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WILD t–NORMS

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Non-continuous triangular norms with continuous diagonal proposed by G. Krause under the name wild t-norms are recalled and investigated. The set of all discontinuity points of wild t-norms is characterized. Fractal-like structure of the diagonal of a linear wild t-norm is shown by means of its derivative.

K e y w o r d s: Cantor set, Farrey sequence, triangular norm
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1 INTRODUCTION

The term triangular norm was for the first time introduced by Menger [1942]. Originally triangular norms were used for generalization of classical triangular inequality for metric spaces (introduced by Fréchet [1906]) on statistic metric spaces (or on probability metric spaces, as we call them today). Triangular norms (shortly t-norms) are operations on the unit interval with special properties. Originally their axioms (Menger [1942]) were relatively weak. Associativity was not demanded and also boundary conditions were weaker then in axioms, which are used today and which were introduced by Schweizer and Sklar [1960].

Definition 1. Triangular norm is a binary operation $T\colon [0,1]^2 \to [0,1]$, where for all $x,y,z \in [0,1]$ the following four axioms are fulfilled:

(1) $T(x,y) = T(y,x)$ (commutativity,)

(2) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),

(3) $T(x,y) \le T(x,z)$ if $y \le z$ (monotonicity),

(4) $T(x,1) = x$ (boundary condition).

Observe that couple $([0,1],T)$ is a special Abelian semigroup with neutral element 1 and anihilator 0.

Example 1. Following are four basic t-norms T_M, T_P, T_L, T_D:

$$T_M(x,y) = \min(x,y) \qquad \text{(minimum)},$$
$$T_P(x,y) = x\cdot y \qquad \text{(product)},$$
$$T_L(x,y) = \max(0,x+y-1) \qquad \text{(Lukasiewicz } t\text{-norm)},$$
$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2 \\ \min(x,y) & \text{otherwise} \end{cases} \qquad \text{(drastic product)}.$$

These four t-norms are important for several reasons. For every t-norm T we have: $T_D \le T \le T_M$. Every continuous t-norm can be constructed from T_M, T_L, T_P by using some suitable transformations and the so-called ordinal sums, see also [2, 6].

Several algebraic properties of t-norms can be derived from their diagonal function ($T(x,x)\colon [0,1] \to [0,1]$), such as Archimedean property, nilpotency, existence of zero divisors, existence of idempotent elements etc. This is the reason why the diagonal function of t-norms is one of important domains of investigation. Following are the diagonal functions of our four basic t-norms:

$$T_M(x,x) = x\,,$$
$$T_P(x,x) = x^2\,,$$
$$T_L(x,x) = \max(0,2x-1)\,,$$
$$T_D(x,x) = \begin{cases} 0 & \text{if } x \in [0,1[\,, \\ \min(x,x) & \text{otherwise.} \end{cases}$$

It is evident that every continuous t-norm has a continuous diagonal but the converse question (mentioned in [6]), whether a t-norm must be continuous when it has a continuous diagonal, was for years an open problem. Counterexample to this problem was found by Gerianne Krause. Krause's construction is still not published and it is known only in rough e-mail form. The aim of this work is a clear description of this construction and investigation of properties of Krause's t-norms. Construction of Krause's t-norms is based on the notions of the Cantor set and the Farrey sequence, which we will now briefly recall.

2 CANTOR SET

Cantor set is a set derived from the unit interval, from which open intervals (so-called middles) are successively deleted:

1. step: $(1/3, 2/3)$, 2. step: $(1/9, 2/9)$, $(7/9, 8/9)$ ... In the n-th step we are deleting exactly 2^{n-1} intervals.

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Each of these deleted intervals can be represented by its left-end point (e.g. $(7/9, 8/9) \to 7/9$). The set of these points together with points $0, 1$ will be denoted by S. So $S = \{0, 1, 1/3, 1/9, 7/9, 1/27, 7/27, 19/27, 25/27, \dots\}$. We will also denote the set of all right-end points of deleted intervals by U. Points from Cantor set which are neither from S nor from U will be called pure Cantor points and the set of all these points will be denoted by C. Every point from $S \setminus \{0, 1\}$ can be represented by a finite sequence of 0's and 2's created by means of triadic expansion in the following way:

$$\begin{aligned} 1/3 = (1)_{1/3} &\quad \simeq \quad \emptyset\,, \\ 1/9 = (0,1)_{1/3} &\quad \simeq \quad (0)\,, \\ 7/9 = (2,1)_{1/3} &\quad \simeq \quad (2)\,, \\ 1/27 = (0,0,1)_{1/3} &\quad \simeq \quad (0,0)\,, \end{aligned}$$

Ending number in triadic expansion of every point from $S \setminus \{0, 1\}$ is 1, so it is not important, and in our representation we will not use it. In this way we can represent each point from $S \setminus \{0, 1\}$ and we will denote the set of such representations by P. We also know that the interval which is represented by current point has a length $3^{-(n+1)}$, where n is the dimension of current 0–2-vector.

We can use triadic expansion also for representation of points from C. These points have infinite triadic expansions, so their representatives will be exactly their triadic expansions, i.e., sequences which contain infinitely many of 0's and infinitely many of 2's (e.g. $1/4 = (0, 2, 0, 2, \dots)_{1/3} \simeq (0, 2, 0, 2, \dots)$).

3 FARREY SEQUENCE

Farrey sequence is also created inductively. At the beginning we have two "fractions": $1/0, 0/1$. In the first step we put between these two fractions a new one: $1/1$. In the second step we put $2/1$ between $1/0$ and $1/1$ and $1/2$ between $1/1$ and $0/1$, and so on. In the n-th step we add to our sequence 2^{n-1} new fractions in such a way, that between every two old neighbours a/b, c/d (in the increasing order) we put new fraction $(a + c)/(b + d)$. So if we denote by F_n the Farrey sequence in the n-th step, we have:

$$\begin{aligned} &F_1 = \{1/0, 1/1, 0/1\}\,, \\ &F_2 = \{1/0, 2/1, 1/1, 1/2, 0/1\}\,, \\ &F_3 = \{1/0, 3/1, 2/1, 3/2, 1/1, 2/3, 1/2, 1/3, 0/1\}\,, \\ &\quad\vdots \end{aligned}$$

For every two neighbours a/b, c/d of the Farrey sequence F_n for each n, $a \cdot d = 1 + c \cdot b$. The whole Farrey sequence is

$$F_\infty = \bigcup_{n=1}^{\infty} F_n\,.$$

We should note that F_∞ is just equal to the set of all rational numbers from $[0, \infty]$ in their basic form.

In the n-the step of the construction of the set S we have added 2^{n-1} new points (corresponding to construction of the Cantor set). Similarly, 2^{n-1} new fractions have been added to the Farrey sequence in the n-th step. So we can map the points from S to the fractions from Farrey sequence. We will do it in the following way:

First we define boundary conditions: $f(0) = 1/0 = \infty$, $f(1) = 0/1 = 0$. Then we define

in the 1$^{\text{st}}$ step: $f(1/3) = 1/1 = 1$,

in the 2$^{\text{nd}}$ step: $f(1/9) = 2/1 = 2$, $f(7/9) = 1/2$,

e.t.c.

A new point from S is mapped to a new fraction from the Farrey sequence preserving the relevant orders of creation. Now we have a one-to-one mapping between $S \setminus \{0, 1\}$ and P and also a one-to-one mapping between S and the fractions from the Farrey sequence. Hence we have also a one-to-one mapping between P and the fractions from the Farrey sequence and thus we can define a function (we will denote it also by f)

$f \colon P \to F_\infty \setminus \{1/0, 0/1\}$ as we have defined it before: $f(1/3) = f(\emptyset) = 1$, $f(1/9) = f((0)) = 2, \dots$

After investigation of f, we have found these properties:

1) $x, y \in P$, $x = (0, x_1, \dots, x_n)$, $y = (x_1, \dots, x_n) \Rightarrow f(x) = 1 + f(y)$,
2) $x, y \in P$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, where $y_i = 2 - x_i$ for $i = 1, \dots, n \Rightarrow f(x) = 1/f(y)$.

Using these properties finitely many times $f(x)$ can be computed for any $x \in P$. In the points from deleted intervals and in the right-end points of deleted intervals we will determine the value of f as follows:

$f(x) = f(m/3^n)$, $x \in [m/3^n, (m + 1)/3^n]$, where $(m/3^n, (m + 1)/3^n)$ is a deleted interval. To define f on the whole interval $[0, 1]$ it remains to determine values of f in pure Cantor points.

Each point $c \in C$ can be expressed as a limit of points from S, so we put

$$f(c) = \lim_{\substack{x \to c^+ \\ x \in S}} f(x)\,, \quad c \in C\,.$$

This definition is equivalent to:

$$f(c) = \inf\{f(s) \in [0, \infty] \mid s \in S, s \le c\}\,.$$

Then, of course, the properties (1), (2) of function f are valid also for $f(c)$. We can also define $f \colon [0, 1] \to [0, \infty]$ by another equivalent expression:

$$f(x) = \inf\{f(s) \mid s \in S, s \le x\}\,.$$

From the construction it is evident that $f \colon S \to Q \cap [0, \infty]$ and $f \colon C \cup S \to [0, \infty]$ are decreasing bijections and $f \colon [0, 1] \to [0, \infty]$ is a decreasing surjection. This means that f is continuous (where the derivative $f' = 0$ in every point, where it exists, i.e. on $[0, 1] \setminus (C \cup S \cup U)$).

Example 2.
$c = 1/4 = (0,2,0,2,\dots) \Rightarrow f(c) = 1 + 1/f(c) \Rightarrow f^2(c) = f(c) + 1 \Rightarrow f^2(c) - f(c) - 1 = 0$. As far as $f(c)$ is nonnegative $f(c) = (1+\sqrt{5})/2$. Point $1/4$ is interesting also for function values of its successive approximations: $1/4 = (0,2,0,2,...)$
$f(\emptyset) = 1,\ f((0)) = 2,\ f((0,2)) = 3/2,\ f((0,2,0)) = 5/3,\ f((0,2,0,2)) = 8/5,\ \dots,\ f((0,2,0,2,\dots)) =$
$f((x_1,\dots,x_n)) = a_{n+1}/a_n$, where numbers a_n are members of Fibonacci sequence ($a_0 = 1,\ a_1 = 1,\ a_n = a_{n-1} + a_{n-2}$). So we can determine $f(1/4)$ also using the explicit form of Fibonacci numbers:

$$a_n = \frac{1}{\sqrt{5}\cdot 2^{n+1}} \cdot [(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}], \text{ and hence}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{2}\cdot\frac{[(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}]}{[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}]} \xrightarrow[n\to\infty]{} \frac{1+\sqrt{5}}{2}.$$

4 WILD t–NORMS

Using the above defined continuous decreasing function $f\colon [0,1] \to [0,\infty]$ we can define a t-norm T.

Definition 2. As for every t-norm we define first

$$T(x,1) = T(1,x) = x,\ T(x,0) = T(0,x) = 0,\ x \in [0,1].$$

Let $G_q\colon [1/3,2/3] \to [i/3^j,(i+1)/3^j]$, where $i/3^j \in S \setminus \{0,1\}$ and $q = f(i/3^j)$, be any system of increasing bijections. Let $x^* = G_p^{-1}(x),\ x \in [i/3^j,(i+1)/3^j]$, $p = f(i/3^j)$ and $y^* = G_q^{-1}(y),\ y \in [m/3^n,(m+1)/3^n]$, $q = f(m/3^n)$. Then we define:

$$T(x,y) = G_{p+q}(\min(x^*,y^*)). \tag{1}$$

If $\{x,y\} \cap C \neq \emptyset$ we put

$$T(x,y) = \inf\{T(c,d) \mid c \in S,\ d \in S,\ x \le c,\ y \le d\} \tag{2}$$

((1) and (2) are equivalent for $x,y \in C \cup S$)).

We should also remark that if $\{x,y\} \cap (C \cup S) \neq \emptyset$ then $T(x,y) \in (C \cup S)$. $T\colon [0,1]^2 \to [0,1]$ is defined correctly. We show now that T is a t-norm.

1) *Commutativity* is evident from the definition of T.

2) *Monotonicity*: We know that G_q is an increasing bijection for $\forall q \in Q \cap]0,\infty[$ and f is a decreasing function. Let $x \in [i/3^j,(i+1)/3^j],\ y \in [m/3^n,(m+1)/3^n],\ y \le z$. We know, that $f(T(x,y)) = p + q = f(i/3^j) + f(m/3^n)$ and from the definition $f(x) = f(i/3^j),\ f(y) = f(m/3^n)$, hence $f(T(x,y)) = f(x) + f(y)$. So if $f(z) < f(y)$ then $T(x,y) < T(x,z)$.

If $f(y) = f(z)$ then y,z are points from the same deleted interval — $[m/3^n,(m+1)/3^n]$. Consequently $y^* \le z^*$. Let $x \in [i/3^j,(i+1)/3^j],\ p = f(m/3^n)$, $q = f(i/3^j) \Rightarrow T(x,y) = G_{p+q}(\min(x^*,y^*))$ $\le G_{p+q}(\min(x^*,z^*)) = T(x,z)$, because G_{p+q} is an increasing bijection. As far as between every point from $C \cup S \cup U$ and another point from C, there are infinitely many points from S, T is monotone also for x,y, where $\{x,y\} \cap C \neq \emptyset$.

3) *Associativity*: We have to prove associativity only for $x,y,z \in]0,1[$ (in other cases it is evident).

a) Let $x \in [i/3^j,(i+1)/3^j],\ y \in [m/3^n,(m+1)/3^n]$, $z \in [r/3^s,(r+1)/3^s],\ p = f(m/3^n),\ q = f(i/3^j),\ t = f(r/3^s)$. Then $T(T(x,y),z) = G_{A+t}(\min(a^*,z^*))$, where $a = T(x,y) \in [L/3^k,(L+1)/3^k],\ A = f(L/3^k)$. Since x,y are points from deleted intervals, from the definition $T(x,y)$ belongs to a deleted interval, as well. So we have:
$T(T(x,y),z) = G_{p+q+t}(\min(a^*,z^*))$,
$a = G_{p+q}(\min(x^*,y^*)),\ a^* = \min(x^*,y^*)$, and hence
$T(T(x,y),z) = G_{p+q+t}(\min(\min(x^*,y^*),z^*))$
$= T(x,T(y,z))$

b) If $x \in C \Rightarrow T(x,y) = \inf\{T(x',y') \mid x',y' \in S, x \le x', y \le y'\} \in C \cup S$. Then
$T(T(x,y),z) = \inf\{T(a',z') \mid a',z' \in S,\ T(x,y) \le a',\ z \le z'\} = \inf\{T(a',z') \mid a',z' \in S, z \le z', \inf\{T(x',y') \mid x',y' \in S, x \le x', y \le y'\} \le a'\}$.

Because T is monotone, the last expression is equal to: $\inf\{T(a',z') \mid a',z' \in S, z \le z', \inf\{T(x',y') \mid x',y' \in S, x \le x', y \le y'\} \le a'\} = \inf\{T(T(x',y'),z') \mid x',y',z' \in S, x \le x', y \le y', z \le z'\} = \inf\{T(x',T(y',z')) \mid x',y',z' \in S, x \le x', y \le y', z \le z'\}$.

If $\{y,z\} \cap C \neq \emptyset$ the proof is finished.

If $y \in [m/3^n,(m+1)/3^n],\ z \in [r/3^s,(r+1)/3^s]$, $p = f(m/3^n),\ t = f(r/3^s),\ T(y,z) = G_{p+t}(\min(y^*,z^*))$ then $T(x,T(y,z)) = \inf\{T(x',b') \mid b',x' \in S, T(y,z) \le b', x \le x'\}$. We know that $\inf\{b' \ge T(y,z), b' \in S\} = \inf\{T(y',z') \mid y',z' \in S, y \le y', z \le z'\}$. Since T is monotone: $T(x,T(y,z)) = \inf\{T(x',b') \mid b',x' \in S, T(y,z) \le b', x \le x'\} = \inf\{T(x',T(y',z')) \mid x',y',z' \in S, x \le x', y \le y', z \le z'\} = T(T(x,y),z)$.

4) The boundary condition was directly introduced in the definition.

We have shown that the introduced mapping T is a t-norm. Following Gerianne Krause, T will be called wild t-norm. This t-norm is continuous on rectangles $]i/3^j,(i+1)/3^j[\times]m/3^n,(m+1)/3^n[$, which are called by G. Krause devil's terraces.

5 WILD t–NORMS ON DIAGONAL

The t-norm T is continuous on the diagonal. We put $D(x) = T(x,x)$.

1) $x \in]i/3^j,(i+1)/3^j[\Rightarrow x$ is inner point of a deleted interval, $p = f(i/3^i)$. On $I =]i/3^j,(i+1)/3^j[$ the function G_p is continuous (increasing bijection) $\Rightarrow$ also G_p^{-1} is continuous. Because x is from a deleted interval we know that: $2p = q$, where $q = f(y),\ y \in S$. Function G_q is also continuous. As far as also $f\colon C \cup S \to [0,\infty]$ is continuous, the whole t-norm is continuous on $I \times I$ and hence $D(x)$ is continuous on I. $D(x)$ is also continuous

from the right in the left-end point and from the left in right-end point of interval I.

2) The continuity of $D(x)$ in points from C follows directly from the monotonicity of $D(x)$, density of S in $S \cup C$ and continuity of bijection $f\colon C \cup S \to [0,\infty]$, as far as for $c \in C$, $D(c) = f^{-1}(2.f(c))$ (where f^{-1} is the inverse function of $f\colon C \cup S \to [0,\infty]$, which is a decreasing bijection).

3) The continuity from the left of diagonal $D(x)$ in points $s \in S \setminus \{0\}$ follows directly from the monotonicity of $D(x)$ and the continuity of $f\colon S \to Q \cap [0,\infty]$ (the same for the continuity from the right in points $u \in U$).

6 NON–CONTINUITY OF WILD t–NORMS

We know that wild t-norms are non-continuous, more precisely every wild t-norm is non-continuous from the left in $(x,1)$, $(1,x)$ where $x \in [0,1] \setminus (C \cup S)$ and non-continuous from the right in (x,y), (y,x)
for $\forall x \in [0,1] \setminus (C \cup S \cup U), \forall y \in U$.

Let $x \in [i/3^j,(i+1)/3^j]$, $p = f(i/3^j)$. We know that $T(x,1) = x$. Then $T(x,1^-) = \lim_{j\to\infty} T(x,1-2/3^j)$, $T(x,1-2/3^j) = G_{p+1/j}(\min(x^*,1/3)) = i/3^j$. This shows that T is continuous in points $(x,1),(1,x)$, $x \in S$, and non-continuous from left in $(x,1),(1,x)$, $x \in [0,1] \setminus (S \cup C)$. We can easily see that in $x \in C$, $T(x,1^-) = \sup\{s_{i,j} \mid s_{i,j} \in S, s_{i,j} \le x\} = x$ $(s_{i,j} = i/3^j)$.

Concerning the non-continuity from the right: if $x \in]i/3^j,(i+1)/3^j[$ and for $k/3^L \in S$, $T(i/3^j,k/3^L) = m/3^n$, this means that $x = i/3^j + \lambda/3^j, \lambda \in]0,1[\Rightarrow T(x,(k+1)/3^L) = m/3^n + \lambda/3^n, T(x,[(k+1)/3^L]^+) = (m+1)/3^n \Rightarrow T(x,[(k+1)/3^L]^+) \ne T(x,(k+1)/3^L)$.

Further we will suppose that G_p is a linear function for $\forall p \in Q \cap]0,\infty[$. The corresponding T will be called a linear wild t-norm.

Example 3. $T(1/2,1) = 1/2$
$T(1/2,1^-) = \lim_{x\to 1^-} T(1/2,x) = \lim_{n\to\infty} T(1/2,x_n)$, where $x_n = (3^n-2)/3^n$. Then $T(1/2,(3^n-2)/3^n = G_{1+1/n}(\min(1/3,1/2)) = G_{1+1/n}(1/3) = (3^n-2)/3^{n+1})$ $\to_{n\to\infty} 1/3, \quad 1/3 \ne 1/2$.

Example 4. $T(1/2,2/3) = 1/6$.
Denote $p = f((2\cdot 3^{n-1}+1)/3^n)$. Then $T(1/2,2/3^+) = T(1/2,(2.3^{n-1}+1)/3^n) = G_{1+p}(min(1/3,1/2))$
$= (2\cdot 3^{n-1}+1)/3^{n+1} \to_{n\to\infty} 2/9. \quad 2/9 \ne 1/6$.

7 DERIVATIVE OF LINEAR WILD t–NORM ON THE DIAGONAL

Suppose that all bijections G_p are linear. This means: $G_{f(m/3^n)}(x) = k\cdot x+q$ for $\forall m/3^n \in S \Rightarrow m/3^n = k/3+q$, $(m+1)/3^n = 2k/3+q \Rightarrow 1/3^n = k/3 \Rightarrow k = 1/3^{n-1}, q = (m-1)/3^n$. Consequently, $G_{f(m/3^n)}(x) = x/3^{n-1} + (m-1)/3^n$ and $G^{-1}_{f(m/3^n)}(x) = 3^{n-1}x - (m-1)/3$.

For the diagonal function related to the linear wild t-norm T we obtain: $D(x) = G_{2f(m/3^n)}\big(3^{n-1}x - (m-1)/3\big)$, $x \in [m/3^n,(m+1)/3^n]$, $m/3^n \in S$. If $f^{-1}(2f(x)) = i/3^j$ then $D(x) = 3^{n-j}x - (m-i)/3^j$.

We have just shown that the value 3^{n-j} is the derivative (slope) of D on the interval $I =]m/3^n,(m+1)/3^n[$.

Let $x \in S$ and let $x = m/3^n$, $f^{-1}\big(2(f(x)\big) = i/3^j$. Then $D(z) = 3^{n-j}z - (m-i)/3^j$ for any $z \in [m/3^n,(m+1)/3^n]$.

Let $s(x)$ be a mapping which assigns to every point $x \in S$ the value $s(x) = (j-n)$. Then the slope of D on $]m/3^n,(m+1)/3^n[$ is $3^{-s(x)}$. So we have only to find out the value of $s(x)$.

Example 5.

$$
\begin{aligned}
s(\emptyset) = s(1/3) &= 2-1 = 1\\
s((0)) = s(1/9) &= 4-2 = 2\\
s((2)) = s(7/9) &= 1-2 = -1\\
s((0,0)) = s(1/27) &= 6-3 = 3\\
s((0,2)) = s(7/27) &= 3-3 = 0\\
s((2,0)) = s(19/27) &= 4-3 = 1\\
s((2,2)) = s(25/27) &= 3-3 = 0
\end{aligned}
$$

The function s has the following properties:

Theorem 1. *Let* $p = (x_1,\dots,x_n)$, $p' = (y_1,\dots,y_n)$, $y_i = 2 - x_i$, $i = 1,\dots,n$, $x_i, y_i \in \{0,2\}$. *Then*

1) $s((0,p)) = 1+s(p)$, *where* $(0,p)$ *means* $(0,x_1,\dots,x_n)$,
2) $s((2,0,p)) = s(p')$,
3) $s((2,2,p)) = s(p)-1$.

Proof. 1) $2f((0,p)) = 2+2f(p) \Rightarrow f^{-1}(2+2f(p)) = (0,0,q)$, where $q = f^{-1}(2f(p))$. $s((0,p)) = j-n \Rightarrow s(p) = (j-2)-(n+1) = s((0,p))-1 \Rightarrow s((0,p)) = 1+s(p)$.

2) $2f((2,0,p)) = \frac{2}{1+1/(1+f(p))} = \frac{2+2f(p)}{2+f(p)} \Rightarrow$
$f^{-1}(2f((2,0,p))) = f^{-1}(1+\frac{f(p)}{2+f(p)}) = f^{-1}(1+\frac{1}{1+2/f(p)})$ $= (0,2,q)$ where $q = f^{-1}(\frac{f(p)}{2})$, $2f(p') = \frac{2}{f(p)} \Rightarrow$ $f^{-1}(2f(p')) = q'$. So $s((2,0,p)) = j-n \Rightarrow s(p') = (j-2)-(n-2) = j-n = s((2,0,p)) \Rightarrow s((2,0,p)) = s(p')$.

3) $2f((2,2,p)) = \frac{2}{2+1/f(p)} = \frac{2f(p)}{2f(p)+1} \Rightarrow$
$f^{-1}(\frac{2f(p)}{2f(p)+1}) = f^{-1}(\frac{1}{1+1/\big(2f(p)\big)}) = (2,q)$ where
$q = f^{-1}(2f(p))$. So $s((2,2,p)) = j-n \Rightarrow s(p) = (j-1)-(n-2) = j-n+1 \Rightarrow s((2,2,p)) = s(p)-1$.

Now, we are able to prove the next interesting result.

Theorem 2. $\forall y, z \in C \cup S,\ y < z,\ \forall k \in Z\ \exists x \in]y, z[$ *such that* $D'(x) = 3^k$ *(this means that there is an interval on which* D *has derivative* 3^k *).*

Proof. Between every two different points $c, d \in C \cup S$ there are infinitely many points from S, so we will prove the theorem only for two different points from S.

Now let $x \in S,\ x = (x_1, \dots, x_n, 1)_{1/3} \simeq (x_1, \dots, x_n)$ and $y \in S,\ y = (y_1, \dots, y_m, 1)_{1/3} \simeq (y_1, \dots, y_m)$. Let $x < y$ and $n < m$. Then for every point $z \simeq (y_1, \dots, y_m, 0, p)$, where $p = (z_1, \dots, z_k),\ z_i \in \{0, 2\}$, $i = 1, \dots, k$ $(z \in S)$ we have $x < z < y \Rightarrow z \in]x, y[$. Similarly, if $x < y$ and $n \geq m$ then we put $z \simeq (x_1, \dots, x_n, 2, p)$.

For any such z, using properties $(1), (2), (3)$, we are able to reduce the value $s(z)$ to one of the following four cases:

$s((y_1, \dots, y_m, 0, p)) = v + s(q)$, where $v \in Z$ is some fixed integer depending on y and q, is either p or p', or $(2, q)$ is either p or p'.

A similar reduction is valid in the case when $z \simeq (x_1, ..., x_n, 2, p)$.

We want to prove that for any $k \in Z$ there is point $z \in]x, y[$ such that $s(z) = k$.

1) If $k - v > 0$ we put
$q = \underbrace{(0, \dots, 0)}_{k-v-1 \text{ times}}$. Then $s(q) = k - v - 1 + s(\emptyset) = k - v$.

2) If $(k - v) \leq 0$
$q = \underbrace{(2, \dots, 2)}_{(2|k-v|+2)} \Rightarrow s(q) = k - v - 1 + s(\emptyset) = k - v$.

Example 6. $x = (0, 2, 1)_{1/3} \simeq (0, 2),\ y = (0, 2, 2, 1)_{1/3} \simeq (0, 2, 2),\ k = 3$, so $z = (0, 2, 2, 0, p),\ s((0, 2, 2, 0, p) = 1 - 1 + 1 + s(p) = 1 + s(p),\ q = p \Rightarrow v = 1,\ (k - v) = 2 \Rightarrow q = (0),\ s(0, 2, 2, 0, 0) = 1 - 1 + 3 = 3$.

If $k = 0$ $(k - v) = -1$, $q = \underbrace{(2, \dots, 2)}_{2|k-v|+2} \Rightarrow q = (2, 2, 2, 2)$, $s((0, 2, 2, 0, 2, 2, 2, 2)) = 1 - 1 + 1 - 2 + 1 = 0$.

Similarly, if $k = -2$, $q = (2, 2, 2, 2, 2, 2, 2, 2)$, $s((0, 2, 2, 0, 2, 2, 2, 2, 2, 2, 2, 2)) = 1 - 1 + 1 - 1 - 1 - 1 - 1 + 1 = -2$.

For $x = (2, 2, 0, 2, 2, 1)_{1/3} \simeq (2, 2, 0, 2, 2)$, $y = (2, 2, 1)_{1/3} \simeq (2, 2),\ k = 0$: $z = (2, 2, 0, 2, 2, 2, p) \Rightarrow s((2, 2, 0, 2, 2, 2, p)) = -1 + 1 - 1 + s(2, p),\ p = (2, q) \Rightarrow s(z) = -2 + s(q)$, so $(k - v) = 0 + 2 = 2$ and so $q = (0)$. $s(2, 2, 0, 2, 2, 2, 2, 0) = -1 + 1 - 1 - 1 + 1 + 1 = 0$.

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