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THE CALCULUS OF THE NON–EXACT RELIEF PERSPECTIVE

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For computer visualization of 3D scenes it is necessary to pay attention to mathematical models of scene, image plane (screen) and viewer (camera, eye, ...) which are represented by data in computer. The alteration of the position of the viewer could be considered under certain assumptions as the alteration of the centre of projection of the relief perspective in Euclidean space $\mathbb{E}^3$. According to the techniques which are used in computer animation, they could be used for solving the problem of continuous deformation of the relief. So the subject of this paper is to ask for these generated reliefs with only little changes within a tolerance ε.

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1 PERSPECTIVE PROJECTION IN THE COMPUTER GRAPHICS

Perspective projection plays an essential role in realistic and naturalistic computer generating of the view of 3D scenes. All 3D objects have to be projected down to two dimensions before they can be displayed on a two-dimensional graphics output device. To explain the continuity between scene-shifting technique used in computer animation and relief perspective, some basic definitions as scene, viewer and view are needed.

Let us define:

Definition 1 (scene)**.** A scene Σ is a triple $\Sigma = (V, S, L)$, where V is the viewer (see Definition 2), S is the set of objects as some $S \in \mathbb{E}^3$ and L is the description of the light in the scene.

The mathematical structure of S and L is not important for the following research and we will not consider them.

Definition 2 (viewer)**.** A viewer V is a 4-tuple $V = (E, D, \omega, U)$, where $E \in \mathbb{E}^3$ is the viewing eye (the camera), $D \in \mathbb{E}^3$ is the viewing direction vector, $\omega \in \mathbb{R}$ is the viewing angle and $U \in \mathbb{E}^3$ is the viewer's up-direction vector.

Definition 3 (normalized viewer)**.** A viewer $V_n = (E_n, D_n, \omega, U_n)$ is called a normalized viewer under the following conditions: $E_n[0,0,0] \in \mathbb{E}^3$, $D_n = \lambda[0,0,1] \in \mathbb{E}^3$, ω is the viewing angle and $U_n = \mu[0,0,1] \in \mathbb{E}^3$, where $\lambda \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$.

The perspective projection ψ on an image plane τ, with $\tau \equiv z - 1 = 0$ of some points $P[x,y,z] \in \mathbb{E}^3$, $z > 1$ in a scene Σ with a normalized viewer V_n is given by

$$\psi\colon \mathbb{E}_3 \to \tau;\ \psi(P[x,y,z>1]) = \Big[r\frac{x}{z}, r\frac{y}{z}, 1\Big], \qquad (1)$$

where r depends on ω: $r = \tan^{-1}(\omega)$.

Definition 4 (view)**.** A view is a pair $(V, \psi(\Sigma))$, where V is the viewer and $\psi(\Sigma)$ is the image of the scene Σ under the perspective projection ψ.

Let ψ_1 be an image due to a normalized viewer V_n. Let then some disnormalization ΔV occur to yield a viewer V_Δ ($V_n \xrightarrow{\Delta V} V_\Delta$), which will cause a modified image ψ_2:

$$(V_n, \psi_1) \xrightarrow{\Delta V} (V_\Delta, \psi_2)\,. \qquad (2)$$

ψ_1 will have some differences to ψ_2. Images ψ_i contain pictures of some objects "seen" by the viewer, because otherwise they could be empty or meaningless. The question is whether the pictures of these objects in images ψ_1 and ψ_2 are "almost the same". "Almost the same" means that they do not differ in their location in the image more than some value ε. These objects will belong to a subspace of the $\mathbb{E}^3$. Let us call this subspace the ε-invariant subspace of two views.

Definition 5. The ε-invariant subspace of two views $\varepsilon[(V_1,\psi_1),(V_2,\psi_2)]$ is determined by
$\varepsilon[(V_1,\psi_1),(V_2,\psi_2)] = \{P \in \mathbb{E}^3 \mid \psi(P) - \psi(\Delta V(P)) < \varepsilon\}$.

Remark. $\psi(\Delta V(P))$ is the image of point P under perspective projection ψ with the viewer V_Δ after some disnormalization ΔV, see (2). Calculation of ε-invariant subspace due to changes ΔV leads to decreasing the computing costs in frame-by-frame animation, see [3]. For every alteration of the viewer there exists a part of space where the frame changes very little, if at all. Based on this calculations, the sequence of single frames may no longer be computed frame by frame. To compute the next frame it is not necessary to use the previous one. No new perspective transformation and rendering is required. It is a shortening of the visualization pipeline.

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2 RELIEF PERSPECTIVE

The relief perspective is a mapping of space into space in order to correspond to the conditions of human seeing and a special case of perspective collineation φ of the extended Euclidean space $\overline{\mathbb{E}}^3$ $\varphi\colon \mathbb{E}^3 \to \overline{\mathbb{E}}^3$ (see e.g. [2], [4]). No mapped objects are ideal and all below presented notions are clear according to context $\overline{\mathbb{E}}^3$ or $\mathbb{E}^3$. A mapping φ is moreover a homology and its determining elements are the centre O, the image plane σ (the set of invariant points) and the vanishing plane ω^r (the image of the plane at infinity).

In the relief perspective the image plane σ has to be placed between the point O and the plane ω^r. The mapped object, the image of which we want to construct, is located in the semi-space, that is opposite to the semi-space $\overset{\longmapsto}{\sigma O}$ ("behind the image plane σ").

Let $\overline{\mathbb{E}}_3$ be a preimage space and $\overline{\mathbb{E}}_3^r$ an image space ($\overline{\mathbb{E}}_3$ with upper index "r") such that $\varphi\colon \overline{\mathbb{E}}_3 \to \overline{\mathbb{E}}_3^r$. The image of the object $\mathcal{U} \in \overline{\mathbb{E}}_3$ under the relief perspective φ is called *the relief* of the object $\mathcal{U}$ and is denoted analogously $\mathcal{U}^r$. The relief perspective is given by the centre O, the image plane σ and the ordered pair of different points $A, A^r = \varphi(A)$ $(A, \varphi(A) \neq O)$ such that $A, A^r \notin \sigma$ and O, A, A^r are collinear.

An analytical representation of the relief perspective (with the centre $O\,[0,0,0]$ and the image plane $z-1=0$) is expressed as follows

$$x' = \frac{(1+k)x}{z+k},\; y' = \frac{(1+k)y}{z+k},\; z' = \frac{(1+k)z}{z+k} \tag{3}$$

where the span $k \in \mathbb{R}^+ - \{1\}$ (for more details see [2]).

3 DEFORMATION OF THE RELIEF

By visualization of 3D objects, in order to increase the perception of their plasticity, it is possible to take the advantage of the relief perspective. In this case the received images of the objects — reliefs are again 3D objects. As it has been already mentioned in the previous section, for determining the relief perspective the location of the centre O and of the planes σ and ω^r is important. The alteration of the position of point O will cause a modification of the relief perspective and also deformation of the generated reliefs.

We would like to investigate the deformation of the relief by a given value ε. For this aim we will start with ideas and terminology described in section 1 applying to the relief perspective.

Because $O = E_n = [0,0,0]$ and $\tau = \sigma \equiv z-1=0$ (compare (1) and (3)), analogously in the meaning of previous designation, we can express the ε-invariant subspace of two reliefs, while $(O, \varphi(\Sigma))$ (in the next (O,φ)) is the relief of the scene Σ under the relief perspective φ with the centre O and $\varphi(\Delta O(X))$ is the relief of the point X under φ with the centre O_Δ after some disnormalization ΔO $(O \xrightarrow{\Delta O} O_\Delta)$

Definition 6. The ε-invariant subspace of two reliefs $\varepsilon[(O_1,\varphi_1),(O_2,\varphi_2)]$ is determined by $\varepsilon[(O_1,\varphi_1),(O_2,\varphi_2)] = \{X \in \mathbb{E}^3 \mid \varphi(X) - \varphi(\Delta O(X)) < \varepsilon\}$.

Now let us discuss different ΔO in detail.

3.1 Alteration of Δx and Δy of the centre O

Alteration of Δx and Δy of the centre of perspectivity O means the shift of the centre O by the value Δx, Δy in the direction of axes x and y.
The alteration of the centre O, which is represented by the shift $-\Delta x$, responds to the shift of the scene Σ by the value Δx.
Let us denote both corresponding reliefs, the first before and the second after the mentioned shift

$$O \xrightarrow{-\Delta x} O_{-\Delta x}$$

as (O,φ_1) and $(O_{-\Delta x},\varphi_2)$. The ε-invariant subspace of these reliefs is expressed as

Proposition 1. $\varepsilon[(O,\varphi_1),(O_{-\Delta x},\varphi_2)] = \{(x,y,z) \in \mathbb{E}^3 \mid z > \frac{|\Delta x|(1+k)-\varepsilon k}{\varepsilon}\}$.

P r o o f . In this case, when the centre O is altered by $-\Delta x$, points in $\mathbb{E}^3$ shift after repeated renormalization, it means after coordinate transformation, by the value Δx (this process also includes the translation of the objects in the scene by Δx without the alteration of the point O). The calculation of the ε-invariant subspace leads to solving the following inequality

$$\left| \frac{(1+k)\,x}{z+k} - \frac{(1+k)(x+\Delta x)}{z+k} \right| < \varepsilon$$

and after carrying out some computations we have

$$|\Delta x| < \frac{\varepsilon(z+k)}{(1+k)} \quad \text{and} \quad z > \frac{|\Delta x|(1+k)-\varepsilon k}{\varepsilon}.$$

Analogously we can obtain

Proposition 2. $\varepsilon[(O,\varphi_1),(O_{-\Delta y},\varphi_2)] = \{(x,y,z) \in \mathbb{E}^3 \mid z > \frac{|\Delta y|(1+k)-\varepsilon k}{\varepsilon}\}$

P r o o f . Trivial by substitution Δx for Δy.

Propositions 1 and 2 have shown that the ε-invariant subspace of two reliefs for the altering of the centre O parallel to the image plane σ is bounded (see Fig. 1 for $\Delta x = 0.1$, $\varepsilon = \frac{1}{1024}$, $k=2$).

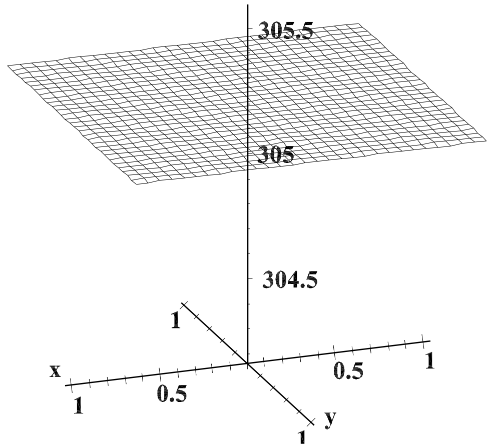

Fig. 1. $\varepsilon[(O,\varphi_1),(O_{-\Delta x},\varphi_2)]$

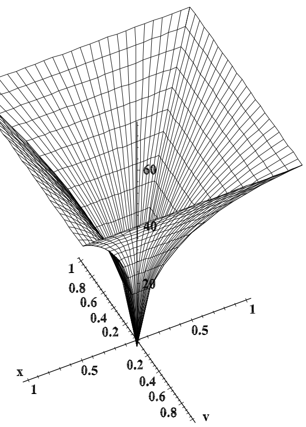

Fig. 2. $\varepsilon[(O,\varphi_1),(O_{-\Delta z},\varphi_2)]$

3.2 Alteration of Δz of the centre O

Alteration of Δz of the centre of perspectivity O means the shift of the point O in the direction orthogonal to the image plane σ.
Let the relief (O,φ_1) be given. The disnormalization $-\Delta z$ of the centre O

$$O \xrightarrow{-\Delta z} O_{-\Delta z}$$

which is equivalent to the shift of the objects by $+\Delta z$ leads to the altered relief $(O_{-\Delta z},\varphi_1)$. For the corresponding ε-invariant subspace we can formulate the next proposition

Proposition 3. $\varepsilon[(O,\varphi_1),(O_{-\Delta z},\varphi_2)] = \varepsilon_x[(O,\varphi_1),(O_{-\Delta z},\varphi_2)] \cap \varepsilon_y[(O,\varphi_1),(O_{-\Delta z},\varphi_2)]$.

(i) *If the arguments of the square-root are positive then* $\varepsilon_x[(O,\varphi_1),(O_{-\Delta z},\varphi_2)] = \{(x,y,z) \in \mathbb{E}^3 \mid z > \max(z_0,z_1);\ z_{0,1} = -\frac{(\Delta z+2k)}{2} + \sqrt{(\frac{\Delta z}{2})^2 \pm \frac{x\Delta z(1+k)}{\varepsilon}}\}$.

(ii) *If the arguments of the square-root are positive then* $\varepsilon_y[(O,\varphi_1),(O_{-\Delta z},\varphi_2)] = \{(x,y,z) \in \mathbb{E}^3 \mid z > \max(z_0,z_1);\ z_{0,1} = -\frac{(\Delta z+2k)}{2} + \sqrt{(\frac{\Delta z}{2})^2 \pm \frac{y\Delta z(1+k)}{\varepsilon}}\}$.

P r o o f . Analogously as the proof of the previous proposition to determine the ε-invariant subspace it is necessary to solve the following inequality

$$\left|\frac{(1+k)\,x}{z+k} - \frac{(1+k)\,x}{z+\Delta z+k}\right| < \varepsilon$$

and after carrying out some computations omitting irrelevant negative z-values (represented cases when the centre O is located between object and the image plane σ) we obtain expression formulated in the proposition.

The proof is completed by trivial textual substitution x for y.

Proposition 3 shows that for translation of the centre O orthogonal to the image plane there exists an ε-invariant subspace with a boundary which has the shape like a "loudspeaker", see Fig. 2 for $\Delta x = \Delta y = 0.2$, $\Delta z = 2$, $\varepsilon = \frac{1}{1024}$, $k = 2$.

3.3 General alteration of $(\Delta x, \Delta y, \Delta z)$ of the centre O

The aim is to generalize the shift of the centre of perspectivity O by arbitrary vectors $(\Delta x, \Delta y, \Delta z)$ (the special cases of $(\Delta x, \Delta y)$ and (Δz) were investigated in the previous subsections).

Let us consider a disnormalization $\Delta O = (-\Delta x, -\Delta y, -\Delta z)$ of the point O

$$(O,\varphi_1) \xrightarrow{(-\Delta x,-\Delta y,-\Delta z)} (O_{(-\Delta x,-\Delta y,-\Delta z)},\varphi_2)$$

which implies the following proposition

Proposition 4. $\varepsilon[(O,\varphi_1),(O_{(-\Delta x,-\Delta y,-\Delta z)},\varphi_2)] = \varepsilon_x[(O,\varphi_1),(O_{(-\Delta x,-\Delta y,-\Delta z)},\varphi_2)] \cap \varepsilon_y[(O,\varphi_1),(O_{(-\Delta x,-\Delta y,-\Delta z)},\varphi_2)]$.

(i) *If the arguments of the square-root are positive then* $\varepsilon_x[(O,\varphi_1),(O_{(-\Delta x,-\Delta y,-\Delta z)},\varphi_2)] = \{(x,y,z) \in \mathbb{E}^3 \mid z > \max(z_0,z_1); z_{0,1} = \frac{-\varepsilon(\Delta z+2k)\pm\Delta x(1+k)}{2\varepsilon} + {} + \sqrt{(\frac{\Delta z}{2})^2 \mp \frac{\Delta z(1+k)(2x+\Delta x)}{2\varepsilon} + \frac{\Delta x^2(1+k)^2}{4\varepsilon^2}}\}$.

(ii) *If the arguments of the square-root are positive then* $\varepsilon_y[(O,\varphi_1),(O_{(-\Delta x,-\Delta y,-\Delta z)},\varphi_2)] = \{(x,y,z) \in \mathbb{E}^3 \mid z > \max(z_0,z_1); z_{0,1} = \frac{-\varepsilon(\Delta z+2k)\pm\Delta y(1+k)}{2\varepsilon} + {} + \sqrt{(\frac{\Delta z}{2})^2 \mp \frac{\Delta z(1+k)(2y+\Delta y)}{2\varepsilon} + \frac{\Delta y^2(1+k)^2}{4\varepsilon^2}}\}$.

P r o o f . To determine the ε-invariant subspace it is necessary to solve the following inequality

$$\left|\frac{(1+k)\,x}{z+k} - \frac{(1+k)(x+\Delta x)}{z+\Delta z+k}\right| < \varepsilon\,.$$

After carrying out some computations omitting some negative z-values (from the same reason as in the proof of Proposition 3) we obtain expression formulated in the proposition.

The proof is completed by simple textual substitution x for y.

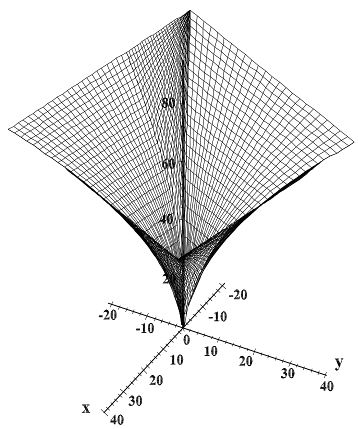

Fig. 3. $\varepsilon[(O,\varphi_1),(O_{(-\triangle x,-\triangle y,-\triangle z)},\varphi_2)]$

Proposition 4 shows that for general shifts of the centre O there exists an ε-invariant subspace with a boundary which has the shape as it is shown in Fig. 3 for $\triangle x=\triangle y=0.01$, $\triangle z=0.1$, $\varepsilon=\frac{1}{1024}$, $k=2$. Propositions 1,2 (with $\triangle z=\triangle y=0$ and $\triangle z=\triangle x=0$ conditions) and Proposition 3 (with $\triangle x=\triangle y=0$ condition) are special cases of the Proposition 4.

4 CONCLUSION

After presented discussions dedicated to the alteration of the centre of perspectivity O some results became evident. First, for all alterations of the point O there exists a part of space where the relief changes very little. Second, for all reliefs mapping an object which is in the ε-invariant subspace of some alteration of the centre O a certain "elasticity" of that relief was derived; the centre O may alter within that "elasticity".

Moving the object in the scene the corresponding relief is continuously deformed. Calculation of the ε-invariant subspace of two reliefs can be useful when the missing relief is computed. The mentioned relief occurs at the alteration of the point O or at the local shift of the given object.

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