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ARCH AND GARCH MODELS APPLIED IN GEODESY AND GEODYNAMICS

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We first introduce some basic theoretical concepts and models used in time series analysis. Then we prepare geodetical data for modeling, fitting models to a given set of data and finally we check the validity of this models.

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1 TIME SERIES

1.1. Introduction

A discrete series consists of a set of observations $\{x_1, x_2, \dots, x_t, \dots, x_n\}$ of some phenomenon. (We assume that x_t is real.) The observations are made at equally spaced time intervals. This assumption enables us to use the interval between two successive observations as the unit of time. The subscript t can be referred to as time, so the x_t is the observed value of the time series at time t. The total number of observations in a time series (here n) is called the length of the time series.

The main purpose of time series analysis is to understand the underlying mechanism that generates the observed data and, in turn, to forecast future values of the series. Given the unknowns that affect the observed values in time series, it is natural to suppose that the generating mechanism is probabilistic and to model time series as stochastic processes. By this we mean that the observation x_t is presumed to be a realized value of some random variable X_t; the time series $\{x_1, x_2, \dots, x_t, \dots\}$, a single realization of a stochastic process $\{X_1, X_2, \dots, X_t, \dots\}$.

In the following we will use the term time series to refer both to the observed data and to the stochastic process.

In general the time series consists of the following components:

1. Trend: The long-therm component that represents the growth or decline in a time series over an extended period of time.
2. Cyclical component: The wavelike fluctuation around the trend.
3. Seasonal component: A pattern of change in quarterly or monthly data that repeats itself from year to year.
4. Irregular component: A measure of the variability of the time series after the other components have been removed.

We can eliminate the first three components in general for example by regression and then we will analyse the irregular component by Box-Jenkins methodology, as it is shown in the following parts.

To "visualize" a time series we plot our observations $\{x_t\}$ as a function of time t. This is called a time plot.

1.2. Univariate ARCH and GARCH Models

In the time series we have considered so far, the disturbances or errors $\{Z_t\}$ are assumed to be homoskedastic, that is, the variance of Z_t is assumed to be independent of t. Autoregressive Conditional Heteroskedasticity (ARCH) models and Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models are used to model the changes in the variance of the errors as a function of time. An ARCH process of order q, ARCH(q), is given by (see Engle (1982))

$$Z_t = \nu_t \sqrt{h_t} \tag{1}$$

where $\{\nu_t\}$ is an independently distributed Gaussian random sequence with zero mean and unit variance; h_t is the conditional variance of Z_t conditional on all the information up to time $t-1$, I_{t-1}:

$$E\big(Z_t^2 \mid I_{t-1}\big) = h_t = \alpha_0 + \alpha_1 z_{t-1}^2 + \alpha_2 z_{t-2}^2 + \dots + \alpha_q z_{t-q}^2 \,. \tag{2}$$

GARCH models are generalizations of ARCH models where h_t, the conditional variance at time t, depends on earlier variances. That is, a GARCH(p,q) process is given by (1) with (see Bollerslev (1986))

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i z_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i} \,. \tag{3}$$

When $p = 0$ we have an ARCH(q) model; when both p and q are zero, Z_t is simply white noise.

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An ARCH(q) model and a GARCH(p,q) model are represented in Time Series package in Mathematica 3.0 by

ARCHModel[$\{\alpha_0, \alpha_1, \dots, \alpha_q\}$] and

GARCHModel[$\{\alpha_0, \alpha_1, \dots, \alpha_q\}, \{\beta_1, \beta_2, \dots, \beta_p\}$],

respectively. Note that since the variance is positive, we usually have $\alpha_0 > 0, \alpha_i \geq 0$, and $\beta_i \geq 0$ for $i > 0$.

The so-called ARCH- or GARCH-regression model is a regression model with the disturbances following an ARCH process (see(1) and (2)) or a GARCH process (see (1), (3)), respectively. That is,

$$Y_t = \mathbf{x}_t' \cdot \mathbf{b} + Z_t \tag{4}$$

where $\mathbf{x}_t$ is a known column vector that can contain lagged values of Y (i.e., y_{t-1}, $y_{t-2}, \dots$ etc.), and $\mathbf{b}$ is a column vector of unknown parameters. The first term on the right-hand side of (4) is the conditional mean of Y_t; that is, $\mathrm{E}(Y_t \mid I_{t-1}) = \mathbf{x}_t' \cdot \mathbf{b} \equiv m_t$, and the conditional variance of Y_t is that of Z_t and is given by (2) or (3).

1.2.1. Estimation of ARCH and GARCH Models

From the definition of the ARCH(q) model it is clear that the correlation $\mathrm{E}(Z_t Z_{t+k})$ ($k \neq 0$) is zero. However, it is easy to see that Z_t^2 follows an AR(q) process. Similarly, if $\{Z_t\}$ is a GARCH(p,q) process, Z_t^2 follows an ARMA(s,p) process, where $s = \mathrm{Max}(p,q)$. This can be used to help identify the orders of ARCH or GARCH models.

The maximum likelihood method is often used to estimate the parameters of an ARCH or GARCH model. The logarithm of the Gaussian likelihood function is given by (apart from an additive constant)

$$\sum_{t=1}^{N} \left(-\frac{1}{2} \ln h_t - \frac{z_t^2}{2h_t} \right), \tag{5}$$

where $z_t((=y)_t - m_t)$ conditional on i_{t-1} is normally distributed with zero mean and variance h_t. The function

LogLikelihood[z, model]

gives the logarithm of the likelihood (5), where model can be ARCHModel or GARCHModel and $z = \{z_1, z_2, \dots, z_N\}$. Once the likelihood function is calculated, we can in principle estimate the model parameters by maximizing the likelihood function (or minimizing its negative). This can be accomplished in some cases by using the built-in function FindMinimum. This gives the maximum likelihood estimate of the ARCH or GARCH parameters:

FindMinimum[-LogLikelihood[z, model].

However, when the number of parameters is large, the function FindMinimum can either be very slow or can go into parameter regions where the LogLikelihood function is complex. In these cases, the function ConditionalMLEstimate should be used to estimate ARCH or GARCH parameters.

ConditionalMLEstimate[data, model]

fits the specified model to data using the maximum likelihood method, where the model can be ARCHModel or GARCHModel.

1.2.2 Testing for ARCH

Various standard procedures are available to test the existence of ARCH or GARCH. A commonly used test is the Lagrange multiplier (LM) test. Consider the null hypothesis that there is no ARCH, that is, $\alpha_1 = \alpha_2 = \dots = \alpha_q = 0$. It is known that (see, for example, Bollerslev (1986), Eqs. (27) and (28)) the LM statistic has an asymptotic χ^2 distribution with q degrees of freedom under the null hypothesis. If the LM statistic evaluated under the null hypothesis is greater than $\chi^2_{1-\alpha}(q)$, the null hypothesis is rejected at level α. The function

LMStatistic[data, model]

gives the LM statistic, where the model is either an ARCH or GARCH model.

2 EXAMPLE OF ANALYSIS OF TIME SERIES USING SOFTWARE MATHEMATICA 3.0

Data used for the analysis

Atmospheric Angular Momentum (AAM) plays a significant role in causing variations in the length of day. Study of AAM can also yield information about the Earth's orientation in space and the large scale behaviour of the atmosphere.

Analysis by Mathemathica 3.0

This contribution is devoted to the time series analysis of AAM data using ARCH and GARCH models by means of Mathematica 3.0.

The timeplot of our series is in Fig. 1.

Mean, variance and length of our series:

mi=Mean[z1]
Variance[z1]
n=Length[z1]

-0.289635
0.00205396
5809

We subtract the mean from our series:

z=z1-mi

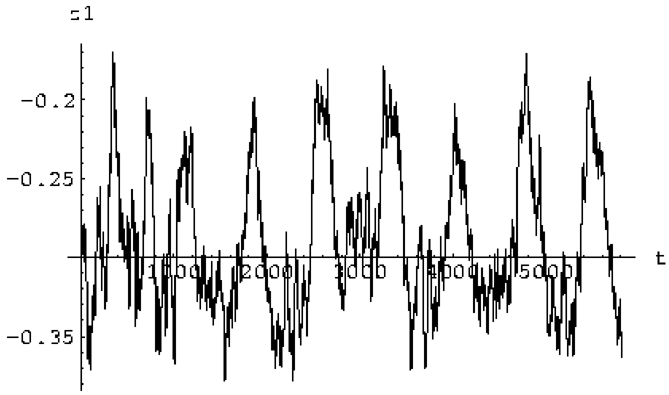

Fig. 1. gd=ListPlot[z1]

Fig. 2. ListPlot[dz]

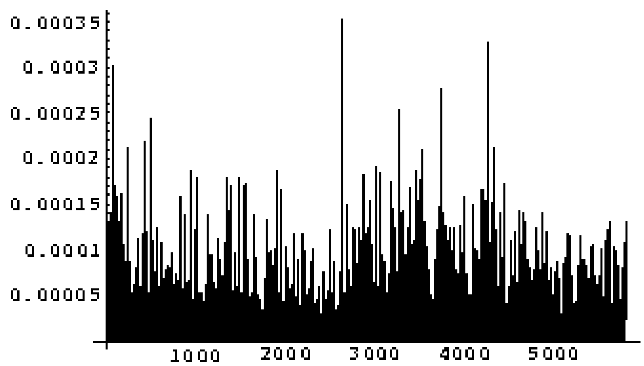

Fig. 3. ListPlot[eps]

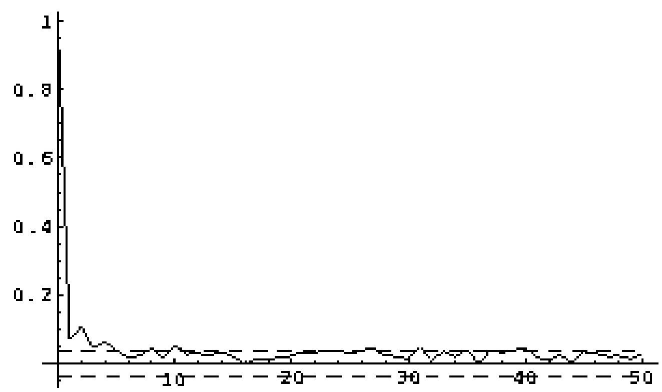

Fig. 4. ListPlot[eps]

We difference the time series to obtain a constant mean series:

dz=Difference[z,1]

The timeplot of series after differencing is in Fig. 2.

The mean and variance after differencing:

mi=Mean[dz]

Variance[dz]

-0.0000164897

0.000020184

We create a new time series to determine the parameters of our ARCH and GARCH models

eps= $\mathbf{dz}^2$

and the time plot of squares is in Fig. 3.

Parameter of MA model for squares (See Fig.4.):

cor=CorrelationFunction[eps,50];

myplotcorr1[cor,3/Sqrt[n],PlotRange→ All]

Parameter of AR model for squares (See Fig. 5):

pkf=PartialCorrelationFunction[eps,50];

myplotcorr1[pkf,3/Sqrt[n]]

We can in principle estimate the model parameters by maximizing the likelihood function (or minimizing its negative). To find the GARCH parameters from the series dz, we can do the following:

The order of selected model is:GARCH(2,1).

r=FindMinimum[-LogLikelihood[dz,
GARCHModel[{a0,a1,a2}, {b1}]],
{a0,0.00000001,0.0000002}, {a1,0.05,0.06},
{a2,-0.02,-0.031}, {b1,0.9,0.91}]

The parameters of our model:

model1=GARCHModel[$1.18003 * 10^{-6}$, 0.04537,0.00228, 0.89385]

Now we compute the LM statistics and qantile $\chi_{0.95}(3)$:

LMStatistic[dz,model1]

1.55121

Quantile[ChiSquareDistribution[3],0.95]

7.81473

Since the computed test LM statistics is not greater than quantile $\chi_{0.95}(3)$, the Lagrange multiplier test has not shown suitability of model GARCH(2,1) for the considered time series on the significance level $\alpha = 0.05$.

Estimation with function ConditionalMLEstimate:

The order of estimated model is: ARCH(2).

ConditionalMLEstimate[dz,
ARCHModel[0.00001,0.04,0.02]]

The parameters of our model:

model2=ARCHModel[0.00001714,0.05474,0.09453]

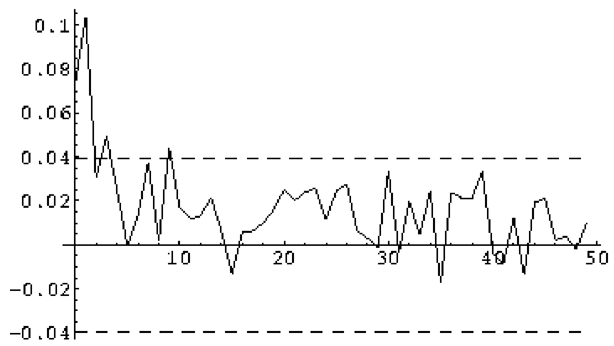

Fig. 5.

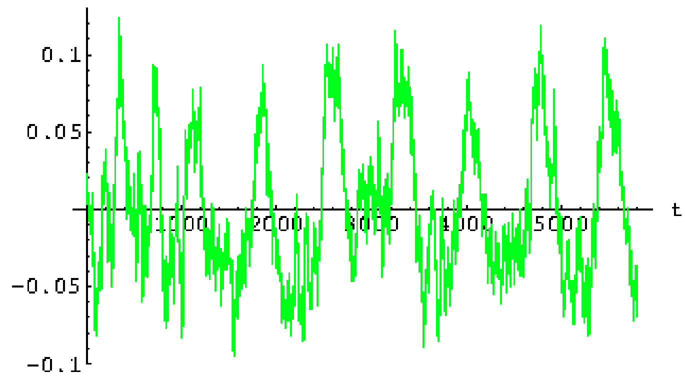

Fig. 6. pts1dz=ListPlot[ts1dz]

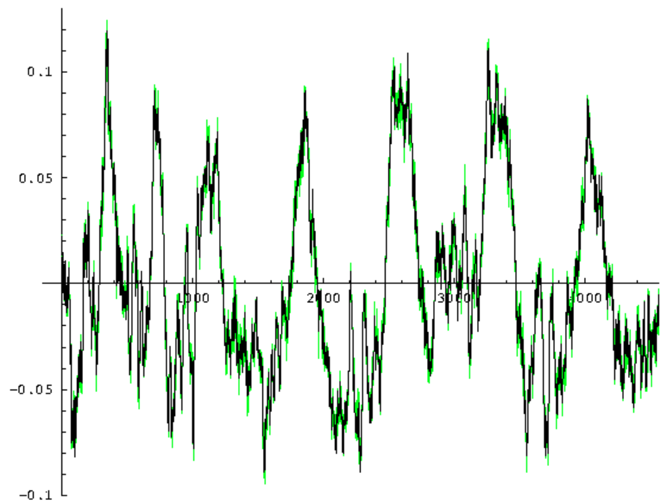

Fig. 7. Show[pts1dz,pdr]

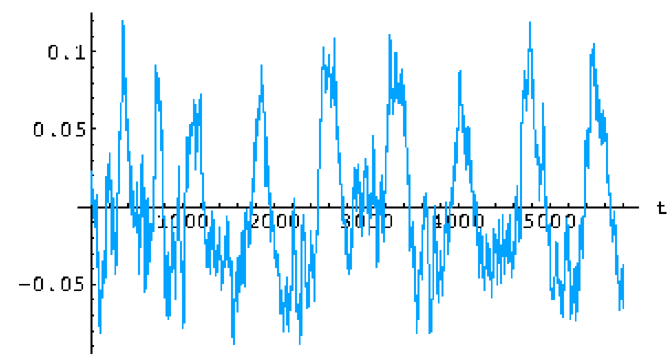

Fig. 8. pts2dz=ListPlot[ts2dz]

Now we compute the LM statistics and quantile $\chi_{0.95}(2)$:

LMStatistic[dz,model2]

$2.6683 * 10^{-7}$

Quantile[ChiSquareDistribution[2],.95]

5.99146

Since the computed test LM statistics is not greater than quantile $\chi_{0.95}(2)$, the Lagrange multiplier test has not shown suitability of model ARCH(2) for the considered time series on the significance level $\alpha = 0.05$.

The next estimation:

Model GARCH(2,2):

r1=FindMinimum[-LogLikelihood[dz, GARCHModel[{a0,a1,a2},{b1,b2}]], {a0,0.00000001,0.0000002},{a1,0.05,0.06}, {a2,-0.02,-0.031},{b1,0.8,0.82},{b2,0.09,0.091}]

The parameters of our model:

model3=GARCHModel[$3.06747 * 10^{-7}$,0.03987, -0.01394,0.82776,0.13098]

We compute the LM statistics and quantile $\chi_{0.95}(4)$:

LMStatistic[dz,model3]

20.0413

Quantile[ChiSquareDistribution[4],0.95]

9.48773

Since the computed test LM statistics is greater than quantile $\chi_{0.95}(4)$, the Lagrange multiplier test has shown suitability of model GARCH(2,2) for the considered time series on the significance level $\alpha = 0.05$.

And finally: Model GARCH(2,1):

ConditionalMLEstimate[dz,GARCHModel[0.0000000000052,0.0045,0.0022,0.89]]

The parameters of our model:

model5=GARCHModel[$-1.64296 * 10^{-9}$, 0.00468,0.00236,0.88984]

We compute the LM statistics and quantile $\chi_{0.95}(3)$:

LMStatistic[dz,model5]

878269,

Quantile[ChiSquareDistribution[3],0.95]

7.81473.

Since the computed test LM statistics is greater than quantile $\chi_{0.95}(3)$, the Lagrange multiplier test has shown suitability of model GARCH(2,2) for the considered time series on the significance level $\alpha = 0.05$.

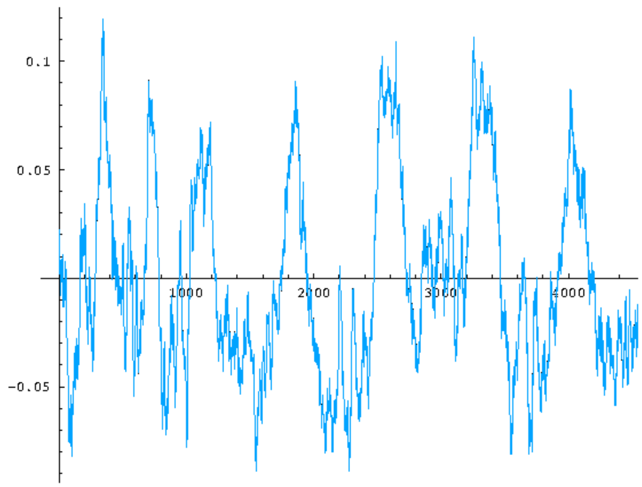

Fig. 9. Show[pdr,pts2dz]

The timeplot of our model3 is in Fig. 6. The timeplot of our model3 together with the original timeplot of our series is in Fig. 7. The timeplot of our model5 is in Fig. 8. The timeplot of our model5 together with the original timeplot of our series is in Fig. 9.

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