Journal of ELECTRICAL ENGINEERING, VOL. 51, NO. 12/s, 2000, 51–53

NON–CONTINUOUS t–NORMS WITH CONTINUOUS DIAGONAL

Dana Smutná *

During the workshops at the 2nd International Conference on Fuzzy Sets Theory and Its Applications (Liptovský Mikuláš, Slovak Republic, January 31-February 4, 1994), several open problems were specified by the participants, see [4]. One of them was the question: If T is an Archimedean t-norm with a continuous diagonal, is T necessarily continuous on $[0,1]\times[0,1]$? (This problem was stated in the book of Schweizer and Sklar [5] and Kimberling [1] has given an example of a continuous Archimedean t-norm which is not uniquely determined by its diagonal.) However, the above problem was already negatively solved by Gerianne Krause, but not yet published. In the paper we show a new class of Archimedean, non-continuous t-norms with a continuous diagonal.

K e y w o r d s: triangular norms, multiplicative generator, non-continuous Archimedean t-norms

2000 *Mathematics Subject Classification*: 03B52,26B99

1 INTRODUCTION

First we recall some well-known definitions and propositions which we will use:

Definition 1. A *triangular norm* (t*-norm* for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T\colon [0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]$ the following four axioms are satisfied:

(T1) *Commutativity*

$$T(x,y) = T(y,x)\,,$$

(T2) *Associativity*

$$T(x,T(y,z)) = T(T(x,y),z)\,,$$

(T3) *Monotonicity*

$$T(x,y) \le T(x,z) \quad \textit{whenever} \;\; y \le z\,,$$

(T4) *Boundary Condition*

$$T(x,1) = x\,.$$

Proposition 1. *A t-norm T is left-continuous if and only if it is left-continuous in its first component, i.e., if for each $y \in [0,1]$ and for each sequence $(x_n)_{n\in N} \in [0,1]^N$ we have*

$$\sup_{n\in N} T(x_n,y) = T(\sup_{n\in N} x_n, y)\,.$$

Proposition 2. *A t-norm T is Archimedean if and only if for each $x \in]0,1[$ we have*

$$\lim_{n\to\infty} x_T^{(n)} = 0\,,$$

where $x_T^{(n)} = \begin{cases} x & \textit{if } n=1 \\ T(x, x_T^{(n-1)}) & \textit{if } n>1. \end{cases}$

Proposition 3. *A t-norm T is strictly monotone if and only if the cancellation law holds, i.e., if $T(x,y)=T(x,z)$ and $x>0$ imply $y=z$.*

Definition 2. Let T be a t-norm. An element $a \in]0,1[$ is called a nilpotent element of T if there exists $n \in N$ such that $a_T^{(n)} = 0$.

Definition 3. A multiplicative generator φ of a triangular norm is a strictly increasing function $\varphi : [0,1] \to [0,1]$ such that $\varphi(1)=1$ and $\varphi(x)\cdot\varphi(y) \in H(\varphi)$ or $\varphi(x)\cdot\varphi(y) < \varphi(0)$. The corresponding t-norm T is defined by means of φ as follows:

$$T(x,y) = \varphi^{(-1)}(\varphi(x)\cdot\varphi(y))\,,$$

where $\varphi^{(-1)}\colon [0,1]\to[0,1]$ is the so-called pseudo-inverse of φ defined by

$$\varphi^{(-1)}(t) = \sup(x \in [0,1]\,;\; \varphi(x) < t)$$

with convention $\sup\emptyset = 0$.

The triangular norms which are generated by the multiplicative generator are Archimedean t-norms. Continuous t-norms which are not Archimedean cannot be generated by means of the multiplicative (additive) generator. However, there are several non-continuous t-norms

* Faculty of Natural Science, Matej Bel University, Department of Mathematics, Tajovského 40, 974 01 Banská Bystrica, SLOVAKIA, E-mail: smutna@fpv.umb.sk

Research supported by grant VEGA 1/7146/20

ISSN 1335-3632

which are generated, e.g., the drastic product T_D. A more general method how to construct t-norms can be found in Viceník [7]. The t-norms which are constructed by this method are not necessarily Archimedean. But assuming the left-continuity of a t-norm, note that the generated t-norms are then continuous and consequently Archimedean.

For $x \in]0,1]$, we can write

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i},$$

which is the unique infinite dyadic expansion of x, where $x_i \in \{0,1\}$ for $i \in N$. The set $\{i;\ x_i = 1\}$ is infinite. It is easy to see that each $x \in\ ]0,1]$ is in a one to one correspondence with $(x_i)_{i\in N}$, where $x_i \in \{0,1\}$ and $\operatorname{card}\{i;\ x_i = 1\}$ is infinite. We will use the following notation:

$$x \approx (x_i)_{i\in N}.$$

Remark 1. Let $x \approx (x_i)_{i\in N}$ and $y \approx (y_i)_{i\in N}$. Then $x < y$ if and only if there exists $k \in N$ such that for all $i \in N,\ i \le k$, we have

$$x_i = y_i \quad \text{and} \quad x_{k+1} < y_{k+1}.$$

We will discuss a t-norm based on the above described dyadic expansion which is generated by a non-continuous multiplicative generator.

2 A NON–CONTINUOUS t–NORM BASED ON DYADIC EXPANSION

In Smutná [6] we have investigated the function $g\colon [0,1] \to [0,1]$ which is based on the dyadic expansion. This function is given by

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sum\limits_{i=1}^{\infty} \frac{2x_i}{3^i} & \text{otherwise,} \end{cases}$$

where if $x \in]0,1]$ then $x \approx (x_i)_{i\in N}$. The function g is strictly increasing and left-continuous. This function is not right-continuous, each finite dyadic rational is a point of discontinuity of the function g.

Remark 2. We can define $f(x) = g^{(-1)}(x) = \sup(z \in [0,1];\, g(z) < x)$ as the pseudo-inverse of the function g. Because of the properties of this function, the new function f is continuous and $f^{(-1)} = g$. Note that f is just the well-known Cantor function.

Example 1. Let $T^*\colon [0,1]^2 \to [0,1]$ be given by

$$T^*(x,y) = f(g(x).g(y)),$$

where g is the function from Smutná [6], and f is its pseudo-inverse.

Then T^* is an operator which is not a t-norm. T^* is commutative, monotone, it fulfils the boundary condition, however, the associativity is violated. For example

$$T^*\big(\tfrac{1}{2}, T^*(\tfrac{3}{4},\tfrac{3}{4})\big) = \frac{1}{4} < \frac{1}{2} = T^*\big(T^*(\tfrac{1}{2},\tfrac{3}{4}),\tfrac{3}{4}\big).$$

On the other hand, the operation

$$T_*(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ g(f(x)\cdot f(y)) & \text{otherwise,} \end{cases}$$

defines a t-norm. The axioms T(1), T(3), T(4) are evidently fulfilled. The associativity follows from the continuity of f and the fact that $f(g(x)) = x$ for all $x \in [0,1]$. For more detail see [6].

Remark 3. As already mentioned, $f(g(x)) = x$ for $x \in [0,1]$. Therefore $T_*(x, T_*(x,x)) = g(f(x)^3)$ for $x \in [0,1]$, which implies $x_{T_*}^{(n)} = g(f(x)^n)$ for $x \in\ [0,1]$.

Remark 4. The t-norm T_* is an example of a t-norm which is left-continuous on $[0,1[^2$ and continuous in point $(1,1)$, but non-left-continuous on $[0,1]^2$ and non-continuous on its diagonal. Moreover, this t-norm is Archimedean but neither strictly monotone nor nilpotent in any point from $]0,1[$.

Indeed, let $a \in]\frac{1}{3^n}, \frac{2}{3^n}]$ for some $n \in N$. Then $a_{T_*}^{(m)} = \frac{1}{3^{m\cdot n}} > 0$ for all $m \in N$. Consequently, no element $b \in]0,1[,\ b > a$, can be a nilpotent element of T_*.

3 THE NON–CONTINUOUS ARCHIMEDEAN t–NORM WITH CONTINUOUS DIAGONAL

Again we deal with the function g and we define the next functions $t^*: [0,1] \to [0,1]$, $t^{**}\colon [0,1] \to [0,1]$ and $h\colon [0,1] \to [0,1]$ as follows:

$$t^*(x) = \sup(g(z); g(z) \le x),$$

$$t^{**}(x) = \inf(g(z); g(z) \ge x),$$

$$h(x) = \frac{x - t^*(x)}{t^{**}(x) - t^*(x)},$$

with convention $\frac{0}{0} = 0$.

Remark 5. Note that $t^*(x) = g \circ g^{(-1)}(x)$. Moreover, if $x \in H(g)$ then $h(x) = 0$.

Now we will investigate an operator based on the above functions and the t-norm T_*. Let the function $T_K\colon [0,1]^2 \to [0,1]$ be given by

$$T_K(x,y) = T_*(x,y) + F(x,y),$$

where the function $F\colon [0,1]^2 \to [0,1]$ is given by

$$F(x,y) = \min(h(x),h(y)) \cdot \big(t^{**}(T_*(x,y)) - t^*(T_*(x,y))\big).$$

Evidently the axioms T(1), T(3) and T(4) are fulfilled. The associativity of T_K follows from the fact that for all $x,y \in [0,1]$ is

$$f(T_K(x,y)) = f(T_*(x,y)) = f(x).f(y).$$

Proposition 4. *The function* $T_K : [0,1]^2 \to [0,1]$ *is a* t*-norm.*

Proposition 5. *The triangular norm T_K is not strictly monotone.*

P r o o f . Let $x \in [0,1]$ and $x \in H(g)$. Then

$$T_*\Big(\frac{1}{2}, x\Big) = g\Big(\frac{1}{2}\cdot x\Big) = T_*\Big(\frac{2}{3}, x\Big) \text{ and } h(x) = 0\,.$$

Therefore $$T_K\Big(\frac{1}{2}, x\Big) = g\Big(\frac{1}{2}\cdot x\Big)$$

$$+ \min\Big(h(x), h(\frac{1}{2})\Big)\cdot\Big(t^{**}(T_*(x,\frac{1}{2})) - t^*(T_*(x,\frac{1}{2}))\Big)$$

$$= g\Big(\frac{1}{2}\cdot x\Big) = T_K\Big(\frac{2}{3}, x\Big)\,,$$

which is violation of the strict monotonicity of T_K.

Proposition 6. *The triangular norm T_K is Archimedean and T_K is not nilpotent in any point from the interval* $]0,1[$.

R e m a r k 6. The t-norm T_K is an example of a t-norm which is non-continuous, but continuous on its diagonal. Moreover, this t-norm is Archimedean but neither strictly monotone nor nilpotent in any point from $]0,1[$.

Theorem 1. *Let $T\colon [0,1]^2 \to [0,1]$ be an arbitrary continuous t-norm. Then*

$$\widetilde{T_K}(x,y) = T_*(x,y) + \widetilde{F}(x,y)\,,$$

where function $\widetilde{F}\colon [0,1]^2 \to [0,1]$ is given by

$$\widetilde{F}(x,y) = T\big(h(x), h(y)\big)\cdot\big(t^{**}(T_*(x,y)) - t^*(T_*(x,y))\big)$$

is a non-continuous Archimedean t-norm with continuous diagonal.

References

[1] KIMBERLING, C.: On a class of associative function, Publ. Math. Debrecen **20** (1973) 21-39.

[2] KLEMENT, E. P.—MESIAR, R.—PAP, E.: Triangular Norms, Kluwer Acad. Publ., Dordrecht, 2000.

[3] MESIAR, R.: Wild t-norms of G. Krause, Abstracts, The fifth International Conference FSTA 2000, Liptovský Ján, 132–133.

[4] MESIAR, R.—NOVÁK, V.: Open problems from the 2nd International Conference on Fuzzy Sets and Its Applications, Fuzzy Sets and Systems **81** (1996), 185–190.

[5] SCHWEIZER, B.—SKLAR, A.: Probabilistic Metric Spaces, North Holland, New York, 1983.

[6] SMUTNÁ, D.: A note on non-continuous t-norms, BUSEFAL **76** (1998), 19–24.

[7] VICENÍK, P.: A note to a construction of t-norms based on pseudo-inverses of monotone functions, Fuzzy Sets and Systems **104** (1999), 15–18.

Received 15 June 2000

Dana Smutná (Mgr) is a graduate student of applied mathematics at the Faculty of Electrical Engineering and Information Technology of the Slovak University of Technology, Bratislava. Her supervisor is Professor Radko Mesiar.