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ON GEOMETRICAL PROPERTIES OF RANDOM TORI AND RANDOM GRAPH MODELS

Martin Nehéz *

In this paper we prove that for a given random d-dimensional torus T of order n^d, the upper bound on the size r of the largest d-dimensional grid that is asymptotically almost surely contained in T is $O([\log n]^{1/d})$. To prove this result we use both probabilistic and Kolmogorov complexity arguments. Possible applications are indicated.

K e y w o r d s: torus, grid, random graph, Kolmogorov complexity.

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1 INTRODUCTION

The theory of random graphs is often used to prove the existence of an object (graph) in a nonconstructive way. There are four known ways to introduce the random graph:

– *static model* [12], [14],
– *dynamic model* [14],
– *probabilistic model* [3],
– *Kolmogorov random model* [7].

The probabilistic model seems to be used most frequently; however, the Kolmogorov model has recently become quite popular as well. In what follows we focus on these two models in our paper. To compare the proving techniques used in both above mentioned models we investigate the geometrical properties of random tori.

We consider a d-dimensional torus T_n^d with randomly deleted edges. Accordingly, we introduce the probabilistic space of all random d-dimensional tori, denoted by $\mathcal{G}(T_n^d, p)$. (The parameter p is called the probability of an edge.)

Important geometrical properties of the random torus are the structure of its subgraphs. The most important convex subgraph is a grid. We estimate the upper bound of the size r of the largest grid $G_r^d \subset T$, where $T \in \mathcal{G}(T_n^d, p)$. This problem is, in some sense, analogous to estimating the largest order of a clique in a random graph (see [9], or [11] for stronger results).

In this paper it is shown that for the size of the largest d-dimensional grid $G_r^d \subseteq T$ the following inequality holds asymptotically almost surely:

$$r \le (\log_{1/p} n)^{1/d} + 1\,.$$

This result holds for both the probabilistic and Kolmogorov random models.

Finally, we comment the obtained results and give some remarks on possible applications.

2 PRELIMINARIES

Let n, d be positive integers, such that $n > 2$. Denote the set of n integers $\{0, 1, \ldots, n-1\}$ by $[n]$. The Cartesian product of d sets $[n]$ is denoted by $[n]^d$. A *d-dimensional vector over* $[n]$ is the d-tuple $\overline{x} = (x_1, \ldots, x_d)$ such that $x_i \in [n]$ for $i = 1, \ldots, d$. We can also write $\overline{x} \in [n]^d$. For two vectors $\overline{x}, \overline{y} \in [n]^d$, their *Hamming distance* is defined as $\rho(x, y) = \sum_{i=1}^d |x_i - y_i|$.

For n, d as above, a *d-dimensional torus* is the undirected graph $T_n^d = (V, E)$, where $V(T_n^d) = [n]^d$ (that is each vertex is labeled by a d-dimensional vector $\overline{x} \in [n]^d$) and $E(T_n^d)$ is defined as follows. Vertices $\overline{x}, \overline{y} \in V(T_n^d)$ are adjacent iff

(i) $\rho(\overline{x}, \overline{y}) = 1$, or
(ii) $\rho(\overline{x}, \overline{y}) = n-1$ and $\overline{x}, \overline{y}$ differ only in one number.

The degree of each vertex of T_n^d is equal to $2d$. The order (number of vertices) of the torus T_n^d is given by $|V(T_n^d)| = n^d$ and its number of edges is $|E(T_n^d)| = dn^d$.

For n, d as above, a *d-dimensional grid* is the undirected graph $G_n^d = (V, E)$, where $V(G_n^d) = [n]^d$ and $E(G_n^d) = \{(\overline{x}, \overline{y}) \mid \overline{x}, \overline{y} \in [n]^d,\ \rho(\overline{x}, \overline{y}) = 1\}$. The parameter n is called the *size of the grid* G_n^d. The order of the grid G_n^d is $|V(G_n^d)| = n^d$ and its number of edges is given by $|E(G_n^d)| = d(n-1)n^{d-1}$. For the torus T_n^d and for a positive integer r, $1 \le r \le n$, we denote the d-dimensional grid on r^d vertices which is a subgraph of the torus T_n^d by $G_r^d \subset T_n^d$. The grid $G_r^d \subset T_n^d$ is induced by the set of vertices $S(\overline{a}) = \{(a_1 + k, \ldots, a_d + k) \mid 0 \le k \le r\}$, for arbitrary $\overline{a} \in V(T_n^d)$, where $+$ denotes addition

* Department of Applied Informatics, University of Economics, Dolnozemská 1/b, 852 35 Bratislava, Slovak Republic,
E-mail: nehez@euba.sk

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mod n. The vertex $\overline{a}$ will be called the *reference vertex of the grid* G_r^d. Hence, each d-dimensional torus T_n^d contains n^d (possibly overlapping) d-dimensional grids $G_r^d \subset T_n^d$. These grids differ in the labels of at least r^{d-1} vertices.

The O, Ω, o notation will be used in a standard way. The notation $\lg x$ denotes the binary logarithm of x. The ceiling of a real number x is denoted by $\lceil x \rceil$.

3 PROBABILISTIC MODEL

In this section we prove the upper bound on the parameter r such that the grid G_r^d is asymptotically almost surely a subgraph of the torus with randomly deleted edges. We use probabilistic arguments.

3.1 Random Tori

Let $p \in \mathcal{R}$ be a constant such that $0 < p < 1$. Consider that for a given d-dimensional torus T_n^d each edge exists in T_n^d independently and with the probability p. It means that $Pr[\{\overline{x}, \overline{y}\} \in E(T_n^d)] = p$ for all adjacent vertices $\overline{x}, \overline{y} \in V(T_n^d)$. The constant p is called the *probability of an edge*. We can introduce the corresponding probabilistic space, now.

Definition 1. Let $(\Omega, \mathcal{F}, Pr)$ be a probabilistic space, where the class Ω consists of all (labeled) graphs T on n^d vertices such that $V(T) = V(T_n^d)$ and $E(T) \subseteq E(T_n^d)$. If T has q edges, $0 \leq q \leq dn^d$, then the probability of obtaining T as a result of random edge generation is given by:

$$Pr[T] = p^q (1-p)^{dn^d - q} \,. \tag{1}$$

The graph T will be called *random d-dimensional torus*. The probabilistic space $(\Omega, \mathcal{F}, Pr)$ will be denoted by $\mathcal{G}(T_n^d, p)$ and called probabilistic space of all random d-*dimensional tori*.

In order to describe a property of random tori we will use the notions from probability theory, e.g. random variables, expectations, etc. We will also use the following property.

Proposition 1(Markov's inequality)**.** *Let X be a non-negative random variable with expectation $E(X)$ and let $\lambda > 0$. Then the following inequality holds*

$$Pr[X \geq \lambda] \leq E(X) \cdot \lambda^{-1} \,. \tag{2}$$

For more details see [5].

3.2 Upper Bound

Important geometrical properties of the random torus are the structure of its subgraphs. The most important convex subgraph of the torus (corresponding to a complete subgraph of a graph) is a grid. We investigate the upper bound of the size of the d-dimensional grid $G_r^d \subseteq T$, where $T \in \mathcal{G}(T_n^d, p)$ is the random d-dimensional torus.

Given n, d, r as above and $\overline{a} \in V(T_n^d)$, let us denote the set of vertices which induce the grid $G_r^d \subset T_n^d$ by $S_r(\overline{a})$, so that $\overline{a}$ is the reference vertex of the grid G_r^d. Let the event that a random torus $T \in \mathcal{G}(T_n^d, p)$ contains the gird $G_r^d \subset T_n^d$ induced by the set $S_r(a)$ be denoted by A. Let X_r be the associated indicator random variable defined on the probabilistic space $\mathcal{G}(T_n^d, p)$. It means that $X_r = 1$ if T contains the grid induced by the set $S_r(a)$, and $X_r = 0$ otherwise. Let us define a random variable Y_r on $\mathcal{G}(T_n^d, p)$ as $Y_r = \sum X_r$, the summation over all sets $S_r(a)$. It means that Y_r is the number of grids G_r^d in the random torus T. We compute the expectation of Y_r.

Lemma 1. *For the expectation of the random variable Y_r the following equality holds*

$$E(Y_r) = n^d p^{d(r-1)r^{d-1}} \,. \tag{3}$$

P r o o f . From the definition of the X_r we have

$$E(X_r) = Pr[A] = p^{d(r-1)r^{d-1}} \,.$$

By linearity of expectation (where the summation ranges over all vertices $\overline{a} \in V(T_n^d)$)

$$E(Y_r) = \sum E(X_r) = n^d p^{d(r-1)r^{d-1}} \,.$$

Using this lemma we estimate the upper bound for the parameter r.

Theorem 1. *Let $T \in \mathcal{G}(T_n^d, p)$ be a random d-dimensional torus. Then for the size of the largest d-dimensional grid $G_r^d \subset T$ the following inequality holds*

$$r \leq (\log_{1/p} n)^{1/d} + 1 \,. \tag{4}$$

with the probability tending to one as $n \to \infty$. Hence the largest grid G_r^d has the maximal order $\log_{1/p} n + o(\log n)$.

P r o o f . By contradiction. The following inequality holds

$$E(Y_r) = n^d p^{d(r-1)r^{d-1}} \leq n^d p^{d(r-1)^d} \,, \tag{5}$$

since $0 < p < 1$. By the substitution $r = (\log_{1/p} n)^{1/d} + 1$ into (5) we have

$$E(Y_r) \leq n^d p^{d \log_{1/p} n} = n^d (1/p)^{-d \log_{1/p} n} = n^d \cdot n^{-d} = 1 \,. \tag{6}$$

From Markov's inequality (2) for $\lambda = 1$ and from (6) follows that

$$Pr[Y_t > 1] \leq E(Y_t) < 1 \tag{7}$$

for arbitrary $t > (\log_{1/p} n)^{1/d} + 1$. The random variable Y_r counts the number of grids G_r^d in the random torus T. The inequality (7) yields that there is no such a grid $G_r^d \subset T$ with the size $r > (\log_{1/p} n)^{1/d} + 1$ with the probability tending to 1 as $n \to \infty$. Hence a contradiction.

4 KOLMOGOROV RANDOM MODEL

In this section we prove the analogous inequality using the Kolmogorov complexity arguments. We introduce the basic notions first.

4.1 Kolmogorov Complexity

Identify natural numbers $\mathcal{N}$ and a set of finite strings (words) over the alphabet $\{0,1\}$ according to the structure of the following form

$$(0,\epsilon),\ (1,0),\ (2,1),\ (3,00),\ (4,01),\ \ldots,$$

where ϵ denotes the empty word, such that the length $|x|$ of a natural number x is the number of bits in a corresponding binary string, $|\epsilon| = 0$.

Let us fix an effective enumeration of Turing machines $T_1, T_2, \ldots$. All Turing machines use a tape alphabet $\{0,1,B\}$. The input to a Turing machine is a program — the binary string over $\{0,1\}$ — delimited by blanks B on both sides, [13]. Using the delimiters, a Turing machine can recognize the beginning and the end of its program. The effective enumeration of Turing machines induces an effective enumeration of partial recursive functions $\phi_1, \phi_2, \ldots$ such that T_i computes ϕ_i for each i. (For more details se [13].)

Let $\langle\cdot,\cdot\rangle : \mathcal{N}\times\mathcal{N}\to\mathcal{N}$ denote a standard computable bijective pairing function of which the inverse is computable too. (It maps the pair (x,y) to the singleton $\langle x,y\rangle$.) An example is $1^{|x|}0xy$. Define $\langle x,y,z\rangle$ inductively by $\langle x,\langle y,z\rangle\rangle$. There exists a universal partial recursive function ϕ_0 which from an input y computes the output x as follows

$$\phi_0\big(\langle y,\langle n,p\rangle\rangle\big) = \phi_n(\langle y,p\rangle) \tag{8}$$

for all n,y,p. The function ϕ_0 is represented by the universal Turing machine T_U.

Intuitively, the Kolmogorov complexity of a binary string x is the shortest description of x. More precisely, the Kolmogorov complexity can be defined in the following way, [7].

Definition 2. Let us fix a universal partial recursive function ϕ with the property (8). Let x,y,p be natural numbers such that $\phi\big(\langle y,p\rangle\big) = x$. The partial recursive function ϕ together with p and y is a description of x. The conditional Kolmogorov complexity of x given y is

$$C(x|y) = \min\{\,|p| : \phi\big(\langle y,p\rangle\big) = x\}\,. \tag{9}$$

$C(x|y) = \infty$ if there is no such a p. We say that the program p computes x by ϕ, given y. The unconditional Kolmogorov complexity of x is defined as $C(x) := C(x|\epsilon)$.

The function ϕ is called the *reference function for* C. We denote $C\big(x|\langle y,z\rangle\big)$ by $C(x|y,z)$. We will use the following property of Kolmogorov complexity.

Proposition 2. *There exists a constant $c \in \mathcal{N}$ such that for all $x,y \in \mathcal{N}$ the following inequality holds:*

$$C(x|y) \le |x| + c\,. \tag{10}$$

On the other hand, for each $y \in \mathcal{N}$ there exists an $x \in \mathcal{N}$ such that $C(x|y) \ge |x|$.

4.2 Kolmogorov random tori

Using the properties of strings with given Kolmogorov complexity we introduce the Kolmogorov random tori. This description can be interpreted in the way that occurrence/absence of each edge of a torus is chosen independently at random.

Let n,d be constants as above. Let us encode (up to automorphism) each d-dimensional torus T_n^d (T for simplicity) by binary string $S(T)$ with dn^d bits. Assume that bits represent the set of edges ordered in standard lexicographical order without repetition. If the ith bit in the string $S(T)$ attains the value 1 (0), the corresponding edge is present (absent) in the torus T. This way we can identify each torus (with possibly absent edges) by corresponding binary string.

Definition 3. A d-dimensional torus T on n^d vertices has *randomness deficiency* at most $\delta(n)$ and is called $\delta(n)$*-random*, if

$$C(S(T)|n,d,\delta) \ge dn^d - \delta(n)\,. \tag{11}$$

The main theorem of this section is the following.

Theorem 2. *Let $\delta(n)$ be a function such that $\delta(n)\to\infty$ as $n\to\infty$. Almost all $\delta(n)$-random tori T on n^d vertices contain the grid $G_r^d \subseteq T$ whose size r satisfies the inequality:*

$$r \le (\lceil \lg n\rceil)^{1/d} + 1\,. \tag{12}$$

Proof. Let us consider the torus T such that the Kolmogorov complexity of its description is given by (11). So it is $\delta(n)$-random. Assume that the torus T contains a d-dimensional grid G_r^d. The size r of the grid can be interpreted as a function of n. The description of $r = r(n)$ has constant length.

Let us construct a new encoding $S^*(T)$ of the torus T as follows.

- Prefix a code of the reference vertex $\overline{a} \in T$ of the grid G_r^d. The length of the code of $\overline{a}$ is $\lceil \lg n^d\rceil$ bits.
- From $S(T)$ let us remove $d(r-1)r^{(d-1)}$ bits corresponding to the edges of the grid G_r^d.
- Keep all bits which correspond to the other edges.
- Save a constant number of bits representing this description.

The length $|S^*(T)|$ of a new code is as follows.

$$|S^*(T)| = |S(T)| - d(r-1)r^{(d-1)} + \lceil \lg n^d \rceil + c_1 \,. \quad (13)$$

This is a new description of the T and from the definition of the Kolmogorov complexity and from (10) it follows that there exists a constant c_2 such that

$$C(S(T)|n,d,\delta) \leq |S^*(T)| + c_2 \,. \quad (14)$$

Equations (11), (13) and (14) yield

$$dn^d - \delta(n) \leq dn^d - d(r-1)r^{(d-1)} + d\lceil \lg n \rceil + c_1 + c_2 \,, \quad (15)$$

which can be simplified to

$$d(r-1)^d \leq d\lceil \lg n \rceil + \delta(n) + O(1) \,. \quad (16)$$

Note that $\delta(n) + O(1) \geq 0$. Hence, for a large enough n, the inequality (16) is satisfied only if $r \leq (\lceil \lg n \rceil)^{1/d} + 1$.

5 CONCLUSION

For a random d-dimensional torus T we studied the sizes of the largest d-dimensional grid $G_r^d \subset T$. We used the probabilistic and Kolmogorov complexity arguments. Both inequalities obtained are very similar, they differ only in the base of the logarithm. (The formulae are the same if the condition $p = 1/2$ holds in the probabilistic version.) Analogous relationships between these two models (namely for the random graphs) are shown in [4], however, a strong formula as in [2] is still not known. (In [2] it is shown that static and probabilistic models of random graphs are closely related if $p \approx |E(G)|/\binom{n}{2}$.)

In our models the random torus can be interpreted as an interconnection network with random faulty/overloaded communicational links. (The case $d = 2$ has a special meaning described bellow.) The faults independently occur with a constant probability. Analogous models (but with faulty processors) are described e.g. in [8] and [10].

Our results can be used

(1) in the design of VLSI circuits with random faults (for $d = 2$),

(2) in the description of the properties of the communication algorithms in the distributed systems with faulty communicational links.

In the first case the inequality $r \leq (\log_{1/p} n)^{1/d} + 1$, especially for $d = 2$ and n sufficiently large, describes e.g. the properties of the layout of fault-free two-dimensional arrays to its faulty version with dilation 1.

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Martin Nehéz (Mgr) is a PhD student of theoretical computer science at Faculty of Mathematics, Physics and Informatics of the Comenius University, Bratislava under the supervision of Professor Daniel Olejár. He is an assistant professor at the University of Economics, Bratislava and also the Silesian University, Opava.