Journal of ELECTRICAL ENGINEERING, VOL. 51, NO. 12/s, 2000, 26–28

ASYMPTOTIC BEHAVIOUR OF n–th ORDER LINEAR DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

Hussain Mohamad *

New sufficient conditions for convergence to 0 of nonoscillatory solutions of some n-th order linear neutral functional differential equations are given.

K e y w o r d s: linear differential equation, nonoscillatory solution

2000 *Mathematics Subject Classification*: 34C10

1 INTRODUCTION

We consider n-th order linear neutral functional differential equation of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t) - p(t)x(\sigma(t))] + q(t)x(\tau(t)) = 0\,, \quad t \geq t_0\,, \quad (1)$$

under the standing hypotheses that $n \geq 2$ and:

(a) $p \in C\,[[t_0,\infty);(0,\infty)]$;

(b) $\sigma,\tau \in C\,[[t_0,\infty);R]$, σ,τ are strictly increasing, $\lim\limits_{t\to\infty}\sigma(t)=\infty$, $\lim\limits_{t\to\infty}\tau(t)=\infty$;

(c) $q \in C\,[[t_0,\infty);R]$, $q(t) \not\equiv 0$.

Our aim is to obtain new sufficient conditions for the nonoscillatory solutions of equation (1) to converge to zero. By a solution of equation (1) we mean a continuous function $x\colon [t_x,\infty) \to R$ such that $x(t)-p(t)x(\sigma(t))$ is n times continuously differentiable and $x(t)$ satisfies Eq. (1) for all sufficiently large $t \geq t_x$. The solutions which vanish for all large t will be excluded from our consideration. A solution of (1) is called nonoscillatory if it is eventually of constant sign in $[t_x,\infty)$ otherwise is oscillatory. The problem of oscillation and nonoscillation for neutral differential equations has received considerable attention in recent years; see the references cited therein. However, most of the works on the subject has been focused on first and higher order equations with constant coefficients and a little has been published on higher order neutral equations with variable coefficients. For some results we refer to [1, 5, 7].

2 SOME BASIC LEMMAS

The following lemmas will be useful in the proof of the main results.

Lemma 1. ([6],Lemmas 2.1 and 2.2) *Suppose that* (a), (b) *hold.*

A_1: *Let* $0 < p(t) \leq 1$, $t \geq t_0$, *and* $x(t)$ *be continuous nonoscillatory solution of the functional inequality* $x(t)[x(t) - p(t)x(\sigma(t))] < 0$ *defined in a neighborhood of infinity.*

(i) *Suppose that* $\sigma(t) < t$ *for* $t \geq t_0$. *Then* $x(t)$ *is bounded. If moreover* $0 < p(t) \leq \lambda^* < 1$, $t \geq t_0$, *for some positive constant* λ^*, *then* $\lim\limits_{t\to\infty} x(t) = 0$.

(ii) *Suppose that* $\sigma(t) > t$ *for* $t \geq t_0$. *Then* $x(t)$ *is bounded away from zero.*

A_2: *Let* $1 \leq p(t)$ *for* $t \geq t_0$, *and* $x(t)$ *be a continuous nonoscillatory solution of the functional inequality* $x(t)\,[x(t) - p(t)x(\sigma(t))] > 0$ *defined in a neighborhood of infinity.*

(i) *Suppose that* $\sigma(t) > t$ *for* $t \geq t_0$. *Then* $x(t)$ *is bounded. If moreover* $1 < \lambda_* \leq p(t)$, $t \geq t_0$, *for some positive constant* λ_*, *then* $\lim\limits_{t\to\infty} x(t) = 0$.

(ii) *Suppose that* $\sigma(t) < t$ *for* $t \geq t_0$. *Then* $x(t)$ *is bounded away from zero.*

The next Lemma can be derived on the base of Theorem 2 in [8].

Lemma 2. *Assume that*

$$g\colon [t_0,\infty) \to [0,\infty), \quad \delta\colon [t_0,\infty) \to R$$

are continuous, $\lim\limits_{t\to\infty}\delta(t) = \infty$,

$$\liminf_{t\to\infty} \int_t^{\delta(t)} g(s)\,ds > \frac{1}{e}\,,$$

where $\delta(t) > t$ *for* $t \geq t_0$. *Then the functional inequality*

$$\dot{x}(t) - g(t)\,x(\delta(t)) \geq 0\,, \qquad t \geq t_0\,,$$

cannot have an eventually positive solution, and

$$\dot{x}(t) - g(t)\,x(\delta(t)) \leq 0\,, \quad t \geq t_0\,,$$

* Department of Applied Mathematics, University of Žilina, 010 26 Žilina, Slovak Republic, E-mail: mohamad@fpv.utc.sk

cannot have an eventually negative solution.

We say that the function $u(\cdot) \in C^n[R;R]$ is of degree $k \in \{0,1,\dots n\}$ if

$$\begin{aligned} u(t)\,u^{(i)}(t) > 0\,, \quad 0 \le i \le k\,,\\ (-1)^{i+k}\,u(t)\,u^{(i)}(t) > 0\,, \quad k \le i \le n\,. \end{aligned} \tag{2_k}$$

Let $\tau^*(t) = \min\{t, \tau(t)\}$.

Lemma 3. ([7], Lemma 3.3) *Suppose that n is odd, and*

$$\int_{t_0}^{\infty} [\tau^*(t)]^{n-2}\,[\tau(t)]^{1-\varepsilon}\,q(t)\,dt = \infty$$

for some $\varepsilon > 0$. Then each nonoscillatory solution of inequality

$$\{v^{(n)}(t) + q(t)\,v(\tau(t))\}\,\operatorname{sgn} v(t) \le 0\,, \quad t \ge t_0\,,$$

is of degree 0.

3 ASYMPTOTIC BEHAVIOUR

In this section we shall study the asymptotic behaviour of the bounded and all nonoscillatory solutions of equation (1). Let $\tau^{-1}(t)$, $\sigma^{-1}(t)$ denote the inverse functions of $\tau(t)$, $\sigma(t)$, and $\alpha\colon [t_0,\infty) \to R$ be a continuous function. We define the function $u(t) = x(t) - p(t)\,x(\sigma(t))$. So Eq. (1) can be written as

$$u^{(n)}(t) = -q(t)\,x(\tau(t)), \quad t \ge t_0, \quad n \ge 2\,. \tag{$1'$}$$

Theorem 1. *Suppose that $0 < p(t) \le \lambda^* < 1$, $(-1)^n q(t) \ge 0$, $\sigma(t) < t < \tau(t)$, $t < \alpha(t)$, and*

$$\liminf_{t\to\infty} \frac{1}{(n-2)!} \int_t^{\tau(t)} \int_v^{\alpha(v)} (\xi - v)^{n-2} |q(\xi)|\, d\xi dv > \frac{1}{e}. \tag{3}$$

Then every nonoscillatory bounded solution of (1) tends to zero as $t \to \infty$.

P r o o f . Without loss of generality we may assume that $x(t)$ is bounded and eventually positive solution of (1). Let n be odd (the proof is similar when n is even) then $u^{(n)}(t) \ge 0$ for all large t. For sufficiently large t_0 we have two cases:
1. $u(t) > 0$, $t \ge t_0$; 2. $u(t) < 0$, $t \ge t_0$.

Case 1. Since $u(t)$ is bounded, and $u^{(n)}(t) \ge 0$ then

$$(-1)^i u^{(i)}(t) < 0\,, \quad t \ge t_1 \ge t_0,\ i = 1,2,\dots,n-1 \tag{4}$$

then from the equality

$$u^{(k)}(t) = \sum_{i=k}^{n-1} (-1)^{i-k} \frac{(s-t)^{i-k}}{(i-k)!} u^{(i)}(s) + \frac{(-1)^{n-k}}{(n-k-1)!} \int_t^s (\xi - t)^{n-k-1} u^{(n)}(\xi)\mathrm{d}\xi\,, \tag{5}$$

where $s > t$, $k = 1$, and with regard to (4), ($1'$) we get

$$u'(t) \ge \frac{-1}{(n-2)!} \int_t^s (\xi - t)^{n-2} q(\xi) x(\tau(\xi))\mathrm{d}\xi\,, \quad t < s\,,$$

we have $u(t) < x(t)$ hence $|q(t)|u(\tau(t)) \le |q(t)|x(\tau(t))$ then

$$u'(t) \ge \frac{1}{(n-2)!} \int_t^s (\xi - t)^{n-2} |q(\xi)| u(\tau(\xi))\mathrm{d}\xi$$

let $s = \alpha(t)$, and so

$$u'(t) - \frac{1}{(n-2)!} \int_t^{\alpha(t)} (\xi - t)^{n-2} |q(\xi)|\mathrm{d}\xi\, u(\tau(t)) \ge 0\,.$$

By Lemma 2 and condition (3) the last inequality can not have eventually positive solution, which is a contradiction.

Case 2. By Lemma 1, A_1 it follows that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. Analogously we can prove the following result.

Theorem 2. *Suppose that $1 < \lambda_* \le p(t)$ is bounded, $(-1)^n q(t) \le 0$, $t < \sigma(t) < \tau(t)$, $t < \alpha(t)$*

$$\liminf_{t\to\infty} \frac{1}{(n-2)!} \int_t^{\sigma^{-1}(\tau(t))} \int_v^{\alpha(v)} (\xi - v)^{n-2} \frac{|q(\xi)|}{p\big(\sigma^{-1}(\tau(\xi))\big)} d\xi dv > \frac{1}{e}. \tag{6}$$

Then every nonoscillatory bounded solution of (1) tends to zero as $t \to \infty$.

Theorem 3. *Suppose that n is odd, $1 < \lambda_* \le p(t)$, $q(t) < 0$, $t < \sigma(t)$ and*

$$\int_{t_0}^{\infty} [\sigma_*^{-1}(\tau(t))]^{n-2} [\sigma^{-1}(\tau(t))]^{1-\varepsilon} \frac{|q(t)|}{p\big(\sigma^{-1}(\tau(t))\big)} dt = \infty\,, \tag{7}$$

for some $\varepsilon > 0$,

$$\int_{t_0}^{\infty} t^{n-1} \frac{|q(t)|}{p\big(\sigma^{-1}(\tau(t))\big)} dt = \infty\,, \tag{8}$$

where $\sigma_*^{-1}(\tau(t)) = \min\{\sigma^{-1}(\tau(t)), t\}$. Then every nonoscillatory solution of (1) tends to zero as $t \to \infty$.

P r o o f . Assume for the sake of contradiction that $x(t)$ is an eventually positive solution of (1), then $u^{(n)}(t) > 0$ for each t large. We consider two cases:
1. $u(t) > 0$, $t \ge t_0$; 2. $u(t) < 0$, $t \ge t_0$, where t_0 is sufficiently large.

Case 1. By Lemma 1, A_2 it follows that $\lim_{t\to\infty} x(t) = 0$, but this case is not possible because $\lim_{t\to\infty} u(t) = 0$

means that $u(t)$ is bounded, and since $u^{(n)}(t)>0$ then $(-1)^i u^{(i)}(t)<0$, $i=1,\dots,n$ which is impossible .

Case 2. We have
$|q(t)|x(\tau(t))> -\dfrac{|q(t)|}{p(\sigma^{-1}(\tau(t)))}u(\sigma^{-1}(\tau(t)))$, so Eq. (1) implies

$$u^{(n)}(t)+\frac{|q(t)|}{p(\sigma^{-1}(\tau(t)))}u(\sigma^{-1}(\tau(t)))>0\,, \qquad (9)$$

By Lemma 3 it follows that all nonoscillatory solutions of (9) are of degree 0. We claim that $\lim\limits_{t\to\infty}u(t)=0$, otherwise $\lim\limits_{t\to\infty}u(t)=L<0$, then $u(t)\le L$, $t\ge t_0$. From (5) we have

$$u(t)=\sum_{i=0}^{n-1}(-1)^i\frac{(s-t)^i}{i!}u^{(i)}(s) + \frac{-1}{(n-1)!}\int_t^s(\xi-t)^{n-1}u^{(n)}(\xi)\mathrm{d}\xi\,, \quad s>t\,. \qquad (5')$$

Then

$$\begin{aligned}
u(t_1)&<\frac{1}{(n-1)!}\int_{t_1}^t(\xi-t_1)^{n-1}q(\xi)x(\tau(\xi))\mathrm{d}\xi\\
&<\frac{1}{(n-1)!}\int_{t_1}^t(\xi-t_1)^{n-1}\frac{|q(\xi)|}{p(\sigma^{-1}(\tau(\xi)))}u(\sigma^{-1}(\tau(\xi)))\mathrm{d}\xi\\
&<\frac{L}{(n-1)!}\int_{t_1}^t(\xi-t_1)^{n-1}\frac{|q(\xi)|}{p(\sigma^{-1}(\tau(\xi)))}\mathrm{d}\xi\,,\\
\frac{u(t_1)}{L}&>\frac{1}{(n-1)!}\int_{t_1}^t(\xi-t_1)^{n-1}\frac{|q(\xi)|}{p(\sigma^{-1}(\tau(\xi)))}\mathrm{d}\xi
\end{aligned}$$

and for $t\to\infty$ we get from the last inequality a contradiction with (8). Then $\lim\limits_{t\to\infty}u(t)=0$. We claim that $x(t)$ is bounded and $\lim\limits_{t\to\infty}x(t)=0$. First suppose that $x(t)$ is not bounded, then there exists a sequence $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}t_n=\infty\,,\qquad \lim_{n\to\infty}x(\sigma(t_n))=\infty\,,\qquad \text{and}\qquad x(\sigma(t_n))=\max_{t_0\le s\le\sigma(t_n)}x(s)\,.$$

Since $u(t)$ is bounded there exists a constant $B<0$ such that $u(t)\ge B$, $t\ge t_1\ge t_0$. Then

$$x(t)\ge p(t)x(\sigma(t))+B\ge\lambda_* x(\sigma(t))+B\,,$$

and so $\lambda_* x(\sigma(t))\le x(t)-B$, then

$$\lambda_* x(\sigma(t_n))\le x(t_n)-B\le x(\sigma(t_n))-B\,,$$
$$x(\sigma(t_n))\le\frac{-B}{\lambda_*-1}$$

which is a contradiction. Then $x(t)$ is bounded.

Next to prove that $\limsup\limits_{t\to\infty}x(t)=0$, suppose that $\limsup\limits_{t\to\infty}x(\sigma(t))=s>0$. Let $\{t_m\}_{m=1}^\infty$ be a sequence such that $\lim\limits_{m\to\infty}t_m=\infty$, $\limsup\limits_{m\to\infty}x(\sigma(t_m))=s$. For m large enough we have
$u(t_m)\le x(t_m)-\lambda_* x(\sigma(t_m))$, $x(t_m)\ge u(t_m)+\lambda_* x(\sigma(t_m))$.
We choose $0<\epsilon<(\lambda_*-1)s$. Then

$$\epsilon+s\ge\limsup_{m\to\infty}x(t_m)\ge\lambda_* s\,,\quad \text{hence}\quad \epsilon\ge(\lambda_*-1)s$$

which is a contradiction. Thus $s=0$.

Example. Consider the neutral differential equation

$$\big(x(t)-2ax(at)\big)'''-\frac{6b}{t^2}x(bt^2)=0\,,\quad t>0\,.$$

(i) Let $a\in(0,\frac12)$, $b>1$, $\alpha(t)=2t$, then the condition of Theorem 1 is satisfied which implies that each nonoscillatory bounded solution of the above equation tends to zero as $t\to\infty$.

(ii) Let $a\in(1,\infty)$, $b>0$, $\varepsilon=1$, then the conditions of Theorem 3 are satisfied, therefore each nonoscillatory solution of the above equation tends to zero as $t\to\infty$. For instance $x(t)=\dfrac{1}{t}$ is such solution.

References

[1] BAINOV, D. D.—MISHEV, D. P.: Oscillation Theory for Neutral Differential Equations with Delay, Adam Hilger, Bristol, Philadelphia and New York, 1991.

[2] BAINOV, D.—PETROV, V.: Asymptotic Properties of the Nonoscillatory Solutions of Second Order Neutral Equations with a Deviating Argment, J. Math. Anal. Appl. **194** (1995), 343–351.

[3] DŽURINA, J.—MIHALIKOVÁ, B.: Oscillation Criterias for Second Order Neutral Differential Equations, Mathematica Bohemica (1998) (to appear).

[4] GRAEF, J. R.—GRAMMATIKOPOULOS, M. K.—SPIKES, P. W. On the Asymptotic: Behavior of Solutions of a Second Order Nonlinear Neutral Delay Differential Equations, Applicable Anal. **22** (1986), 1–19.

[5] GYÖRI, I.—LADAS, G.: Oscillation Theory of Delay Differential Equations, Clarendon Press, Oxford, 1991.

[6] JAROŠ, J.—KUSANO, T.: On a Class of First Order Nonlinear Functional Differential Equations of Neutral Type, Czechoslovak Math. J. **40** (1990), 475–490.

[7] JAROŠ, J.—KUSANO, T.: On Oscillation of Linear Neutral Differential Equations of Higher Order, Hiroshima Math. J. **20** (1990), 407–419.

[8] KOPLATADZE, R. G.—ČANTURIA, T. A.: On Oscillatory and Monotonic Solutions of First Order Differential Equations with Retarded Arguments, Differencianye Uravnenija **8** (1982), 1463–1465. (in Russian)

[9] LADAS, G.—QIAN, C.: Oscillation in Differential Equations with Positive and Negative Coefficients, Cand. Math. Bull. **33** (1990), 442–451.

Received 1 June 2000

Hussain Ali Mohamad (Dr), born in Baghdad, Iraq, awarded MSc degree in applied mathematics at the College of Science, University of Baghdad, and PhD degree in applied mathematics in the year 2000 at the Faculty of Science, University of Žilina, Slovak Republic.