Journal of ELECTRICAL ENGINEERING, VOL. 51, NO. 12/s, 2000, 33–35

EXTENSIONALITY AND CONTINUITY OF FUZZY RELATIONS

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In this paper, we investigate the connection between the extensionality of fuzzy relations and its continuity. Some of the conclusions follow from the investigation of similarity relations and pseudo-metrics, precisely, from the investigation of their interrelations. We will see that in a very special case of fuzzy relations we do not need the continuity for obtaining extensionality.

K e y w o r d s: fuzzy relations, extensionality, continuity
2000 *Mathematics Subject Classification*: 03B50, 03B52

1 GL–MONOID, SIMILARITY AND EXTENSIONALITY

In the following text we will concentrate on a concrete structure in order to obtain the general formal framework of our investigations. For this purpose we choose GL-monoid.

Definition 1. $(L, \leq, t)$ is a GL-monoid if
1. $(L, \leq)$ is a complete lattice,
2. $(L, \leq, t, 1, 0)$ is a commutative monoid with unit 1 and zero element 0,
3. t is isotone,
4. $(L, \leq, t)$ is integral, divisible and residuated monoid,
5. the infinite distributive law holds.

Assume, $(L, \leq, t)$ be a GL-monoid. In the theory of fuzzy sets L is used to be a set of truth values. Normally, it is supposed that L is the unit interval $[0, 1]$ with linear ordering. Furthermore, we suppose that it is so, and moreover, that $\vee$, $\wedge$ are lattice operations and t is any continuous operation which fulfils Definition 1. In the case $L = [0, 1]$ it is called a t-norm. The binary operation $\rightarrow_t$ on L w.r.t. a given operation t can be computed by

$$x \rightarrow_t y = \bigvee\{z \mid x\ t\ z \leq y\},$$

and is called a residuation operation. In a straightforward manner we define the biresiduation operation $\leftrightarrow_t$ by

$$x \leftrightarrow_t y = (x \rightarrow_t y) \wedge (y \rightarrow_t x).$$

Now we are able to give other important definitions.

Definition 2. A binary fuzzy relation $E\colon D^2 \to L$ is said to be a similarity relation on a set D w.r.t. the operation t, if it is reflexive, symmetric and t-transitive, i.e. for any $x, y, z \in D$:
1. $E(x, x) = 1$;
2. $E(x, y) = E(y, x)$;
2. $E(x, y)\, t\, E(y, z) \leq E(x, z)$.

Definition 3. A fuzzy relation $P\colon D^n \to L$ is extensional w.r.t. a similarity E on D if for all $x_1, \dots, x_n, y_1, \dots, y_n \in D$,

$$E(x_1, y_1)\, t\ \dots\ t\, E(x_n, y_n) \leq P(x_1, \dots, x_n) \leftrightarrow_t P(y_1, \dots, y_n).$$

Definition 4. A mapping $\mathrm{d}\colon D^2 \to [0, \infty)$ is called a pseudo-metric on D if for all $x, y, z \in D$:
1. $\mathrm{d}(x, x) = 0$;
2. $\mathrm{d}(x, y) = \mathrm{d}(y, x)$;
3. $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$.

We can interpret a similarity relation as the dual concept to a distance. For example, a pseudo-metric d on a set D induces the similarity relation $E(x, y) = 1 - \min(\mathrm{d}(x, y), 1)$ w.r.t. the Łukasiewicz t-norm $\otimes$. Vice versa, a similarity relation E w.r.t. $\otimes$ induces the pseudo-metric $\mathrm{d}(x, y) = 1 - E(x, y)$.

In what follows we will consider a special class of t-norms, namely t-norms with additive generators.

Let $f\colon [0, 1] \to [0, \infty]$ be a continuous strictly decreasing mapping such that $f(1) = 0$. Then f is an additive generator of a t-norm t if

$$x\, t\, y = f^{(-1)}\big(f(x) + f(y)\big) \tag{1}$$

holds for all $x, y \in [0, 1]$, where $f^{(-1)}$ denotes the pseudoinverse of f, i.e.

$$f^{(-1)}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)], \\ 0 & \text{if } y \in (f(0), \infty]. \end{cases} \tag{2}$$

In the sequel we will deal with continuous additive generators only.

For a given t-norm t generated by a continuous additive generator f, the corresponding residuation is determined by

$$x \rightarrow_t y = f^{(-1)}\big(\max(0, f(y) - f(x))\big),$$

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ISSN 1335-3632

and the biresiduation operation corresponding to the above chosen t-norm is

$$x \leftrightarrow_t y = f^{(-1)}\big(|f(x) - f(y)|\big)\,.$$

Generally, a pseudo-metric d on a set D induces the similarity relation $E(x,y) = f^{(-1)}\big(\mathrm{d}(x,y)\big)$ w.r.t. a t-norm t generated by f. Vice versa, a similarity relation E w.r.t. t generated by continuous additive generator f induces the pseudo-metric $\mathrm{d}(x,y) = f\big(E(x,y)\big)$. The proof see in [3].

2 FROM EXTENSIONALITY TO CONTINUITY OF THE FUZZY RELATIONS AND VICE VERSA

In this section, we will present results concerning the extensionality of fuzzy relations and the relationship to the continuity property of their membership functions. The main result is formulated in the following Theorem which is proved for some special subclass of t-norms. This subclass is contingent on their additive generators. Moreover, these generators play a leading role in the construction of pseudo-metrics based on similarity relations, and vice versa.

The structure of truth values will be supposed to be a GL-monoid with the support L. In a special case we take $L = [0,1]$ and the monoidal operation t will be Łukasiewicz t-norm $\otimes$ or some other continuous t-norm.

Theorem 1. *Let f be an additive generator of a t-norm t such that*

1. *for all $x, y \in [0,1]$:*

$$|x - y| \le |f(x) - f(y)|\,. \qquad (3)$$

If a fuzzy relation $P\colon D^n \to [0,1]$ is extensional w.r.t. some similarity E then the membership function of P is Lipschitz continuous w.r.t. the pseudo-metric d, *where $\mathrm{d}(x,y) = f\big(E(x,y)\big)$, and moreover*

$$\big|P(x_1,\dots,x_n) - P(y_1,\dots,y_n)\big| \le \sum_{i=1}^{n} \min\big(f(0), \mathrm{d}(x_i,y_i)\big)\,. \qquad (4)$$

2. *for all $x, y \in [0,1]$:*

$$|x - y| \ge |f(x) - f(y)|\,. \qquad (5)$$

Suppose that a fuzzy relation P fulfils (4) w.r.t. some pseudo-metric d, *then P is extensional w.r.t. the similarity E defined as follows: $E(x,y) = f^{(-1)}\big(\mathrm{d}(x,y)\big)$.*

P r o o f . 1. Assume $n = 2$. Then

$$E(x_1,y_1)\,t\,E(x_2,y_2) \le f^{(-1)}\big(|f(P(x_1,x_2)) - f(P(y_1,y_2))|\big)\,.$$

Since f is decreasing then

$$f(E(x_1,y_1)\,t\,E(x_2,y_2)) \ge \big|f(P(x_1,x_2)) - f(P(y_1,y_2))\big|\,. \qquad (6)$$

Furthermore, using (6), (1), (2) and decreasing property of an additive generator f, we obtain

$$\begin{aligned}\min\big(f(0), f(E(x_1,y_1)) + f(E(x_2,y_2))\big) &= f\big(f^{(-1)}(f(E(x_1,y_1)) + f(E(x_2,y_2)))\big)\\ &= f\big(E(x_1,y_1)\,t\,E(x_2,y_2)\big)\\ &\ge \big|f(P(x_1,x_2)) - f(P(y_1,y_2))\big|\,,\end{aligned}$$

and hence by (3)

$$\begin{aligned}\min\big(f(0), \mathrm{d}(x_1,y_1)\big) + \min\big(f(0), \mathrm{d}(x_2,y_2)\big) &\ge \big|f(P(x_1,x_2)) - f(P(y_1,y_2))\big|\\ &\ge \big|P(x_1,x_2) - P(y_1,y_2)\big|\,,\end{aligned}$$

which gives the condition (4). The Lipschitz continuity w.r.t. the pseudo-metric d follows from the inequality

$$\begin{aligned}\min\big(f(0), \mathrm{d}(x_1,y_1) + \mathrm{d}(x_2,y_2)\big) &\le \min\big(f(0), \mathrm{d}(x_1,y_1)\big) + \min\big(f(0), \mathrm{d}(x_2,y_2)\big)\\ &\le \mathrm{d}(x_1,y_1) + \mathrm{d}(x_2,y_2)\,.\end{aligned}$$

2. Suppose that $n = 2$, and the conditions (4), (5) are valid, then

$$\begin{aligned}f^{(-1)}\big(|f(P(x_1,x_2)) - f(P(y_1,y_2))|\big) &\ge f^{(-1)}\big(\min(f(0), \mathrm{d}(x_1,y_1)) + \min(f(0), \mathrm{d}(x_2,y_2))\big)\\ &= f^{(-1)}\big(f(f^{(-1)}(\mathrm{d}(x_1,y_1))) + f(f^{(-1)}(\mathrm{d}(x_2,y_2)))\big)\\ &= E(x_1,y_1)\,t\,E(x_2,y_2)\,.\end{aligned}$$

This shows that P is extensional w.r.t. the similarity

$$E(x,y) = f^{(-1)}\big(\mathrm{d}(x,y)\big)\,.$$

Using (3), (5) and the definition of additive generators we conclude that $f(x) = C(1-x)$, where $C > 0$. This f generates Łukasiewicz t-norm $\otimes$. We will formulate following Corollary for this t-norm.

Corollary 1. *Let $L = [0,1]$ and monoidal operation t is Łukasiewicz t-norm $\otimes$. A fuzzy relation P on a set D is extensional w.r.t. a similarity E iff it is Lipschitz continuous w.r.t. the pseudo-metric* d, *where $\mathrm{d}(x,y) = 1 - E(x,y)$, and moreover*

$$\big|P(x_1,\dots,x_n) - P(y_1,\dots,y_n)\big| \le \sum_{i=1}^{n} \min\big(\mathrm{d}(x_i,y_i), 1\big)\,. \qquad (7)$$

3 EXTENSIONALITY OF THE SPECIAL RELATIONS

We will consider an unary fuzzy relation $P\colon D \to L$ as a special case. If there exists an element $\hat{x} \in D$: $P(\hat{x}) = 1$ then the choosing of a similarity relation E, such that P is extensional w.r.t. E and $P(x) = E(x, \hat{x})$, is not resolutely caused by a continuity of the membership function P. We choose two different similarities E_G, E_F ([2]) as follows

$$E_G(x,y) = P(x) \leftrightarrow_t P(y)\,, \tag{8}$$

$$E_F(x,y) = \begin{cases} 1 & \text{if } x = y\,, \\ P(x)\,t\,P(y) & \text{otherwise,} \end{cases} \tag{9}$$

such that E_G is the greatest and E_F is the finest similarity relation on D and P is extensional w.r.t. both E_G, E_F. We can prove that $E_G(x,y) \geq E(x,y) \geq E_F(x,y)$ for all E such that $P(x) = E(x, \hat{x})$. Now we can give more precise formulations of the assertions introduced above.

Proposition 1. *Let $P\colon D \to L$ be a fuzzy relation on D, then there exists the greatest similarity relation E_G on D such that P is extensional w.r.t. E_G.*

P r o o f . Consider $E_G(x,y) = P(x) \leftrightarrow_t P(y)$. It is easy to see that E_G is a similarity relation. Indeed, transitivity of E_G follows from

$$(a \leftrightarrow_t b)\,t\,(b \leftrightarrow_t c) \leq a_t c\,,$$

which is valid in any GL-monoid.

The extensionality of P w.r.t. E_G follows from the Definition 3 and the definition of E_G.

Now we will show that E_G is the greatest similarity relation on D such that P is extensional w.r.t. E_G. Let $\overline{E}_G$ be a similarity such that P is extensional w.r.t. $\overline{E}_G$. By Definition 3 we have

$$\overline{E}_G(x,y) \leq P(x) \leftrightarrow_t P(y) \text{ implies } \overline{E}_G(x,y) \leq E_G(x,y)\,,$$

which proves that E_G is the greatest similarity relation making P extensional.

Proposition 2. *Let $P\colon D \to L$ be a fuzzy relation on D and let there exist at least one element $\hat{x} \in D$ such that $P(\hat{x}) = 1$. Then there exists the finest similarity relation E_F on D such that for all $x \in D$*

$$P(x) = E_F(x, \hat{x})\,. \tag{10}$$

P r o o f . Consider E_F defined by (9). It is easy to see that E_F is a similarity relation on D, e.g. transitivity of E_F follows from

$$P(x)\,t\,P(y) \leq P(x) \leftrightarrow_t P(y)\,. \tag{11}$$

Now, we have to show that P is extensional w.r.t. E_F, i.e.

$$E_F(x,y) \leq P(x) \leftrightarrow_t P(y)\,.$$

This easily follows from (11) and the definition of E_F.

Finally, let $\overline{E}_F$ be a similarity relation making P extensional and also satisfying the condition (10). From the inequality

$$\overline{E}_F(x,y) \geq \overline{E}_F(x,\hat{x})\,t\,\overline{E}_F(\hat{x},y) = P(x)\,t\,P(y)$$

we conclude that $E_F(x,y) \leq \overline{E}_F(x,y)$ for all $x, y \in D$.

R e m a r k 1. Consider $t = \wedge$, $L = [0,1]$ and $P\colon D \to [0,1]$ being a fuzzy relation on D such that there exists at least one element $\hat{x} \in D$: $P(\hat{x}) = 1$. Then there is at most one similarity relation E satisfying condition $P(x) = E(x, \hat{x})$.

4 CONCLUSIONS

In this paper, we investigated the extensional property of fuzzy relations. This property seems to be useful in approximate deduction. Extensionality of the concrete fuzzy relations depends on the similarity relations. The criteria for the extensionality of fuzzy relations are established using the fact that similarity relations can be regarded dual to pseudo-metrics. Our main result is Theorem 1. The proof of this Theorem has been given and some important corollaries were exemplified.

References

[1] DE BAETS, B.—MESIAR, R.: Pseudo-Metrics and T-Equivalences, The Journal of Fuzzy Mathematics **5(2)**, Los Angeles 1997.

[2] KLAWONN, F.: Fuzzy Points, Fuzzy Relations and Fuzzy Functions, Discovering the World with Fuzzy Logic, Heidelberg, to appear.

[3] MESIAR, R.—NOVÁK, V.: Operations Fitting Triangular-Norm-Based Biresiduation, Fuzzy Sets and Systems **104** (1999), 77–84.

[4] NOVÁK, V.—PERFILIEVA, I.—MOČKOŘ, J.: Mathematical Principles of Fuzzy Logic, Kluwer, Boston, 1999.

Received 13 June 2000

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