Journal of ELECTRICAL ENGINEERING, VOL. 51, NO. 12/s, 2000, 41–45

THE SHEAR CORRECTION COEFFICIENT IN THE VISCOELASTIC MINDLIN–TIMOSHENKO THIN PLATE MODEL

Dávid Pancza *

The Mindlin-Timoshenko Model allows us to describe the vertical motion (bending) of a viscoelastic thin plate by an operator equation

$$\mathcal{K}(W'', V) + \langle \mathcal{A}(0)W, V\rangle + \langle \mathcal{A}' * W, V\rangle = \mathcal{F}(V) + \mathcal{G}(V)\,.$$

The bilinear form $\mathcal{A}$ can be written as a sum of two members which are differently dependent on the thickness h of the plate:

$$\mathcal{A} = h\mathcal{A}_1 + h^3\mathcal{A}_3\,.$$

To correct the inexactness of the MT model, a factor k called the shear correction coefficient is introduced into $\mathcal{A}$:

$$\mathcal{A} = kh\mathcal{A}_1 + h^3\mathcal{A}_3\,.$$

The term $kh\mathcal{A}_1$ plays here the role of a penalty term. We shall deal with the properties of the MT model in the special cases of $k \to 0$ and $k \to \infty$.

K e y w o r d s: stress and strain, viscoelastic material, Mindlin-Timoshenko thin plate model, shear correction coefficient, coercivity

2000 *Mathematics Subject Classification*: 35Q72

INTRODUCTION

In this paper we use a mathematical description of the viscoelastic stress-strain relations which can be found in the works of S.Shaw, M.K. Warby and J.R. Whiteman [3], [4].

The basic formulation of the viscoelastic Mindlin-Timoshenko thin plate model and a theorem about the existence and uniqueness of its solution can be found in [5]. Here we recall these results.

The Mindlin-Timoshenko thin plate theory for the values $k = 0$ and $k = \infty$ in the case of an elastic material can be found in [1]. Our aim is to generalize these results to the viscoelastic case.

THE VISCOELASTIC MATERIAL

Consider a thin plate of a constant thickness h. Its points will be represented by rectangular coordinates (x_1, x_2, x_3). We assume that the middle surface of the plate occupies a region Ω of the plane $x_3 = 0$.

Let (u_1, u_2, u_3) denote the displacement vector of the point which, when the plate is in equilibrium, has coordinates (x_1, x_2, x_3). The strain tensor is denoted by $\epsilon_{ij}(u)$ and the stress tensor by $\sigma_{ij}(\epsilon)$. In small displacement theory [2]

$$\epsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \qquad (1)$$

The plate is assumed here to be homogeneous and isotropic. In this case the viscoelastic stress-strain relations are given by a modified Hooke's law

$$\sigma_{ij}(\epsilon) = \lambda(0)[\epsilon_{kk}]\delta_{ij} + 2\mu(0)\epsilon_{ij} + \lambda' * [\epsilon_{kk}]\delta_{ij} + 2\mu' * \epsilon_{ij}\,, \quad (2)$$

where the Lamén coefficients λ and μ are considered being positive, sufficiently smooth, nonincreasing functions dependent on t, and $*$ is the convolution product

$$f * g = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau\,.$$

We are using the Einstein summation convention.

THE MINDLIN–TIMOSHENKO THIN PLATE MODEL

The Mindlin-Timoshenko thin plate model is based on a hypothesis that the linear filaments of the plate, initially perpendicular to the middle surface, remain straight and

* Department of Mathematics, Faculty of Electrical Engineering and Information Technology STU, Ilkovičova 3, 812 19 Bratislava 1, Slovak Republic, È-mail: pancza@kmat.elf.stuba.sk

Supported by the grant 1/5094/98 of the Grant Agency of the Slovak Republic.

ISSN 1335-3632

undergo neither contraction nor extension. Let ϕ_1 and ϕ_2 denote the angles between a filament and the planes $x_1 = 0$ and $x_2 = 0$ respectively. Initially they are both zero. This hypothesis allows us to linearize the dependence of the displacement vector u on x_3:

$$\begin{aligned} u_1(x_1,x_2,x_3) &= w_1(x_1,x_2) + x_3\,\phi_1(x_1,x_2) \\ u_2(x_1,x_2,x_3) &= w_2(x_1,x_2) + x_3\,\phi_2(x_1,x_2) \\ u_3(x_1,x_2,x_3) &= w_3(x_1,x_2)\,. \end{aligned} \tag{3}$$

We shall assume that the plate is subjected to a volume distribution of forces (f_1, f_2, f_3). The motion of the plate is determined by the balance law, according to which the displacement function u must satisfy

$$\rho u_i'' = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i\,. \tag{4}$$

THE VARIATIONAL FORMULATION OF THE PROBLEM

Let us consider a plate which is clamped along a portion $\Gamma_0 \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ and simply supported on $\Gamma_1 \times \left[-\frac{h}{2}, \frac{h}{2}\right]$, where $\Gamma_0 \neq \varnothing$ and $\Gamma_1 = \Gamma \backslash \Gamma_0$.

Multiplying both sides of (4) by a vector of test functions $z = (z_1, z_2, z_3)$ from the space

$$\left\{(v_1 + x_3\psi_1, v_2 + x_3\psi_2, v_3);\ v_1, v_2, v_3, \psi_1, \psi_2 \in H^1_{\Gamma_0}(\Omega)\right\},$$

where

$$H^1_{\Gamma_0}(\Omega) = \left\{v \in H^1(\Omega);\ v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0\right\}, \tag{5}$$

we obtain a variational formulation

$$\iint_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho u_i'' z_i + \sigma_{ij}\epsilon_{ij}(z)\,\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 = \iint_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i z_i\ dx_3\mathrm{d}x_2\mathrm{d}x_1\,. \tag{6}$$

After introducing (1), (2), (3) into (6) and carrying out the integration in x_3 it is possible to uncouple the stretching members w_1, w_2, v_1, v_2 from the bending members $w_3, v_3, \phi_1, \phi_2, \psi_1, \psi_2$. So the equation splits into two independent equations describing separately the energy of stretching and the energy of bending. The bending equation is:

$$\mathcal{K}(W'', V) + \langle \mathcal{A}(0)W, V\rangle + \langle \mathcal{A}' * W, V\rangle = \mathcal{F}(V)\,, \tag{7}$$

where

$$W = (w_3, \phi_1, \phi_2), \quad V = (v_3, \psi_1, \psi_2) \tag{8}$$

are vectors of the unknown and test functions,

$$\mathcal{K}(W,V) = \iint_\Omega \rho h w_3 v_3 + \rho\frac{h^3}{12}(\phi_1\psi_1 + \phi_2\psi_2)\mathrm{d}x \tag{9}$$

is a bilinear form, for $V = W$ describing the kinetic energy in bending of the plate,

$$\mathcal{A} = kh\mathcal{A}_1 + h^3\mathcal{A}_3\,, \tag{10}$$

$$\begin{aligned} \langle \mathcal{A}_3(t)W, V\rangle = \frac{1}{12}\iint_\Omega &(\lambda + 2\mu)(t)\Big(\frac{\partial\phi_1}{\partial x_1}\frac{\partial\psi_1}{\partial x_1} + \frac{\partial\phi_2}{\partial x_2}\frac{\partial\psi_2}{\partial x_2}\Big) \\ &+ \lambda(t)\Big(\frac{\partial\phi_1}{\partial x_1}\frac{\partial\psi_2}{\partial x_2} + \frac{\partial\phi_2}{\partial x_2}\frac{\partial\psi_1}{\partial x_1}\Big) \\ &+ \mu(t)\Big(\frac{\partial\phi_1}{\partial x_2} + \frac{\partial\phi_2}{\partial x_1}\Big)\Big(\frac{\partial\psi_1}{\partial x_2} + \frac{\partial\psi_2}{\partial x_1}\Big)\,\mathrm{d}x\,, \end{aligned} \tag{11}$$

$$\begin{aligned} \langle \mathcal{A}_1(t)W, V\rangle = \mu(t)\iint_\Omega &\Big(\phi_1 + \frac{\partial w_3}{\partial x_1}\Big)\Big(\psi_1 + \frac{\partial v_3}{\partial x_1}\Big) \\ &+ \Big(\phi_2 + \frac{\partial w_3}{\partial x_2}\Big)\Big(\psi_2 + \frac{\partial v_3}{\partial x_2}\Big)\,\mathrm{d}x\,, \end{aligned} \tag{12}$$

is a bilinear form, for $V = W$ describing the strain energy in bending of the plate: $\langle \mathcal{A}(0)W, W\rangle$ describes the energy of the immediate elastic reaction of the plate, $\langle \mathcal{A}' * W, W\rangle$ describes the energy of the previous deformations, decreasing because of the viscous creeping of the plate. Their sum characterizes the viscoelastic properties of the plate. The factor k is introduced to correct the inexactness of the model and is called the shear correction coefficient (see [1] for details).

The operator

$$\mathcal{F}(V) = \iint_\Omega F_3 v_3 + M_1\psi_1 + M_2\psi_2\,\mathrm{d}x\,, \tag{13}$$

where

$$F_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_3\,\mathrm{d}x_3\,, \quad M_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i x_3\,\mathrm{d}x_3\,, \tag{14}$$

describes the work done by the force f.

The equation of stretching can be written using the just defined operators:

$$\mathcal{K}(X'', Z) + h\langle \mathcal{A}_3(0)X + \mathcal{A}_3' * X, Z\rangle = \mathcal{F}(Z)\,, \tag{15}$$

where

$$X = (0, w_1, w_2)\,, \quad Z = (0, v_1, v_2)\,.$$

R e m a r k: T h e K i r c h h o f f m o d e l . The Kirchhoff thin plate model supposes, in addition to the conditions required by the Mindlin-Timoshenko hypothesis, that the linear filaments remain all the time perpendicular to the middle surface. The linearization of the displacement vector is now:

$$u_1(x_1,x_2,x_3) = w_1(x_1,x_2) - x_3 \frac{\partial w_3(x_1,x_2)}{\partial x_1}\,,$$
$$u_2(x_1,x_2,x_3) = w_2(x_1,x_2) - x_3 \frac{\partial w_3(x_1,x_2)}{\partial x_2} \tag{16}$$
$$u_3(x_1,x_2,x_3) = w_3(x_1,x_2)\,.$$

In the same way as before we obtain the weak formulation for bending:

$$\mathcal{K}(U'',Y) + h\langle \mathcal{A}_3(0)U,Y\rangle + h\langle \mathcal{A}_3' * U,Y\rangle = \mathcal{F}(Y), \tag{17}$$

where

$$U = (w_3, \nabla w_3)\,, \quad Y = (v_3, \nabla v_3)$$

and the operators are those defined above.

The stretching equations for the Kirchhoff model are exactly the same as those of the Mindlin-Timoshenko model.

EXISTENCE AND UNIQUENESS OF AN APPROXIMATE WEAK SOLUTION

For every $t \in [0,T]$ we have a weak formulation

$$\mathcal{K}(W'',V) + \langle \mathcal{A}(0)W,V\rangle + \langle \mathcal{A}' * W,V\rangle = \mathcal{F}(V) \quad \forall V \in \mathcal{V}_3\,, \tag{18}$$

with the initial conditions

$$W(0) = W^0\,,\ W'(0) = W^1\,, \quad W^0, W^1 \in \mathcal{V}_3\,, \tag{19}$$

where

$$\mathcal{V}_n = \left(H^1_{\Gamma_0}(\Omega)\right)^n\,. \tag{20}$$

$\mathcal{V}_n$ is a closed subspace of the Hilbert space $\left(H^1(\Omega)\right)^n$ with a scalar product

$$(U,V)_{(H^1(\Omega))^n} = \iint_\Omega u_i v_i + \nabla u_i \nabla v_i \,\mathrm{d}x$$

and norm

$$\|U\|_{(H^1(\Omega))^n} = \left((U,U)_{(H^1(\Omega))^n}\right)^{1/2}\,.$$

In order to analyze the initial value problem (18), (19) we add a penalty member

$$\mathcal{J}_\theta(W,V) = \iint_\Omega \theta(\nabla w_3 \nabla v_3 + \nabla\phi_1\nabla\psi_1 + \nabla\phi_2\nabla\psi_2)\,\mathrm{d}x \tag{21}$$

to the bilinear form $\mathcal{K}$ and denote the new form by $\mathcal{K}_\theta$:

$$\mathcal{K}_\theta(W,V) = \mathcal{K}(W,V) + J_\theta(W,V)\,. \tag{22}$$

We get a new system

$$\mathcal{K}_\theta(W'',V) + \langle \mathcal{A}(0)W,V\rangle + \langle \mathcal{A}' * W,V\rangle = \mathcal{F}(V) \quad \forall V \in \mathcal{V}_3\,, \tag{23}$$

with unchanged initial conditions.

The parameter θ and the shear correction coefficient k will be considered now as penalty terms and the solution of the system (23), (19), which corresponds to given $\theta > 0$, $k \geq 0$, shall be denoted by $W_{k\theta}$. About its existence and uniqueness it holds (see [5]):

Theorem 1. *Let $\lambda, \mu \in C^1([0,T],\mathbb{R})$ and $f_i \in C([0,T],L_2(\Omega))$, $i = 1,2,3$. Then there exists a unique solution $W_{k,\theta} \in C^2([0,T],\mathcal{V}_3)$ of the initial value problem* (23), (19).

THE MINDLIN–TIMOSHENKO MODEL FOR $k \to 0$

Theorem 1 holds for $k = 0$, too. But in the proof (see [5]) of the existence and uniqueness of the solution of (18), (19) coercivity and boundedness of the operator $\mathcal{A}$ were used. The coercivity holds for every fixed positive k, but for $k \to 0$ the unknown w_3 vanishes from the operator $\mathcal{A}$ and some kind of degeneracy appears. The properties of the operator $\mathcal{A}$ must be reformulated, more exactly:

Lemma 1. *$\exists \alpha_0, \alpha_1 \in \mathbb{R}^+$, such that $\forall k$, $0 < k \leq \frac{1}{2}$ it holds:*

$$\langle \mathcal{A}(0)W,W\rangle \geq k\alpha_1\|W\|^2_{\mathcal{V}_3} + \alpha_0\|\hat{W}\|^2_{\mathcal{V}_2}\,, \tag{24}$$

where $\hat{W} = (\phi_1,\phi_2)$.

Lemma 2. *$\exists \alpha_2, \alpha_3 \in \mathbb{R}^+$ such that $\forall k$, $0 < k \leq \frac{1}{2}$ it holds:*

$$\langle \mathcal{A}' * W,W\rangle \leq k\alpha_3\|W\|^2_{C([0,T],\mathcal{V}_3)} + \alpha_2\|\hat{W}\|^2_{C([0,T],\mathcal{V}_2)}\,. \tag{25}$$

In the following we shall assume that

$$\lambda,\ \mu \in C^3([0,T],\mathbb{R}) \text{ and } f_i \in C^1([0,T],L_2(\Omega))\,, \quad i = 1,2,3\,. \tag{26}$$

To get an estimate of W_k we shall put $V = W'_{k\theta}$ and integrate in t both sides of equation (23). Using the per partes method and inequalities valid for bilinear forms, applying Lemmas 1, 2 and initial conditions (19), we obtain an inequality

$$\|W'_{k\theta}\|^2_{L^3_2(\Omega)} + k\|W_{k\theta}\|^2_{\mathcal{V}_3} + \|\hat{W}_{k\theta}\|^2_{\mathcal{V}_2} \leq C_1 + C_2\int_0^t \|W'_{k\theta}\|^2_{L^3_2(\Omega)} + k\|W_{k\theta}\|^2_{\mathcal{V}_3} + \|\hat{W}_{k\theta}\|^2_{\mathcal{V}_2} + k\|W_{k\theta}\|_{C([0,s],\mathcal{V}_3)}\|W_{k\theta}\|_{\mathcal{V}_3} + \|\hat{W}_{k\theta}\|_{C([0,s],\mathcal{V}_2)}\|\hat{W}_{k\theta}\|_{\mathcal{V}_2}\mathrm{d}s\,. \tag{27}$$

This inequality holds also (with other constants) when instead of the used norms $\|\cdot\|_{L^3_2(\Omega)}$, $\|\cdot\|_{\mathcal{V}_n}$ the norms $\|\cdot\|_{C([0,T],L^3_2(\Omega))}$, $\|\cdot\|_{C([0,T],V_n)}$ are taken. To prove this, we use the following lemma, which can be easily verified.

Lemma 3. *Let* $p,q : [0,T] \to \mathbb{R}\backslash\mathbb{R}^-$ *satisfy* $p(t) \le c + \int_0^t q(s)ds$. *Then*

$$\|p(t)\|_{C[0,t]} \le c + \int_0^t \|q(s)\|_{C(0,s)} ds\,.$$

Applying Lemma 3 and Gronwall's lemma, we get from (27) an apriori estimate of the solution $W_{k\theta}$:

$$\|W'_{k\theta}\|^2_{C([0,T],L^3_2(\Omega))} + k\|W_{k\theta}\|^2_{C([0,T],\mathcal{V}_3)} + \|\hat{W}_{k\theta}\|^2_{C([0,T],\mathcal{V}_2)} \le B_1\,, \quad B_1 \in \mathbb{R}\,. \quad (28)$$

Setting $t = 0$ and $V = W''_{k\theta}(0)$ into the equation (23) we get

$$\mathcal{K}_\theta(W''_{k\theta}(0), W''_{k\theta}(0)) \le C_3\,. \quad (29)$$

In order to achieve other estimates of $W_{k\theta}$ we have to differentiate the equation (23). After introducing into it $V = W''_{k\theta}$, carrying out the integration in t and applying (29), in a similar way as above we can get an estimate

$$\|W''_{k\theta}\|^2_{C([0,T],L^3_2(\Omega))} + k\|W'_{k\theta}\|^2_{C([0,T],\mathcal{V}_3)} + \|\hat{W}'_{k\theta}\|^2_{C([0,T],\mathcal{V}_2)} \le B_2\,, \quad B_2 \in \mathbb{R}\,. \quad (30)$$

After setting $k = 0$ we get from the estimates (28) and (30) different results for $\hat{W}$ and for w_3. Applying the Banach-Alaoglu theorem and the continuity properties (26) we can prove that

1) there exists a function

$$\begin{aligned} &\hat{W} \in C_1\big([0,T], L^2_2(\Omega)\big) \cap C\big([0,T],\mathcal{V}_2\big) \quad \text{with} \\ &\hat{W}' \in C([0,T], L^2_2(\Omega)) \cap L_\infty\big([0,T],\mathcal{V}_2\big)\,, \\ &\hat{W}'' \in L_\infty\big([0,T], L^2_2(\Omega)\big)\,, \end{aligned} \quad (31)$$

and a sequence $\{\hat{W}_{0\theta}\}$, $\theta \to 0$ such that

$$\begin{aligned} &\hat{W}_{0\theta} \overset{*}{\rightharpoonup} \hat{W} \text{ in } L_\infty([0,T],\mathcal{V}_2)\,, \\ &\hat{W}'_{0\theta} \overset{*}{\rightharpoonup} \hat{W}' \text{ in } L_\infty([0,T],\mathcal{V}_2)\,, \\ &\hat{W}''_{0\theta} \overset{*}{\rightharpoonup} \hat{W}'' \text{ in } L_\infty([0,T], L^2_2(\Omega))\,; \end{aligned} \quad (32)$$

2) there exists

$$\begin{aligned} &w_3 \in C^1\big([0,T], L_2(\Omega)\big) \quad \text{with} \\ &w'_3 \in C\big([0,T], L_2(\Omega)\big)\,, \\ &w''_3 \in L_\infty\big([0,T], L_2(\Omega)\big)\,, \end{aligned} \quad (33)$$

and a sequence $\{(w_3)_{0\theta}\}$, $\theta \to 0$ such that

$$(w'_3)_{0\theta} \overset{*}{\rightharpoonup} w'_3\,,\ (w''_3)_{0\theta} \overset{*}{\rightharpoonup} w''_3 \text{ in } L_\infty\big([0,T], L_2(\Omega)\big)\,. \quad (34)$$

Using the estimate (30) we can show that for $\forall t \in [0,T]$

$$\|\nabla W''_\theta\|_{L_2(\Omega)} \le \theta^{-\frac{1}{2}} M_3\,.$$

It implies that for $\theta_n \to 0$ the penalty member $\mathcal{J}_{\theta_n}(W,V)$ vanishes and the limit $W = (w_3, \hat{W})$ is a solution of the initial value problem (18), (19).

Using Gronwall's lemma we can prove the uniqueness of the limit W:

Lemma 4. *If* W_1 *and* W_2 *are solutions of* (18), (19) *then* $W_1 = W_2$.

$\hat{W}$ is a solution of the reduced system

$$\hat{\mathcal{K}}(\hat{W}'', \hat{V}) + h^3\langle \hat{\mathcal{A}}_3(0)\hat{W} + \hat{\mathcal{A}}'_3 * \hat{W}, \hat{V}\rangle = \hat{\mathcal{F}}(\hat{V}) \qquad \forall \hat{V} \in \mathcal{V}_2\,, \quad (35)$$

$$\hat{W}(0) = \hat{W}^0\,,\ \hat{W}'(0) = \hat{W}^1\,,\ \hat{W}^0, \hat{W}^1 \in \mathcal{V}_2\,, \quad (36)$$

where the hat operators are $\mathcal{K}, \mathcal{A}_3, \mathcal{F}$ restricted to their second and third variable (for $k = 0$ they are represented by bloc-matrices, which allow such a restriction).

The function w_3 is a solution of a system complementary to (35), (36):

$$\langle \rho h w''_3, v_3\rangle = \langle F_3, v_3\rangle\,, \quad \forall v_3 \in H^1_{\Gamma_0}(\Omega)\,, \quad (37)$$

$$w_3(0) = w^0_3\,,\ w'_3(0) = w^1_3\,,\ w^0_3, w^1_3 \in H^1_{\Gamma_0}(\Omega)\,. \quad (38)$$

Now we shall summarize our results in the following theorem:

Theorem 2. *Let the assumptions* (26) *hold and* $k = 0$. *Then the initial value problem* (18),(19) *can be treated as two independent systems* (35),(36) *and* (37),(38), *and it has a unique solution* $(w_3, \hat{W})$ *with properties* (31),(33).

R e m a r k . The equation (35), (36) is formally identical (differing only in the constants) to that of the stretching (15). The operators are the same, only (ϕ_1, ϕ_2) have to be changed by (w_1, w_2). So we have answered also the question about the existence and uniqueness of a solution of the stretching equation.

THE MINDLIN–TIMOSHENKO MODEL FOR $k \to \infty$

For k large enough the coercivity and boundedness of the operator $\mathcal{A}$ can be reformulated as follows:

Lemma 5. *There exist* $\alpha_0, \alpha_1 \in \mathbb{R}^+$, *such that for every* k, $k > 1$ *it holds:*

$$\langle \mathcal{A}(0)W, W\rangle \ge \alpha_1\|W\|^2_{\mathcal{V}_3} + (k-1)\alpha_0\langle \mathcal{A}_1 W, W\rangle\,. \quad (39)$$

Lemma 6. *There exist* $\alpha_2, \alpha_3 \in \mathbb{R}^+$, *such that for all* k, $k > 1$ *it holds:*

$$\langle \mathcal{A}' * W, W\rangle \le \alpha_3\|W\|^2_{C([0,T],\mathcal{V}_3)} + (k-1)\alpha_2\langle \mathcal{A}_1 W, W\rangle\,. \quad (40)$$

Assuming $k > 1$ and conditions (26), using Lemmas 5 and 6, in a similar way as in the previous case we get the following estimates:

$$\|W'_{k\theta}\|^2_{C([0,T],L^3_2(\Omega))} + \|W_{k\theta}\|^2_{C([0,T],\mathcal{V}_3)} + (k-1)\|\langle A_1 W_{k\theta}, W_{k\theta}\rangle\|_{C[0,T]} \le M_3\,, \quad (41)$$

$$\|W''_{k\theta}\|^2_{C([0,T],L^3_2(\Omega))} + \|W'_{k\theta}\|^2_{C([0,T],\mathcal{V}_3)} + (k-1)\|\langle A_1 W'_{k\theta}, W'_{k\theta}\rangle\|_{C[0,T]} \le M_4 \,. \quad (42)$$

For $k \to \infty$ the terms $\langle A_1 \cdot, \cdot\rangle$ in our inequalities must tend to zero and it implies that

$$(\phi_i + \frac{\partial w_3}{\partial x_i})_{k\theta} \to 0 \,, \quad i = 1,2 \,. \quad (43)$$

This leads us to use a special space of test functions

$$\overline{\mathcal{V}}_3 := \{(v_3, \psi_1, \psi_2) \in \mathcal{V}_3 \,;\ \psi_i + \frac{\partial v_3}{\partial x_i} = 0 \,,\ i = 1,2\} \quad (44)$$

which causes reduction of the equation (23) to

$$\mathcal{K}_\theta(W'', V) + \langle h^3 \mathcal{A}_3(0) W, V\rangle + \langle h^3 \mathcal{A}'_3 * W, V\rangle = \mathcal{F}(V) \,, \quad \forall V \in \overline{\mathcal{V}}_3 \,, \quad (45)$$

and it allows us to pass to the limit $k = \infty$.

For $\theta \to 0$ there exists a sequence, weak star convergent in $L_\infty([0,T], \overline{\mathcal{V}}_3)$ $\{W_{\infty\theta}\}$, whose limit is

$$W \in C_1([0,T]\,,\ L^3_2(\Omega)) \cap C([0,T]\,,\ \mathcal{V}_3) \ \text{ with}$$
$$W' \in C\big([0,T], L^3_2(\Omega)\big) \cap L_\infty\big([0,T], \mathcal{V}_3\big) \,, \quad (46)$$
$$W'' \in L_\infty\big([0,T], L^3_2(\Omega)\big) \,,$$

and it holds

$$\phi_1 + \frac{\partial w_3}{\partial x_1} = 0 \,, \quad \phi_2 + \frac{\partial w_3}{\partial x_2} = 0 \,.$$

It can be easily verified that the limit W is unique.

We have got a simple dependence of ϕ_1, ϕ_2 on w_3, and so the weak formulation of the Mindlin-Timoshenko model for $k = \infty$ can be formulated as a system:

$$\phi_1 + \frac{\partial w_3}{\partial x_1} = 0 \,, \quad \phi_2 + \frac{\partial w_3}{\partial x_2} = 0 \,, \quad (47)$$

$$\mathcal{K}(U'', Y) + h^3\langle \mathcal{A}_3(0) U, Y\rangle + h^3 \langle \mathcal{A}'_3 * U, Y\rangle = \mathcal{F}(Y)$$

$$\forall Y \in \overline{\mathcal{V}}_3 \,, \ \text{where} \ U = (w_3, \nabla w_3) \,, \quad Y = (v_3, \nabla v_3) \,,$$

with changed initial conditions:

$$w_3(0) = w_3^0 \,,\ w'_3(0) = w_3^1 \,,\ w_3^0, w_3^1 \in H^2_{\Gamma_0}(\Omega) \,; \quad \phi_i(0) + \frac{\partial w_3}{\partial x_i}(0) = 0 \,, \quad i = 1,2 \,. \quad (48)$$

We can summarize now the results in the following theorem:

Theorem 3. *Let the assumptions* (26) *hold. For* $k = \infty$ *the initial value problem* (18), (19) *is transformed into a system* (47), (48) *and it has a unique solution* $(w_3, \nabla w_3)$, *where*

$$w_3 \in C^1([0,T], H^1_{\Gamma_0}(\Omega)) \cap C\big([0,T], H^2_{\Gamma_0}(\Omega)\big) \quad (49)$$
$$w'_3 \in \ C([0,T], H^1_{\Gamma_0}(\Omega)) \cap L_\infty\big([0,T], H^2_{\Gamma_0}(\Omega)\big) \,,$$
$$w''_3 \in \ L_\infty\big([0,T], H^1_{\Gamma_0}(\Omega)\big) \,.$$

Remark. Formally equation (46) is identical (differing only in constants) to the equation of the Kirchhoff model (17). Solving the Mindlin-Timoshenko problem for $k = \infty$ we have proved the existence and uniqueness of a solution of the Kirchhoff model equation too.

References

[1] LAGNESE, J.: Modelling analysis and control of thin plates, Masson, Springer-Verlag, Paris, Berlin, 1989.

[2] NEČAS, J.—HLAVÁČEK, I.: Mathematical theory of elastic and elasto-plastic bodies, Elsevier, Amsterdam, 1981.

[3] SHAW, S.—WARBY, M. K.—WHITEMAN, J. R.—DAWSON, C.—WHEELER, M. F.: Numerical techniques for the treatment of quasistatic solid viscoelastic stress problems, Bicom **93/2** (1993).

[4] SHAW, S.—WARBY, M. K.—WHITEMAN, J. R.: Numerical techniques for problems of quasistatic and dynamic viscoelasticity, The mathematics of finite elements and applications, John Wiley, Chichester, 1993.

[5] PANCZA, D.: The viscoelastic Mindlin-Timoshenko thin plate model, 2nd Workshop on Functional Analysis and its Applications in Mathematical Physics and Optimal Control (proceedings), Nemecká, 1999.

Received 13 June 2000

Dávid Pancza (Mgr), born in Bratislava in 1970, has studied mathematical analysis at Comenius University in Bratislava. Now he is a lecturer at the Faculty of Electrical Engineering and Information Technology of the Slovak University of Technology. His PhD-thesis supervisor (applied mathematics) is Professor Igor Bock.