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COMMUNICATIONS

APPLICATION OF ATYPICAL CURVES FOR THE FORMATION SURFACES OF REFLECTORS

Elena Pastuchová — Jana Galanová — Alfonz Smola *

This article deals with two cycloidal curves: Ceva's cycloid and trochoid. Formulas that allow to apply these curves to construction of rotary symmetric reflectors are derived.

K e y w o r d s: incident light ray, reflected light ray, Ceva's cycloid, trochoid

1 INTRODUCTION

Modelling and calculation of reflectors in luminaires belongs to the most important area of lighting technology research. There exist two ways how to solve the problem of calculation: analysis and synthesis of reflectors surfaces. Differential equation of reflection is the basic equation for both of these methods. We decided to solve this problem by an analytical approach, i.e., first we model the reflector surface and then we explore its optical properties. The differential equation describes the relationship between the incident and reflected light rays on an elementary strip of the reflector. The universal form of the differential equation of a rotary symmetric reflector is

$$\frac{\mathrm{d}\rho}{\mathrm{d}\varphi} = \rho \operatorname{tg} \frac{\varphi \pm \alpha}{2} \tag{1}$$

Reflectors with forming curves based on conic sections are relatively satisfactorily explored in [1] and [2]. Recently, the use of atypical curves seems to be interesting for technical applications: spirals, Pascal's shell, cardioid and other curves have already been used in lighting, and their solutions have been published in [3–7]. In spite of this fact, there are many other curves that belong to a big group of curves called cycloids. Their optical properties have not been studied yet. This article deals with two cycloidal curves: Ceva's cycloid and trochoid.

2 CEVA CYCLOID

Let a circle roll along a horizontal line without slipping. Point P on the circle will trace out a curve called a cycloid. Using a rectangular co-ordination system, we begin with the circle of radius a centred at the origin at co-ordinates $(0,0)$. To describe the points of Ceva's cycloid we have to trace line q from the origin. Intersection of q and of the circle will give point M. Let us construct point A on the x axes the distance of which from point M is a and point N on the line q the distance of which from point A is the same, a. For various lines q we obtain a set of points N that form the Ceva cycloid (see Fig. 1). For distance MN within ΔMAN we have

$$MN = 2a \cos 2\varepsilon \tag{2}$$

and

$$ON = a + 2a \cos 2\varepsilon \,. \tag{3}$$

Using polar co-ordinates ρ, φ we can rewrite Eq. (3) to the form

$$\rho = a + 2a \cos 2\varphi \,. \tag{4}$$

Substituting Eq. (4) to Eq. (1) and solving this equation the following expression is valid for the angles of incident and reflected light rays within the Ceva cycloid:

$$\alpha = \pm\Big(\varphi + 2 \operatorname{arctg} \frac{4a \sin 2\varphi}{\rho}\Big) \qquad \text{where } \varphi \pm \alpha \in (-\pi, \pi) \,. \tag{5}$$

3 TROCHOID

Trochoid is a special case of a cycloid. The points which trace out the curve called a trochoid do not originate from the circumference of the moving circle. They are traced out by a fixed point inside or outside of a circle rolling along another circle (see [8]).

Let h denote the distance of this point from the centre of the circle (see Fig. 2). Let us take a co-ordinate system with its origin at the centre of the fixed circle with radius R and let the radius of the rolling circle be r. If $h > r$, the trochoid is called elongated, if $h < r$, we get it truncated. Trochoids are divided into epitrochoids or hypotrochoids. When the moving circle rolls inside a fixed circle, we get a hypotrochoid, when it rolls outside, we get an epitrochoid.

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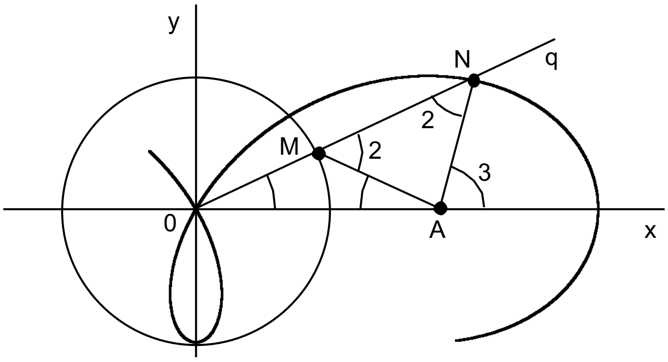

Fig. 1.

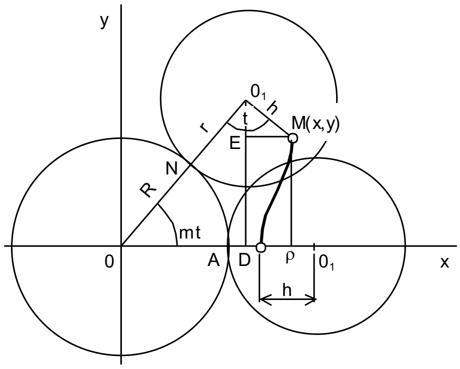

Fig. 2.

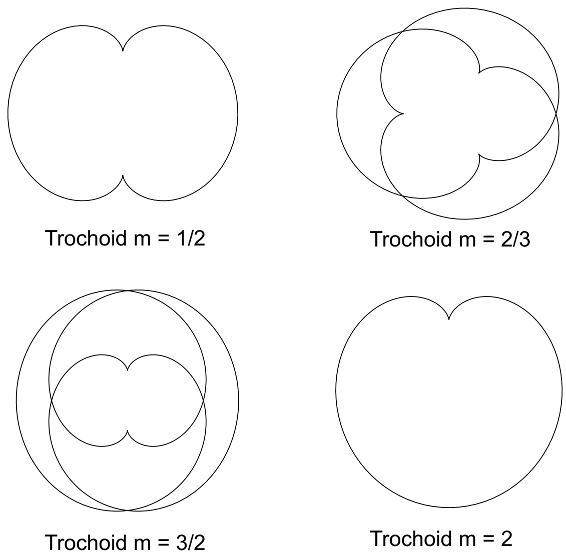

Fig. 3.

For co-ordinates of point M of the epitrochoid we have following equations:

$$x = (R + mR)\cos mt - h\cos(t + mt)\,, \tag{6}$$

$$y = (R + mR)\sin mt - h\sin(t + mt)\,, \tag{7}$$

where m and t are arbitrarily chosen.

Similarly for the hypotrochoid we obtain

$$x = (R - mR)\cos mt + h\cos(t - mt)\,, \tag{8}$$

$$y = (R - mR)\sin mt - h\sin(t - mt)\,, \tag{9}$$

where parameter $m = \frac{r}{R}$ is called a module and $mt = \frac{r}{R}t = \angle NOA$.

It is evident that Eqs. (8) and (9) can be obtained from Eqs. (6) and (7) substituting $m \to -m$, $h \to -h$ and inverting the axis direction. That is why we can consider $m > 0$.

Let $h = R + r = R + mR$, then we obtain a special type of trochoid called a rosette with co-ordinates

$$x = (R + mR)2\sin\Big(mt + \frac{t}{2}\Big)\sin\frac{t}{2}\,, \tag{10}$$

$$y = -(R + mR)2\cos\Big(mt + \frac{t}{2}\Big)\sin\frac{t}{2}\,. \tag{11}$$

Using polar co-ordinates and formulas for conversion from rectangular to polar co-ordinates we get

$$\rho^2 = 4(R + mR)^2\sin^2\frac{t}{2} \tag{12}$$

and

$$\frac{y}{x} = \operatorname{tg}\varphi = -\operatorname{tg}\Big[\frac{\pi}{2} - \big(mt + \frac{t}{2}\big)\Big]\,. \tag{13}$$

Point M can be expressed in polar co-ordinates as follows:

$$\rho = 2(R + mR)\Big|\sin\frac{1}{2m+1}\big(\varphi + \frac{\pi}{2}\big)\Big|\,. \tag{14}$$

After rotating the polar axis through an angle $\frac{\pi}{2}$ counterclockwise we obtain

$$\rho = 2(R + mR)\Big|\sin\frac{\varphi}{2m+1}\Big|$$

or

$$\rho = 2R(m+1)\Big[\big(\operatorname{sign}\sin\frac{\varphi}{2m+1}\big)\sin\frac{\varphi}{2m+1}\Big] \tag{15}$$

and the path of light rays in the meridian plane for the trochoid is described by the following equation:

$$\alpha = \pm\Big[\varphi - 2\operatorname{arctg}\Big|\frac{1}{2m+1}\frac{(2R+2mR)}{\rho}\cos\frac{1}{2m+1}(\varphi)\Big|\Big]. \tag{16}$$

4 CONCLUSION

Atypical curves are usually described in parametric or analytic forms. For our purpose, expression in polar co-ordinates is more suitable.

However, for some types of mathematical curves, for instance for cycloids, it may be difficult to express them in polar co-ordinates. For two types of cycloids, Ceva cycloid and trochoid, this was found and presented in this paper. These curves, Ceva cycloids and trochoids, seem to be applicable in lighting technology as shown in Fig. 3.

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