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# SOME CONSTRUCTIONS OF AGGREGATION OPERATORS

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The motivation of developing new methods of constructing aggregation operators is a need for flexible classes of aggregation operators for numerous applications. Some of these constructing methods are discussed in this work.

K e y w o r d s: Aggregation operator, quasi-arithmetic mean, $t$-norm, $t$-conorm, ordinal sum of aggregation operators

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## 1 INTRODUCTION

Aggregation of inputs into a single output is a central problem of several modern intelligent systems. There is a need for new effective aggregation methods. Before describing some constructing methods, we recall the definition of an aggregation operator.

**Definition 1.** An aggregation operator $A\colon \bigcup_{n\in N}[0,1]^n \to [0,1]$ is a mapping fulfilling the following conditions:

(i) $A(x) = x$ for each $x \in [0,1]$ (identity)

(ii) $A(x_1,\dots,x_n) \le A(y_1,\dots,y_n)$ (monotonicity) whenever $x_i \le y_i$ for each $i = 1,\dots,n$, $n \in N$

(iii) $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$ (boundary conditions)

One of the often required properties is the idempotency of an aggregation operator.

**Definition 2.** An aggregation operator $A$ is called idempotent if

$$A(x,\dots,x) = x\,,\ \forall x \in [0,1]\,.$$

## 2 TRANSFORMATION

There are several ways how to construct new aggregation operators. One of them is a method by means of transformation. If we have an aggregation operator $A$ and some transformation $\varphi\colon [0,1] \to [0,1]$, which is bijection, we can define a new aggregation operator $A_\varphi$ as follows: $A_\varphi = \varphi^{-1} \circ A \circ \varphi$. By the transformation we change the scale of inputs. When we apply this method on the arithmetic mean $M$ (on extended real line, with convention $-\infty + \infty = -\infty$) we obtain an important class of aggregation operators called quasi-arithmetic means.

**Definition 3.** Let $f\colon [0,1] \to [-\infty,\infty]$ be a continuous strictly monotone mapping. A quasi-arithmetic mean is an operator $M_f\colon \cup_{n\in N}\,[0,1]^n \to [0,1]$ defined by $M_f(x_1,\dots,x_n) = f^{-1}(M(f(x_1),\dots,f(x_n)))$, where $f^{-1}$ is the inverse function of $f$. The function $f$ is called an additive generator of the quasi-arithmetic mean $M_f$.

**Example 1.**

(i) $M_x = M$ (arithmetic mean)

(ii) $M_{\log x} = G$ (geometric mean)

(iii) $M_{\frac{1}{x}} = H$ (harmonic mean)

(iv) $M_{x^2} = Q$ (quadratic mean)

(v) $M_{x^p} = M_p\,,\ p \in (-\infty,0) \cup (0,\infty)$ (power-root operator)

It is easy to see that $M_p(x_1,\dots,x_n) = \left(\frac{1}{n}\sum\limits_{i=1}^{n} x^p\right)^{\frac{1}{p}}$

Directly from the definition it follows that $M_{af+b} = M_f$ for each $a,b \in R$, $a \ne 0$ (up to the case when $\operatorname{Ran} f = [-\infty,\infty]$; then $a > 0$ should be required). For quasi-arithmetic means generated by powers $f^\lambda$, $\lambda \in (0,\infty)$, of a given generator $f$ with $\operatorname{Ran} f \subseteq [0,\infty]$, the limit properties were studied. Considering the situation when $\lambda$ approaches to infinity, the limit operators are maximum or minimum and depend only on the monotonicity of the generator $f$.

$$\lim_{\lambda\to\infty} M_{(f^+)^\lambda} = \mathrm{Max}, \quad \lim_{\lambda\to\infty} M_{(f^-)^\lambda} = \mathrm{Min}\,.$$

The function $f^+$ ($f^-$) is an increasing (decreasing) generator of $M_{f^+}$ ($M_{f^-}$). When $\lambda$ approaches zero from the right, the limit operators are $f$-transformations of the geometric mean, depending on the generator $f$.

$$\lim_{\lambda\to 0^+} M_{(f^+)^\lambda} = G_{f^+}, \quad \lim_{\lambda\to 0^+} M_{(f^-)^\lambda} = G_{f^-}.$$

They are called quasi-geometric means, see [9].

Also for quasi-arithmetic means generated by functions $f_\alpha$, $\alpha \in (0,\infty)$, $f_\alpha(x) = f(x^\alpha)$, the limit operators are known. Namely, for $\alpha$ approaching zero from the right the limit operator (if it exists) is the geometric

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mean $G$ and for $\alpha$ approaching infinity the limit operator is maximum or minimum, depending on the type of monotonicity of $f$.

A special type of this transformation method is dualization. The transformation in this case is the function $\varphi\colon [0,1] \to [0,1]$, $\varphi(x) = 1 - x$. The dual operator $DA$ to a given aggregation operator $A$ is given by

$$DA(x_1, \dots, x_n) = 1 - A(1 - x_1, \dots, 1 - x_n)\,.$$

A special case is duality between $t$-norms and $t$-conorms. For the definition of $t$-norms and $t$-conorms and their properties we refer to [8]. An aggregation operator $A$ is called self-dual if $DA = A$. Self-dual aggregation operators were called symmetric sums in [13]. The construction of a self-dual aggregation operator is given by

$$A(x_1, \dots, x_n) = \frac{g(x_1, \dots, x_n)}{g(x_1, \dots, x_n) + g(1 - x_1, \dots, 1 - x_n)}\,,$$

where the function $g$ is non-decreasing with $g(0, \dots, 0) = 0$, $g(1, \dots, 1) > 0$ and with convention $\frac{0}{0} = \frac{1}{2}$. As a function $g$ it can be also used an arbitrary $t$-norm or $t$-conorm.

**Example 2.**

(i) If $g(x_1, \dots, x_n) = \sum\limits_{i=1}^{n} x_i$, then

$$A(x_1, \dots, x_n) = M(x_1, \dots, x_n)\,.$$

(ii) For $g(x_1, \dots, x_n) = \prod\limits_{i=1}^{n} x_i$ we get

$$A(x_1, \dots, x_n) = \begin{cases} \frac{1}{2} & \{0,1\} \subseteq \{x_1, \dots, x_n\}, \\ \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} (1 - x_i)} & \text{otherwise.} \end{cases}$$

(iii) Let $g(x,y) = T_L(x,y) = \max(x + y - 1, 0)$, where $T_L$ is the Lukasiewicz $t$-norm, see [8]. Then

$$A(x,y) = \begin{cases} 0 & \text{if } x + y < 1, \\ 1 & \text{if } x + y > 1, \\ \frac{1}{2} & \text{if } x + y = 1. \end{cases}$$

(iv) Put $g(x,y) = S_L(x,y) = \min(x + y, 1)$. $S_L$ is $t$-conorm called bounded sum, which is dual to the $t$-norm $T_L$. Then $A(x,y) = \begin{cases} \frac{x+y}{x+y+1} & \text{if } x + y \le 1, \\ \frac{1}{3-x-y} & \text{if } x + y > 1\,. \end{cases}$

There are also some open problems in this domain. For instance what kinds of aggregation operators are invariant with respect to all increasing (decreasing) transformations. For example, operators maximum and minimum are invariant with respect to all increasing transformations. Or, what is the class of transformations for which a given aggregation operator is invariant. It can be shown that the product (on [0,1]) is invariant with respect to the class of transformations $P = \{\varphi;\, \varphi(x) = x^r, r \in R \setminus \{0\}\}$ and the sum (on $R$) is invariant with respect to the class of transformations $S = \{g;\, g(x) = rx, r \in R \setminus \{0\}\}$.

## 3 COMPOSED AGGREGATION

Take some aggregation operators $B, A_1, \dots, A_m$ (not necessarily different), where $B$ is an idempotent aggregation operator. We can define a new aggregation operator $C$ by $C = B(A_1, \dots, A_m)$, given by

$$C(x_1, \dots, x_n) = B(A_1(x_1, \dots, x_n), \dots, A_m(x_1, \dots, x_n))\,.$$

We speak about two-step aggregation if all applied operators are from the same class. Some classes of aggregation operators are closed under this construction, for example weighted means. An extension on $k$-step aggregation is possible by induction.

In the next example we will work with a special type of aggregation operators, called weighted ordered averages (OWA operators [15]), and we will show that OWA operators (Choquet integrals) are not closed under two-step aggregation.

**Definition 4.** A mapping $F\colon [0,1]^n \to [0,1]$ is called an ordered weighted average associated with a weighting vector $w = (w_1, \dots, w_n)$ such that

(i) $w_i \in [0,1]$,

(ii) $\sum\limits_{i=1}^{n} w_i = 1$, $n \in N$

if $F(a_1, \dots, a_n) = \sum\limits_{i=1}^{n} w_i b_i$, where $(b_1, \dots, b_n)$ is a non-increasing permutation of input arguments $(a_1, \dots, a_n)$.

Some properties of OWA operators are discussed in [15]. Special types of OWA operators are, e.g., minimum, maximum, order statistics, but also the arithmetic mean.

**Example 3, Two-step Choquet integral.** Any OWA operator with weights $w_1, \dots, w_n$ is the Choquet integral with respect to the fuzzy measure $\mu$ on $\{1, \dots, n\}$ defined by

$$\mu(Y) = \sum_{j=0}^{i-1} w_{n-j}, \quad \forall\, Y \text{ such that } |Y| = i\,,$$

where $|Y|$ denotes the cardinality of $Y$.

Consider the operators $A_1, A_2, B$ which are OWA operators and therefore Choquet integrals:

$$A_1(x,y,z) = \frac{\max(x,y,z) + \min(x,y,z)}{2}\,,$$

$$A_2(x,y,z) = \frac{x + y + z}{3}\,,$$

$$B(u,v) = \frac{\max(u,v) + 2\min(u,v)}{3}\,.$$

Let $x \le y \le z$. Then $A_1(x,y,z) = u = \frac{z+x}{2}$, $A_2(x,y,z) = v = \frac{x+y+z}{3} = \frac{2u+y}{3}$. Consequently,

$$C(x,y,z) = \begin{cases} \frac{4x+y+4z}{9} & \text{if } y \ge \frac{x+z}{2}, \\ \frac{7x+y+7z}{18} & \text{if } y < \frac{x+z}{2}\,. \end{cases}$$

Now, evidently $C$ is not an OWA operator. Moreover, it is neither a Choquet integral, because for a given order of input arguments ($x \le y \le z$ in the discussed case) we have two possible different output formulae, which contradicts the definition of the Choquet integral [1].

**Example 4.** Aggregation operators in the following examples are based on a $t$-norm $T$ and a $t$-conorm $S$ and their role is to compensate some defects of the $t$-norm and $t$-conorm aggregation.

(i) $A_1 = T$, $A_2 = S$, $B(x,y) = x^{1-\gamma}y^{\gamma}$ (weighted geometric mean)
$C(x_1,\ldots,x_n) = (T(x_1,\ldots,x_n))^{1-\gamma}(S(x_1,\ldots,x_n))^{\gamma}$
This operator is customary denoted by $E_{\gamma,T,S}(x_1,\ldots,x_n)$ and called the exponential convex compensatory operator [13]. In the case when $T = T_P$ (product), $S = S_P$ (probabilistic sum) we obtain so called gamma-operators, see [11, 15].

(ii) $A_1 = T$, $A_2 = S$, $B(x,y) = (1-\gamma)x + \gamma y$ (weighted mean). Now we obtained so called linear convex compensatory operator [13] $L_{\gamma,T,S}(x_1,\ldots,x_n)$
$$= (1-\gamma)T(x_1,\ldots,x_n) + \gamma S(x_1,\ldots,x_n).$$

## 4 CARTESIAN PRODUCT BASED METHOD

This method is based on two aggregation operators $A$ and $B$, and the order $k$. For $k = 2$ we define a new aggregation operator as follows:

$$C_2(x_1,\ldots,x_n)$$
$$= A\big(B(x_1,x_1)\ldots,B(x_1,x_n),B(x_2,x_1),\ldots,B(x_n,x_n)\big).$$

For $k > 2$ we aggregate by means of $A$ all outputs of $B$ applied to elements of $\{x_1,\ldots,x_n\}^k$ formally distinguishing all $x_i$'s. It is convenient to require the idempotency of the operator $A$ because of the definition of an aggregation operator.

**Example 5.** Examples below are again considered for $k = 2$. For a general $k$ it is enough to replace power (root) 2 by power (root) $k$.

(i) Let $A = M$, $B = \mathrm{Min}$. Then
$$C_2(x_1,\ldots,x_n) = \sum_{i,j=1}^{n} \min(x_i,x_j)\frac{1}{n^2} = \sum_{i=1}^{n} \frac{(n-i+1)^2-(n-i)^2}{n^2} y_i,$$
where $(y_1,\ldots,y_n)$ is a non-decreasing permutation of $(x_1,\ldots,x_n)$. The result is an OWA operator.

(ii) Put $A(x_1,\ldots,x_n) = \dfrac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n}(1-x_i)}$ (with convention $\frac{0}{0} = \frac{1}{2}$), $B = G$.
Then the resulting operator is defined by
$$C_2(x_1,\ldots,x_n) = A(\sqrt{x_1x_1},\ldots,\sqrt{x_1x_n},$$
$$\sqrt{x_2x_1},\ldots,\sqrt{x_nx_n}) = \frac{\left(\prod_{i=1}^{n} x_i\right)^n}{\left(\prod_{i=1}^{n} x_i\right)^n + \prod_{i,j=1}^{n}(1-\sqrt{x_ix_j})}.$$

(iii) For $A = M$, $B = G$ we obtain the operator
$$C_2(x_1,\ldots,x_n) = \sum_{i,j=1}^{n} \frac{\sqrt{x_ix_j}}{n^2} = \left(\sum_{i=1}^{n} \frac{\sqrt{x_i}}{n}\right)^2 = M_p(x_1,\ldots,x_n),$$
which is the power-root operator with parameter $p = \frac{1}{2}$.

## 5 ORDINAL SUMS OF AGGREGATION OPERATORS

Ordinal sums of aggregation operators are extensions of given aggregation operators acting on inputs from some given intervals to an aggregation operator acting on any inputs from unit interval $[0,1]$, see [10] and [11]. We can speak about two boundary extensions, so called lower and upper extensions.

- **Lower extension** is a minimal aggregation operator on $[0,1]$ preserving $A$ (on $[a,b] \subseteq [0,1]$) defined as follows:

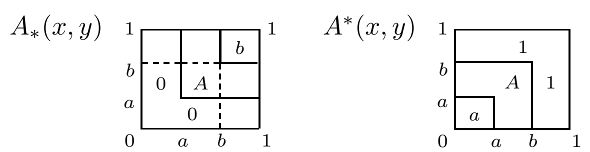
$$A_* \sim (\langle a,b,A\rangle)_*\colon \bigcup_{n\in N}[0,1]^n \to [0,1],\ A_*(x_1,\ldots,x_n) =$$
$$= \begin{cases} 1 & \text{if } \min(x_1,\ldots,x_n) = 1, \\ b & \text{if } b \le \min(x_1,\ldots,x_n) < 1, \\ A(\bar{x}_1,\ldots,\bar{x}_n) & \text{if } a \le \min(x_1,\ldots,x_n) < b, \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{x} = \min(x,b)$.

- **Upper extension** is a maximal aggregation operator on $[0,1]$ preserving $A$ (on $[a,b]$) defined as follows:

$$A^* \sim (\langle a,b,A\rangle)^*\colon \bigcup_{n\in N}[0,1]^n \to [0,1],\ A^*(x_1,\ldots,x_n) =$$
$$= \begin{cases} 0 & \text{if } \max(x_1,\ldots,x_n) = 0, \\ a & \text{if } 0 < \max(x_1,\ldots,x_n) \le a, \\ A(\tilde{x}_1,\ldots,\tilde{x}_n) & \text{if } a < \max(x_1,\ldots,x_n) \le b, \\ 1 & \text{otherwise,} \end{cases}$$

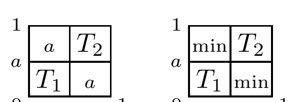
where $\tilde{x} = \max(x,a)$.




We can easily extend this type of construction to an arbitrary (countable) system of aggregation operators and corresponding system of pairwise non-overlapping subintervals of the unit interval $[0,1]$. For more details see [11].

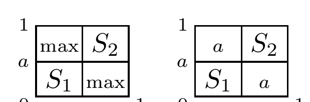
**Example 6.**

(i) $A_1 = T_1$, $A_2 = T_2$



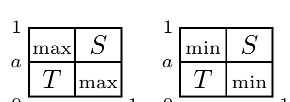

upper ext. (not a $t$-norm) lower ext. (a $t$-norm)

(ii) $A_1 = S_1$, $A_2 = S_2$



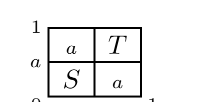

upper ext. (not a $t$-norm) lower ext. (not a $t$-norm)

(iii) $A_1 = T$, $A_2 = S$




upper ext. lower ext.

(iv) $A_1 = S_1$, $A_2 = S_2$

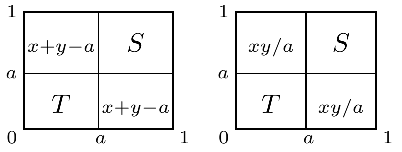



lower ext.=upper ext.

Note that operators obtained in (iii) and (iv) are called uninorms and nullnorms, respectively. For more information see [6, 7] and [3], respectively.

**Example 7.** Lower (upper) extensions of continuous aggregation operators need not be continuous, in general. We present some continuous aggregation operators as ordinal sums (note that $T$ and $S$ are supposed to be continuous).

(i) $A_1 = T$, $A_2 = S$




## 6 SOME OTHER METHODS

From other construction methods we recall the symmetrisation methods only. For an arbitrary (in general non-symmetric) aggregation operator $A$ we can always construct a new operator $A^s$ forcing the symmetry as follows:

$$A^s(x_1, \dots, x_n) = A(y_1, \dots, y_n)\,,$$

where $(y_1, \dots, y_n)$ is a non-increasing permutation of $(x_1, \dots, x_n)$. Applying this method on a weighted mean $W$ we obtain the corresponding symmetric operator $W^s$ (OWA-operator). Similarly, from a weighted geometric mean we obtain an ordered weighted geometric mean (OWG), see [5].

Obviously, if we use a non-decreasing permutation $(z_1, \dots, z_n)$ of $(x_1, \dots, x_n)$ and define

$$A_s(x_1, \dots, x_n) = A(z_1, \dots, z_n)\,,$$

again we obtain a symmetric aggregation operator $A_s$. Note that OWA operators $W^s$ and $W_s$ differ only in the opposite order of weights.

Finally, we can force the symmetry for a given aggregation operator $A$ as follows: for given input vector $(x_1, \dots, x_n)$, take all possible permutations $(x_{\alpha(1)}, \dots, x_{\alpha(n)})$. For a preselected symmetric idempotent aggregation operator B we define

$$A^B(x_1, \dots, x_n) = B(u_1, \dots, u_{n!})\,,$$

where $u_i$, for $i = 1, \dots, n!$, are the outputs of $A(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ obtained for different permutations $\alpha$ of $(1, \dots, n)$, see also Section 4. Evidently, if $A$ is a symmetric aggregation operator, for any $B$ (symmetric and idempotent) $A^B = A$.

**Example 8.**

(i) To illustrate this last method, let $A = W$ be a weighted mean with weights $w = (w_1, \dots, w_n)$ and $B = min$. Then $W^{min}$ is the OWA operator with weights $\bar{w} = (\bar{w}_1, \dots, \bar{w}_n)$, where $\bar{w}$ is a non-increasing permutation of $w$. Similarly, $W^{\max}$ is an OWA operator corresponding to a non-decreasing permutation of $w$.

(ii) Next, $W^M = M$.

(iii) Finally, let $A(x, y) = \sqrt[3]{xy^2}$ and $B(x, y) = \frac{x+y}{2}$. Then $A^B = \frac{\sqrt[3]{xy}(\sqrt[3]{x} + \sqrt[3]{y})}{2}$.

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