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SECOND–ORDER TWO–DIMENSIONAL SYSTEMS: COMPUTING THE TRANSFER FUNCTION

George E. Antoniou — Marinos T. Michael *

In this paper the discrete Fourier transform is used to determine the coefficients of a transfer function of a new two-dimensional model of second-order: $x(i_1+2,i_2+2) = \mathbf{A}_0 x(i_1+1,i_2+1) + \mathbf{A}_1 x(i_1+1,i_2) + \mathbf{A}_2 x(i_1,i_2+1)$. The algorithm is straightforward and has been implemented using the software package $Matlab^{TM}$. Two step-by-step examples illustrating the application of the algorithm are presented.

K e y w o r d s: two-dimensional (2D) systems, second-order systems, transfer function, Fourier transform

1 INTRODUCTION

During the past two decades there has been extensive research in two dimensional (2D) systems. This is due to the extensive range of applications, especially in engineering and computing [1]–[3]. 2D systems can be represented with a transfer function in polynomial form or using state space like structures [3]. State space based techniques play a very crucial role in the analysis and synthesis of 2D systems. An important problem is to determine the coefficients of a transfer function from its model representation and vice versa. Leverrier-Fadeeva, discrete Fourier transform (DFT) algorithms and Vanderlmode matrices can be modified to be used for various models. The DFT has been used for the evaluation of the transfer function coefficients for linear, singular, and multidimensional state space systems [4]–[7].

In this paper a computer implementable algorithm is proposed for the computation of the 2D transfer function for a new 2D system that is also of second-order. Second-order one-dimensional systems have been used in circuit theory, linear control systems, filtering, mechanical system modeling and applied mathematics [8]–[10]. The proposed algorithm determines the coefficients of the determinantal polynomial and the coefficients of the adjoint polynomial matrix, using the DFT. The computational speed of the method can be improved by using fast Fourier transform techniques. It is noted that the proposed algorithm easily can be modified to be used with second-order 2D systems of other types [11].

2 SECOND–ORDER TWO–DIMENSIONAL SYSTEM

A second-order two-dimensional (SO2D) system has the following structure [11]:

$$\begin{aligned} x(i_1+2,i_2+2) &= \mathbf{A}_0 x(i_1+1,i_2+1) \\ &\quad + \mathbf{A}_1 x(i_1+1,i_2) + \mathbf{A}_2 x(i_1,i_2+1) \\ &\quad + \mathbf{B}_1 u(i_1+1,i_2) + \mathbf{B}_2 u(i_1,i_2+1)\,, \\ y(i_1,i_2) &= \mathbf{C}\, x(i_1,i_2) \end{aligned} \tag{1}$$

where, $x(i_1,i_2) \in \mathcal{R}^\lambda$, $u(i_1,i_2) \in \mathcal{R}^m$, $y(i_1,i_2) \in \mathcal{R}^p$, i_1, i_2 are integer-valued vertical and horizontal coordinates, respectively, $x(i_1,i_2)$ is the local vector at (i_1,i_2), $u(i_1,i_2)$ is the input vector and $y(i_1,i_2)$ is the output vector. $\mathbf{A}_k$, for $k=0,1,2$ and $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{C}$, are real matrices of appropriate dimensions denoting the characteristics of the SO2D system that can also be represented by a 2D transfer function, as in the regular 2D systems [12]. It is noted that this particular SO2D system (1) is an extension of the regular 2D Fornasini-Marchesini model [12] to cover systems of second-order. For more 2D second-order structures the reader can refer to [11].

Applying the 2D z_i, $i=1,2$ transform to system (1), with zero initial conditions, the transfer function is found to be:

$$\mathbf{T}(z_1,z_2) = \mathbf{C}\,[\mathbf{I}z_1^2z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2]^{-1} \cdot [\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2]\,. \tag{2}$$

In the following section an interpolative approach is developed for determining the transfer function $\mathbf{T}(z_1,z_2)$, given the matrices $\mathbf{A}_k$, $k=0,1,2$ and $\mathbf{B}_1$ $\mathbf{B}_2$, $\mathbf{C}$ using the 2D DFT. For the sake of completeness a brief description of the 2D DFT follows.

* Image Processing and Systems Laboratory, Department of Computer Science, Montclair State University, Montclair, N.J 07043

3 2D DISCRETE FOURIER TRANSFORM

Consider the finite sequences $X(k_1,k_2)$ and $\tilde{X}(r_1,r_2)$, $k_i, r_i = 0,\dots,M_i$, $\forall i = 1,2$. In order for the sequences $X(k_1,k_2)$ and $\tilde{X}(r_1,r_2)$, to constitute a 2D DFT pair the following relations should hold [13]:

$$\tilde{X}(r_1,r_2) = \sum_{k_1=0}^{M_1}\sum_{k_2=0}^{M_2} X(k_1,k_2) W_1^{-k_1 r_1} W_2^{-k_2 r_2}, \tag{3}$$

$$X(k_1,k_2) = \frac{1}{R}\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2} \tilde{X}(r_1,r_2) W_1^{k_1 r_1} W_2^{k_2 r_2} \tag{4}$$

where

$$R = (M_1+1)(M_2+1)\,, \tag{5}$$

$$W_i = e^{(2\pi j)/(M_i+1)},\ i=1,2\,, \tag{6}$$

X, $\tilde{X}$ are discrete argument matrix valued functions, with dimensions $p\times m$.

In the following section an interpolative approach is developed for determining the transfer function $\mathbf{T}(s)$, given the matrices $\mathbf{A}_i$, $i=0,1,2$ and $\mathbf{B}_1$, $\mathbf{B}_2$ $\mathbf{C}$, using the 2D DFT.

4 ALGORITHM

The transfer function of a SO2D system (1) is

$$\mathbf{T}(z_1,z_2) = \frac{\mathbf{N}(z_1,z_2)}{d(z_1,z_2)} \tag{7}$$

where

$$\mathbf{N}(z_1,z_2) = \mathbf{C}\,\mathrm{adj}\big[\mathbf{I}z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2\big]\cdot(\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2)\,, \tag{8}$$

$$d(z_1,z_2) = \det\big[\mathbf{I}z_1^2 z_2^2 - \mathbf{A}_0 z_1 z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2\big]\,. \tag{9}$$

Equations (8) and (9) can be written in polynomial form as follows:

$$\mathbf{N}(z_1,z_2) = \sum_{\lambda_1=0}^{n_{\max}^P}\sum_{\lambda_2=0}^{n_{\max}^P} \mathbf{P}_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \tag{10}$$

with $n_{\max}^P := \max\big((2\lambda-1),(2\lambda-1)\big)$. The numerator coefficients $\mathbf{P}_{\lambda_1,\lambda_2}$ are matrices with dimensions $(p\times m)$.

$$d(z_1,z_2) = \sum_{\lambda_1=0}^{n_{\max}^q}\sum_{\lambda_2=0}^{n_{\max}^q} q_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2} \tag{11}$$

where $n_{\max}^q := \max(2\lambda, 2\lambda)$. The denominator coefficients q_{λ_1,λ_2} are scalars.

The numerator polynomial matrix $\mathbf{N}(z_1,z_2)$ and the denominator polynomial $d(z_1,z_2)$ can be numerically computed at $R=(r+1)^2$ points equally spaced on the unit 2D disc. The R points are chosen as $(z_1,z_2) = [v(i_1), v(i_2)]$, $i_1,i_2 = 0,\dots,r$ with $r=2\lambda$, according to definition as:

$$v_1(i) = v_2(i) = W^{-i},\ \forall\, i=0,\dots,r \tag{12}$$

where

$$W_i = e^{(2\pi j)/(r+1)},\ \ i=1,2\,. \tag{13}$$

The values of the transfer function (7) at the R points are the corresponding 2D DFT coefficients.

4.1 Denominator Polynomial

To evaluate the denominator coefficients q_{λ_1,λ_2} define

$$a_{i_1,i_2} = \det\big[\mathbf{I}v_1^2(i_1)v_2^2(i_2) - \mathbf{A}_0 v_1(i_1)v_2(i_2) - \mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2)\big]\,. \tag{14}$$

Therefore, using equations (11) and (14), a_{i_1,i_2} can be defined as

$$a_{i_1,i_2} = d[v_1(i_1), v_2(i_2)]\,. \tag{15}$$

Provided that at least one of $a_{i_1,i_2} \neq 0$.

Equations (11), (12) and (15) yield

$$a_{i_1,i_2} = \sum_{\lambda_1=0}^{r}\sum_{\lambda_2=0}^{r} q_{\lambda_1,\lambda_2} W^{-(i_1\lambda_1 + i_2\lambda_2)} \tag{16}$$

In the above equation (16) it is obvious that $[a_{i_1,i_2}]$ and $[q_{\lambda_1,\lambda_2}]$ form a 2D DFT pair. Therefore the coefficients $[q_{\lambda_1,\lambda_2}]$ can be computed using the inverse 2D DFT, as follows:

$$q_{\lambda_1,\lambda_2} = \frac{1}{R}\sum_{i_1=0}^{r}\sum_{i_2=0}^{r} a_{i_1,i_2} W^{(i_1\lambda_1 + i_2\lambda_2)} \tag{17}$$

4.2 Numerator Polynomial

To evaluate the numerator matrix polynomial $\mathbf{P}_{\lambda_1,\lambda_2}$, define

$$\mathbf{F}_{i_1,i_2} = \mathbf{C}\,\mathrm{adj}[\mathbf{I}v_1^2(i_1)v_2^2(i_2) - \mathbf{A}_0 v_1(i_1)v_2(i_2) - \mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2)]\cdot\big[\mathbf{B}_1 v_1(i_1) + \mathbf{B}_2 v_2(i_2)\big]\,. \tag{18}$$

Using equations (10) and (18), $\mathbf{F}_{i_1,i_2}$ can be defined as

$$\mathbf{F}_{i_1,i_2} = \mathbf{N}[v_1(i_1), v_2(i_2)]\,. \tag{19}$$

Equations (10), (12) and (19) yield

$$\mathbf{F}_{i_1,i_2} = \sum_{\lambda_1=0}^{r-1}\sum_{\lambda_2=0}^{r-1} \mathbf{P}_{\lambda_1,\lambda_2} W^{-(i_1\lambda_1 + i_2\lambda_2)}. \tag{20}$$

In the above equation (20), $[\mathbf{F}_{i_1,i_2}]$, $[\mathbf{P}_{\lambda_1,\lambda_2}]$ form a 2D DFT pair. Therefore the coefficients $\mathbf{P}_{\lambda_1,\lambda_2}$ can be computed, using the inverse 2D DFT, as follows

$$\mathbf{P}_{\lambda_1,\lambda_2} = \frac{1}{R}\sum_{i_1=0}^{r-1}\sum_{i_2=0}^{r-1} \mathbf{F}_{i_1,i_2} W^{(i_1\lambda_1 + i_2\lambda_2)}. \tag{21}$$

Two salient examples, simple yet illustrative of the theoretical concepts presented in this work, follow below

5 NUMERICAL EXAMPLES

5.1 Single-Input Single-Output

Consider the following single-input single-output SO2D system

$$x(i_1+2,i_2+2) = \mathbf{A}_0 x(i_1+1,i_2+1) + \mathbf{A}_1 x(i_1+1,i_2) + \mathbf{A}_2 x(i_1,i_2+1) + \mathbf{B}_1 u(i_1+1,i_2) + \mathbf{B}_2 u(i_1,i_2+1)\,,$$
$$y(i_1,i_2) = \mathbf{C}\, x(i_1,i_2) \qquad (22)$$

where

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix},\ \mathbf{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix},\ \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$
$$\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = [0 \quad 1]\,.$$

Since $\lambda = 2$, the $r = 2\lambda = 4$. Therefore $R = (r+1)^2 = 25$. The direct application of the proposed algorithm yields

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} =$$
$$\begin{bmatrix} -6.0000+0.0000j & 3.0451+2.4899j & -2.5451+0.2245j & -2.5451-0.2245j & 3.0451-2.4899j \\ -0.3090+4.7553j & 1.9271+3.3022j & 0.1180-2.7144j & 3.3090+1.4001j & -1.0000-3.8042j \\ 0.8090+2.9389j & 2.1910-4.3920j & -1.4271-3.2164j & -1.0000-2.3511j & -2.1180+2.2654j \\ 0.8090-2.9389j & -2.1180-2.2654j & -1.0000+2.3511j & -1.4271+3.2164j & 2.1910+4.3920j \\ -0.3090-4.7553j & -1.0000+3.8042j & 3.3090-1.4001j & 0.1180+2.7144j & 1.9271-3.3022j \end{bmatrix}$$

and

$$\begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} & F_{04} \\ F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \\ F_{20} & F_{21} & F_{22} & F_{23} & F_{24} \\ F_{30} & F_{31} & F_{32} & F_{33} & F_{34} \\ F_{40} & F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} =$$
$$\begin{bmatrix} 2.0000+0.0000j & 0.6180-0.7265j & -1.6180-3.0777j & -1.6180+3.0777j & 0.6180+0.7265j \\ -0.5000-1.5388j & 0.1910-1.7634j & -3.4271+1.7634j & 1.7361+1.5388j & 2.0000+0.0000j \\ -0.5000+0.3633j & -2.7361-0.3633j & 1.3090+2.8532j & 2.0000+0.0000j & -0.0729-2.8532j \\ -0.5000-0.3633j & -0.0729+2.8532j & 2.0000-0.0000j & 1.3090-2.8532j & -2.7361+0.3633j \\ -0.5000+1.5388j & 2.0000-0.0000j & 1.7361-1.5388j & -3.4271-1.7634j & 0.1910+1.7634j \end{bmatrix}.$$

Using (17), the denominator coefficients are

$$\begin{bmatrix} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} \\ q_{10} & q_{11} & q_{12} & q_{13} & q_{14} \\ q_{20} & q_{21} & q_{22} & q_{23} & q_{24} \\ q_{30} & q_{31} & q_{32} & q_{33} & q_{34} \\ q_{40} & q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using (21), the numerator matrix polynomials are

$$\begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} & P_{04} \\ P_{10} & P_{11} & P_{12} & P_{13} & P_{14} \\ P_{20} & P_{21} & P_{22} & P_{23} & P_{24} \\ P_{30} & P_{31} & P_{32} & P_{33} & P_{34} \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Once the denominator and the adjoint matrix coefficients have been computed, the transfer function $T(z_1,z_2)$ is determined as

$$T(z_1,z_2) = \big(P_{23}z_1^2z_2^3 + P_{12}z_1z_2^2 + P_{11}z_1z_2\big)\big(q_{44}z_1^4z_2^4 + q_{33}z_1^3z_2^3 + q_{23}z_1^2z_2^3 + q_{21}z_1^2z_2 + q_{20}z_1^2 + q_{12}z_1z_2^2 + q_{11}z_1z_2 + q_{02}z_2^2\big)^{-1}$$

or

$$T(z_1,z_2) = \frac{z_1^2z_2^3 - z_1z_2^2 + z_1z_2}{z_1^4z_2^4 - z_1^3z_2^3 - z_1^2z_2^3 + z_1^2z_2 - z_1^2 - 2z_1z_2^2 - 2z_1z_2 - z_2^2}.$$

The above result can be verified using (2).

5.2 Multiple-Input Single-Output

Consider the following two-input single-output SO2D system:

$$x(i_1+2,i_2+2) = \mathbf{A}_0 x(i_1+1,i_2+1) + \mathbf{A}_1 x(i_1+1,i_2) + \mathbf{A}_2 x(i_1,i_2+1) + \mathbf{B}_1 u(i_1+1,i_2) + \mathbf{B}_2 u(i_1,i_2+1)\,,$$
$$y(i_1,i_2) = \mathbf{C}\, x(i_1,i_2) \qquad (23)$$

where

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix},\ \mathbf{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix},\ \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\mathbf{B}_1 = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix},\ \mathbf{C} = [1 \ \ 0]\,.$$

Since $\lambda = 2$, the $r = 2\lambda = 4$. Therefore $R = (r+1)^2 = 25$. The direct application of the proposed algorithm yields

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} =$$

$$\begin{bmatrix} -6.0000+0.0000j & 3.0451+2.4899j & -2.5451+0.2245j \\ -0.3090+4.7553j & 1.9271+3.3022j & 0.1180-2.7144j \\ 0.8090+2.9389j & 2.1910-4.3920j & -1.4271-3.2164j \\ 0.8090-2.9389j & -2.1180-2.2654j & -1.0000+2.3511j \\ -0.3090-4.7553j & -1.0000+3.8042j & 3.3090-1.4001j \end{bmatrix.$$

$$\begin{matrix}. & -2.5451-0.2245j & 3.0451-2.4899j \\ & 3.3090+1.4001j & -1.0000-3.8042j \\ & -1.0000-2.3511j & -2.1180+2.2654j \\ & -1.4271+3.2164j & 2.1910+4.3920j \\ & 0.1180+2.7144j & 1.9271-3.3022j \end{bmatrix} .$$

and

$$\begin{aligned}
\mathbf{F}_{00} &= [\,-19.0000 \quad 2.0000\,], \\
F_{01} &= [\,-9.7533+18.1558j \quad -7.4721-0.4490j\,], \\
\mathbf{F}_{02} &= [\,9.2533+4.6493j \quad 1.4721+4.9798j\,], \\
\mathbf{F}_{03} &= [\,9.2533-4.6493j \quad 1.4721-4.9798j\,], \\
\mathbf{F}_{04} &= [\,-9.7533-18.1558j \quad -7.4721+0.4490j\,], \\
\mathbf{F}_{10} &= [\,2.1180+13.1230j \quad -0.5000+0.8123j\,], \\
\mathbf{F}_{11} &= [\,22.6074+2.9389j \quad 3.8090+4.1145j\,], \\
\mathbf{F}_{12} &= [\,1.9549-12.0005j \quad 3.8090-0.5878j\,], \\
\mathbf{F}_{13} &= [\,-7.6180-3.3552j \quad -0.5000-1.5388j\,], \\
\mathbf{F}_{14} &= [\,-2.8820+11.0494j \quad 1.4721+3.0777j\,], \\
\mathbf{F}_{20} &= [\,-0.1180-6.3471j \quad -0.5000-3.4410j\,], \\
\mathbf{F}_{21} &= [\,-5.3820-7.3309j \quad -0.5000+0.3633j\,], \\
\mathbf{F}_{22} &= [\,-3.1074-4.7553j \quad 2.6910-6.6574j\,], \\
\mathbf{F}_{23} &= [\,-5.1180+5.5146j \quad -7.4721-0.7265j\,], \\
\mathbf{F}_{24} &= [\,7.5451-6.1024j \quad 2.6910+0.9511j\,], \\
\mathbf{F}_{30} &= [\,-0.1180+6.3471j \quad -0.5000+3.4410j\,] \\
\mathbf{F}_{31} &= [\,7.5451+6.1024j \quad 2.6910-0.9511j\,], \\
\mathbf{F}_{32} &= [\,-5.1180-5.5146j \quad -7.4721+0.7265j\,], \\
\mathbf{F}_{33} &= [\,-3.1074+4.7553j \quad 2.6910+6.6574j\,], \\
\mathbf{F}_{34} &= [\,-5.3820+7.3309j \quad -0.5000-0.3633j\,], \\
\mathbf{F}_{40} &= [\,2.1180-13.1230j \quad -0.5000-0.8123j\,], \\
\mathbf{F}_{41} &= [\,-2.8820-11.0494j \quad 1.4721-3.0777j\,], \\
\mathbf{F}_{42} &= [\,-7.6180+3.3552j \quad -0.5000+1.5388j\,], \\
\mathbf{F}_{43} &= [\,1.9549+12.0005j \quad 3.8090+0.5878j\,], \\
\mathbf{F}_{44} &= [\,22.6074-2.9389j \quad 3.8090-4.1145j\,].
\end{aligned}$$

Using (17), the denominator coefficients are

$$\begin{bmatrix} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} \\ q_{10} & q_{11} & q_{12} & q_{13} & q_{14} \\ q_{20} & q_{21} & q_{22} & q_{23} & q_{24} \\ q_{30} & q_{31} & q_{32} & q_{33} & q_{34} \\ q_{40} & q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using (21), the numerator matrix polynomials are

$$\begin{bmatrix} P_{00} & P_{01} & \mathbf{P}_{02} & P_{03} & P_{04} \\ P_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} & P_{13} & P_{14} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & P_{22} & \mathbf{P}_{23} & P_{24} \\ P_{30} & P_{31} & \mathbf{P}_{32} & P_{33} & P_{34} \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & [-3 \;\; 0] & 0 & 0 \\ 0 & [-7 \;\; -2] & [0 \;\; 3] & 0 & 0 \\ [-4 \;\; -2] & [-9 \;\; 0] & 0 & [3 \;\; 1] & 0 \\ 0 & 0 & [1 \;\; 2] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Once the denominator and the adjoint matrix coefficients have been computed, the transfer function $\mathbf{T}(z_1, z_2)$ is determined as

$$\begin{aligned}
\mathbf{T}(z_1, z_2) = \big(&\mathbf{P}_{02} z_2^2 + \mathbf{P}_{11} z_1 z_2 + \mathbf{P}_{12} z_1 z_2^2 + \mathbf{P}_{20} z_1^2 \\
&+ \mathbf{P}_{21} z_1^2 z_2 + \mathbf{P}_{23} z_1^2 z_2^3 + \mathbf{P}_{32} z_1^3 z_2^2\big)\big(q_{02} z_2^2 + q_{11} z_1 z_2 + q_{12} z_1 z_2^2 \\
&+ q_{20} z_1^2 + q_{21} z_1^2 z_2 + q_{23} z_1^2 z_2^3 + q_{33} z_1^3 z_2^3 + q_{44} z_1^4 z_2^4\big)^{-1}
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{T}(z_1, z_2) = \Big(&z_2^2\,[-3 \quad 0] + z_1 z_2\,[-7 \quad -2] + z_1 z_2^2\,[0 \quad 3] \\
&+ z_1^2\,[-4 \;\; -2] + z_1^2 z_2\,[-9 \;\; 0] + z_1^2 z_2^3\,[3 \;\; 1] + z_1^3 z_2^2\,[1 \;\; 2]\Big) \\
&\times \Big(-z_2^2 - 2z_1 z_2 - 2z_1 z_2^2 - z_1^2 + z_1^2 z_2 - z_1^2 z_2^3 - z_1^3 z_2^3 + z_1^4 z_2^4\Big)^{-1}
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{T}(z_1, z_2) = \Big(&\big[-3z_2^2 - 7z_1 z_2 - 4z_1^2 - 9z_1^2 z_2 + 3z_1^2 z_2^3 \\
&+ z_1^3 z_2^2 \mid -2z_1 z_2 + 3z_1 z_2^2 - 2z_1^2 + z_1^2 z_2^3 + 2z_1^3 z_2^2\big]\Big) \\
&\times \Big(-z_2^2 - 2z_1 z_2 - 2z_1 z_2^2 - z_1^2 + z_1^2 z_2 - z_1^2 z_2^3 - z_1^3 z_2^3 + z_1^4 z_2^4\Big)^{-1}.
\end{aligned}$$

The above result can be verified using (2).

6 COMPLEXITY OF THE ALGORITHM

The proposed algorithm has two parts. In the first part the matrices $\mathbf{F}_{i_1,i_2}$ and the scalars a_{i_1,i_2} are evaluated with a cost of $pmR\lambda^3$ operations. In the second part the coefficients of $\mathbf{P}_{\lambda_1,\lambda_2}$ and q_{λ_1,λ_2} are evaluated using the DFT with a cost of $pmR^2 + R^2$ operations. For more efficient computation, especially for high order systems, fast Fourier methods can be used to implement the DFT [13].

Due to the inherent modularity and the algorithmic structure of the presented method high parallelism is permitted. In this case the computation of each determinant a_{i_1,i_2}, (16), and each matrix product $\mathbf{F}_{i_1,i_2}$, (20), can be distributed over a number of processing elements, considerably reducing the computation time of the algorithm.

7 CONCLUSIONS

An algorithm was presented for the computation of the transfer function for a new 2D system model of second order. The technique is using the DFT algorithm and has been implemented with the software package $Matlab^{TM}$. To further improve the computational speed of the algorithm, fast Fourier techniques and VHDL/FPGA based implementations can be used. The presented algorithm can be easily modified to be used for regular, singular and positive SO2D models of other types [11]. Also the presented model/algorithm can easily be extended to systems with higher order and dimensions. It is noted that known problems such as stability, coefficient sensitivity, filtering *etc*, can be studied using the SO2D system model.

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George Antoniou (PhD) Awarded his Doctor of Electrical Engineering (Informatics) degree in 1998 from National Technical University of Athens, Athens, Greece. He is currently a professor in the department of Computer Science at Montclair State University, Montclair, New Jersey, USA. His present research interests include system theory, multidimensional digital signal processing and FPGA/VHDL applications. He is a Senior member of the IEEE.

Marinos T. Michael (BSCS) Awarded his Bachelor of Science in Computer Science degree (Magna Cum Laude) in 2003 from Montclair State University, Montclair, New Jersey, USA. He is currently a graduate student in Computer Science at the same university. He is a student member of the IEEE.