two descendants of the original problem, one with x_j (or y_i^j) $= 0$ or the other with x_j (or y_i^j) $= 1$. Since x_j (or y_i^j) is a binary variable and the descendant problems are exactly the same as the original problem except of the assignment of a specific value to x_j (or y_i^j) $= 0$, the solution to one of the these two descendant problems must in fact be the solution to the original problem. Thus the original problem is replaced by these descendant problems, leaving two BILP problems to solve instead of one. Since each new BILP problem is more restricted than the original problem, and has fewer variables that must be assigned binary values, it is expected that it may be solved somewhat more readily than its parent.

The process then repeats, selecting one of the problems that remain to be solved as the current BILP problem, and treating it exactly as the original problem. This in turn may create two new problems to replace the one under consideration unless, for example, the current problem is 'infeasible'. Eventually, by this process an integer solution is obtained for the current BILP problem, whereupon this solution provides a candidate for the optimal solution to the original problem. Keeping track of the current best of these candidates and their associated $Z(\boldsymbol{X})$ values provides an additional way to weed out the descendant problems of the original BILP problem that are unprofitable for exploration. The current best candidate is called the *incumbent*.

The possibilities for this type of approach can be itemized as follows:

Let S^* denote the objective function value of the incumbent solution. Then, whenever a current BILP problem is solved as an LP problem, one of the following four alternative may arise:

(a) The RLP problem has no feasible solution (that is associated BILP problem is also infeasible).

(b) The LP problem has an optimal solution with $S_0 \le S^*$ (*ie*, the current BILP problem optimum must also yield $S_0' \le S^*$).

(c) The optimal solution to the LP problem is both integer feasible and yields $S_0 > S^*$ (in which case the solution provides an improved incumbent for original BILP problem, S^* is set to S_0)

(d) None of the foregoing occurs; *ie*, the optimal solution exists, satisfies $S_0 > S^*$, and is not integer feasible.

In case of (a) to (c), the current BILP problem is disposed off simply by solving the associated LP problem. Otherwise the problem requires further exploration.

The computational experience shows that the selection of right branching variables in the B&C algorithm has a considerable influence on the convergence of the solution to optimality. During the enumeration process, the special structure of the problem is exploited to assign priorities to various branching variables. It is obvious that the elements of $\boldsymbol{X}$ have a clear precedence over the elements of $\boldsymbol{Y}$. Therefore, the elements of $\boldsymbol{X}$ are assigned higher priorities. Among different elements of $\boldsymbol{X}$, the priorities are assigned arbitrarily.

Table 3. Derivation of logical implications

Implication	Derivation
$y_{12}^j = 1 \Rightarrow$ $y_2^j = y_{14}^j = y_{22}^j = y_{24}^j = 0$ $y_{12}^j = 0\,,\ j' \neq j \in \boldsymbol{X}$	$y_2^j + y_4^j + y_{12}^j + y_{22}^j + y_{24}^j \leq 1$ $y_{12}^j = 1 \Rightarrow y_2^j + y_4^j + y_{14}^j + y_{22}^j y_{24}^j \leq 0$ $\therefore y_i^j \in \{1,0\} \Rightarrow y_2^j = y_4^j = y_{14}^j = y_{22}^j = y_{24}^j = 0$
$y_{12}^j = 1 \Rightarrow y_{10}^j = y_{20}^j = 0$	$y_{10}^j + y_{12}^j + y_{20}^j \leq 1$ $y_{12}^j = 1 \Rightarrow y_{10}^j + y_{20}^j \leq 0$ $\therefore y_i^j \in \{1,0\} \Rightarrow y_{10}^j = y_{20}^j = 0$
$y_{12}^j = 1 \Rightarrow y_9^j = y_{19}^j = 0$	$y_{12}^j + y_9^j + y_{19}^j \leq 1$ $y_{12}^j = 1 \Rightarrow y_9^j + y_{19}^j \leq 0$ $\therefore y_i^j \in \{1,0\} \Rightarrow y_9^j = y_{19}^j = 0$
$y_{12}^j = 1 \Rightarrow y_1^j = y_3^j = y_5^j$ $= y_{15}^j = y_{21}^j = y_{23}^j = y_{25}^j = 0$	$y_1^j + y_3^j + y_5^j + y_{12}^j + y_{15}^j + y_{21}^j + y_{23}^j + y_{25}^j \leq 1$ $y_{12}^j = 1 \Rightarrow y_1^j + y_3^j + y_5^j + y_{15}^j + y_{21}^j + y_{23}^j + y_{25}^j \leq 0$ $\therefore y_i^j \in \{1,0\} \Rightarrow y_1^j = y_3^j = y_5^j = y_{15}^j = y_{21}^j = y_{23}^j = y_{25}^j = 0$

The elements of $\boldsymbol{Y}$ are assigned priorities on the basis of corresponding node-weights. In the case of a tie, the selection is made arbitrarily.

The success of the B&C algorithm also depends on as how effectively the formulation is developed to reduce the integrality gap between the objective function values of the integer program problem and the corresponding relaxed linear programming problem. The inequalities used in the proposed formulation to model various constraints discard large uninteresting sections of the enumeration tree. The branching on a chosen variable results in discarding of many variables, and thus reduces the number of sub-problems to be solved substantially. As an example, consider the topology graph shown in Fig. 3. If in the B&C enumeration process a variable y_{12}^j is branched on, then the logical implications listed in Tab. 3 can be derived.

Before embarking on the enumeration process, a preprocessing [9] technique is used to eliminate redundancies. *Clique Cuts* were generated during the enumeration process. A *root rounding* [10] heuristic is used at the *root node* of the B&C enumeration tree to construct an initial solution. The *best estimate search* [10] was adopted for the next node selection, which chooses the node with the best estimate of integer objective values that would be obtained once all integer unfeasibilies are removed. Depending on the structure of a given problem, the steepest edge or devex pricing [11] algorithm is used while solving the relaxed linear programming problems. Cutting planes were generated during the optimization process.

5 APPLICATION EXAMPLES AND COMPUTATIONAL RESULTS

In order to test the formulation of the CLDP as a BILP problem, the model has been applied into a range of network design problems. The mathematical modeling was carried out in general algebraic modeling system GAMS [12] and Cplex [9] was used as LP solver. Various heuristics were implemented using C programming language. The tests were performed on a 360 MHz dual processor Pentium machine.

The specifications of the problems examined and the associated solution results are given in Tab. 4. The number of grid-elements (M) for each problem is listed in column (2) of Tab. 4. The maximum independent sets (MIS) developed to generate the contiguity constraints for each problem are listed in column (3). The upper bound (N) on the number of Base Stations is listed in the column (4). The optimal number of BS's found by the B&C algorithm is listed in column (7). For each problem, the number of the search solution tree nodes processed by the B&C algorithm and the reported CPU time are also listed.

In order to check the performance of the presented methodology for the real-world problems, the model has been applied to a typical benchmark mobile radio network design problem for the area of East Anglia in the UK (and is listed as Problem No. 13 in Tab. 4). The area of East Anglia is considered to be divided into 154 square-shaped grid elements, of side dimension 10 km, according to the National Geographical Grid for the UK. In the absence of more accurate data for large-scale network design, the assumption has been made that the expected traffic is proportional to population density. The population data used in this illustrative example come from the census of 1991 for the same area.

In each grid element, one channel is added to the associated demand node for every 5000 people. This rule has been used to create typical requirement data. In the case of the East-Anglia network design problem, the channel demands in various grid elements vary from 1 channel to 25 channels. The range of channel-demands for the grid-elements is shown in the column (6) of Tab. 4. In this example, it is assumed that a total of 30 channels are allocated to the network operator. The application of Hata's radio propagation model means that no more than two grid elements can be merged together to form

Table 4. Problem specifications and computational results

Problem No (1)	No of grid elements (M) (2)	Number of MIS's developed (3)	Upper bound (N) on No. of BS's (4)	Available bandwidth (b) (No. of channels) (5)	Weights of demand nodes (Min – Max) (6)	No of BS's determined by Branch and Cut algorithm (Z) (7)	No of nodes processed by Branch and Cut algorithm (8)	Solution time (sec.) (9)
1	144	16	30	150	1–15	9	Root node only	35
2	144	16	39	110	1–75	14	9	5260
3	100	16	37	200	1–200	20	29	4230
4	64	16	18	100	1–13	7	29	92
5	64	16	44	160	1–160	14	51	3788
6	121	9	30	100	1–15	16	10	1101
7	100	9	30	100	1–100	16	18	498
8	81	9	25	200	1–15	10	8	529
9	64	9	20	190	2–25	9	42	353
10	49	9	16	100	1–15	9	16	54
11	36	9	20	100	1–16	4	Root node	237
12	25	9	15	50	4–14	4	12	175
13	154	4	96	30	1–25	44	3	15480

a cell. In order to develop the stronger valid inequalities, four maximum independent set (MIS) are developed using the methodology described earlier in the paper. Using Eq. (31), an upper bound of 96 base stations was found for the East Anglia design problem. The B&C algorithm solved the East Anglia design problem in 15480 seconds and converged to optimal solution in 3 nodes only yielding an optimal value of 44 BSs.

As another example, consider the problem No. 1 listed in Tab. 4. In this case, the network design area is partitioned into a grid of 144 elements of side dimension 1 km. The minimum acceptable signal strength at the boundary of the cell was set to -93 dBm. Using Hata's propagation model, the maximum radio coverage of BS was found to be 3.5 km. Using the methodology presented in Section 3, 16 MIS are developed for this example. Note the additional MIS are needed because, in this example, several sets of surrounding nodes can be combined with a given node to form a cell. In order to ensure the contiguity of the nodes included in a cell, additional contiguity constraints were developed using the 'ladder' mechanism described in the Section 3.2. Using Eq. (26), the upper bound (N) on the number of base stations was found to be 30. Therefore, the number of binary variables for this problem can be found using the following equation:

$$\text{No. of binary variables defined} = N + N \cdot M = 30 + 30 \times 144 = 4350$$

Therefore, the total searchable solution space for Problem No. 1 consists of 2^{30} combinations of $\boldsymbol{X}$ and $2^{30\times 144}$ combinations of $\boldsymbol{Y}$. In order to reduce the size of this problem, a number of binary variables were prefixed using Eqs. (28)–(30). The demand variation in the grid-elements was assumed to vary from 1 channel to 15 channels. The total numbers of available channels were assumed to be 150. In this example, the B&C algorithm found that 9 BS's are required to cover the design area.

In order to further demonstrate the effectiveness of the proposed design methodology, several other CLDP were considered. The detailed specifications and results obtained for these problems are also listed in Tab. 4. In all the problems considered, the minimum acceptable signal strength at the cell boundary was set from -90 dBm to -95 dBm.

For all the network design problems listed in Tab. 4, a pre-processing [Savelsbergh, 1994] [CPLEX] technique is used before applying the B&C algorithm. A *root rounding* [Avriel, 1996] heuristic is used at the *root node* of the B&C enumeration tree to construct an initial solution.

The computational results obtained demonstrate that the proposed solution methodology can be used effectively to solve fairly large-size CLDPs. Since the mathematical programming methodology is developed in a generalized way, it can be tailored to incorporate various radio propagation and traffic models.

6 CONCLUSIONS

In this paper, an analytical approach is proposed for finding the minimum number of base stations required to cover a given network design area. The solution also yields topographical configurations of the cells associated with base stations, and the best cell sites. The cellular layout design problem is formulated as binary integer linear programming problem involving various constraints. In order to solve the resulting mathematical programming problem, a special branch and cut algorithm is applied.

In order to increase the increase the computational efficiency of the associated branch and cut algorithm, some special techniques are presented. These techniques play an important role in the faster convergence of the branch and bound algorithm to optimality. The techniques developed have been applied to a range of network problems. The results obtained show that the proposed technique can be applied effectively to solve fairly large size cellular network design problems.

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