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SOME PROPERTIES OF SYSTEMS WITH QUADRATIC HAMILTONIANS

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For a quadratic Hamiltonian the classical evolution, the evolution of mean values of quantum position and momentum, classical projection, nonlinear evolution and the projection of quantum evolution by the momentum mapping are investigated. All these evolutions can be, in a sense, regarded as being identical. For some of them the Planck constant is considered to have a varying value.

K e y w o r d s: quadratic Hamiltonian, analytic vector,classical projection, nonlinear evolution, momentum mapping

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1 INTRODUCTION

Quadratic Hamiltonians can be considered both in classical mechanics and in quantum mechanics (QM). Examples of systems given by quadratic Hamiltonians are the free particle and harmonic oscillator. In the present paper, we shall investigate some properties of systems with quadratic Hamiltonians. We shall consider various constructions of time evolutions based on quadratic Hamiltonians, e.g. classical evolution, evolution of mean values of quantum position and momentum, classical projection, nonlinear evolution. We shall show that these evolutions can be in a natural way regarded as being identical. We shall also consider the Planck constant to be a variable in some of these cases, and investigate the system given by a quadratic Hamiltonian with the changed value of the Planck constant. In the case $\hbar \to 0$ this will lead to the classical limit of QM.

In Section 1, some preliminary considerations are presented. In Section 2, the time evolution of the mean values of quantum position and momentum is investigated where the Planck constant is a variable. This is related to the approach to the classical limit of QM given in [3]. In Section 4, the classical projections [1] are considered where the Planck constant is also a variable. This also has a connection to the classical limit of QM. In Section 5, the nonlinear evolution [2] is considered and in Section 6 the projection of the quantum evolution by the momentum mapping is investigated.

2 PRELIMINARIES

$$\text{Let}\quad \mathbb{R}^{2n} \ni x = (x_1, x_2, \dots, x_{2n}) = (q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$$

be the $2n$-tuple of coordinates of position and momentum in the $2n$-dimensional classical "flat" phase space. Let $\mathcal{H}$ be the state space of a quantum mechanical system with n degrees of freedom and

$$X = (X_1, \dots, X_{2n}) = (Q_1, \dots, Q_n, P_1, \dots, P_n)$$

be the corresponding operators of position and momentum. They generate an irreducible representation of the Heisenberg canonical commutation relations (CCR).

The Planck constant $\hbar$ is a fundamental constant of Nature, but it is useful to investigate the limit transition $\hbar \to 0$ for some purposes. Therefore we shall consider varying values of this constant. The changed value will be denoted by $\lambda^2\hbar$ where $\lambda \neq 1$. The original value of the Planck constant will be obtained by setting $\lambda = 1$. The limit transition $\hbar \to 0$ will be represented by $\lambda \to 0$ in this context. Now, for the possibly changed value of Planck constant $\lambda^2\hbar$ we have

$$X^\lambda = (X_1^\lambda, \dots, X_{2n}^\lambda) = (Q_1^\lambda, \dots, Q_n^\lambda, P_1^\lambda, \dots, P_n^\lambda)$$

the operators of position and momentum where we define

$$Q_i^\lambda = \lambda Q_i\,, \quad P_i^\lambda = \lambda P_i\,, \quad i = 1, \dots, n\,.$$

It can be straightforward checked that the operators Q_i^λ, P_i^λ form an irreducible representation of CCR with the value $\lambda^2\hbar$ of the Planck constant. Obviously, for $\lambda = 1$ we obtain $X^1 = X$. Let us consider a projective representation of the additive group $\mathbb{R}^{2n}$ given by the well-known Weyl operators

$$U_x = \exp\Big(\frac{i}{\hbar} X \cdot S \cdot x\Big)$$

where S is the standard symplectic matrix $2n \times 2n$ with elements

$$S_{j\,j+n} = -S_{j+n\,j} = 1\,, \qquad j = 1, 2, \dots, n;$$
$$S_{jk} = 0 \qquad \text{otherwise.}$$

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For λ possibly different from 1 we have also the Weyl operators

$$U_x^\lambda = \exp\Big(\frac{i}{\lambda^2\hbar} X^\lambda \cdot S \cdot x\Big)\,.$$

Obviously $U_x^\lambda = U_{\frac{x}{\lambda}}$.

We shall consider the quantum quadratic Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^{2n} a_{ij} X_i X_j \tag{1}$$

where the real constants satisfy the relation $a_{ij} = a_{ji}$ in order to make the operator H symmetric. In fact, in this case the operator H is essentially selfadjoint on a common domain of all operators X_j. For the changed value of Planck constant $\lambda^2\hbar$ it is natural to take the following version of the quantum quadratic Hamiltonian:

$$H^\lambda = \frac{1}{2}\sum_{i,j=1}^{2n} a_{ij} X_i^\lambda X_j^\lambda\,. \tag{2}$$

Obviously $H^1 = H$. We shall also use the classical counterpart of these quantum Hamiltonians:

$$h(x) = \frac{1}{2}\sum_{i,j=1}^{2n} a_{ij} x_i x_j\,, \tag{3}$$

which can be considered to be the their “classical limit”.

Let

$$\begin{aligned} x(0) &= \big(x_1(0),\dots,x_{2n}(0)\big) = \big(q(0),p(0)\big)\\ &= \big(q_1(0),\dots,q_n(0),p_1(0),\dots,p_n(0)\big) \end{aligned} \tag{4}$$

be the initial state of the system with classical Hamiltonian h given by (3),

$$\begin{aligned} x(t) &= \big(x_1(t),\dots,x_{2n}(t)\big) = \big(q(t),p(t)\big)\\ &= \big(q_1(t),\dots,q_n(t),p_1(t),\dots,p_n(t)\big) \end{aligned} \tag{5}$$

be the state evolved at time t.

Let us consider the symplectic reformulation of QM given, e.g. in the work [1]. The role of phase space will be played by the projective Hilbert space $P(\mathcal{H})$. If $\psi \in \mathcal{H}$ then $P_\psi \in P(\mathcal{H})$ is the corresponding onedimensional projector. Given a $\lambda > 0$ and $\psi \in \mathcal{H}$, the orbit of generalized coherent states will be

$$O_\psi^\lambda = \{U_x^\lambda \psi; x \in \mathbb{R}^{2n}\}.$$

Its natural projection into $P(\mathcal{H})$ is

$$P\big(O_\psi^\lambda\big) = \{P_\eta; \eta \in O_\psi^\lambda\} = \{U_x^\lambda P_\psi \big(U_x^\lambda\big)^{-1} = P_{U_x^\lambda\psi}; x \in \mathbb{R}^{2n}\}.$$

The orbits O_ψ^λ are all identical sets if ψ is fixed, only the parametrization by x is different for different λ. On each orbit $P(O_\psi^\lambda)$ there is a unique $P_\phi \in P(\mathcal{H})$ such that for all $i = 1,2,\dots,2n$, $\mathrm{Tr}\big(P_\phi X_i\big) = 0$. Now if λ is fixed, for each $P_\psi \in P(O_\phi^\lambda)$ there exist unique $x \in \mathbb{R}^{2n}$ such that

$$P_\psi = P_x^\lambda = U_x^\lambda P_\phi \big(U_x^\lambda\big)^{-1}\,.$$

It would be correct to denote $P_x^\lambda = P_{x,\phi}^\lambda$ as this projector depends also on the choice of ϕ. We shall not do this but always if we deal with projectors P_x^λ where x or λ is a variable, we consider ϕ to be fixed. From now on ϕ will always denote such a natural “initial” element of the orbit, i.e. with zero mean values of the observables X_i, $i = 1,\dots,2n$. It can easily be seen now that the orbit $P(O_\phi^\lambda)$ can be naturally identified with the “flat” space $\mathbb{R}^{2n}$. It is $\mathrm{Tr}\big(P_x^\lambda X_i^\lambda\big) = x_i$ for all $i = 1,\dots,2n$.

Within this framework, we can consider classical projections [1]. These are classical systems given on the orbits $P(O_\phi^\lambda)$ in the following way: If $\lambda > 0$ be given, let H_{q}^λ be an arbitrary quantum Hamiltonian then the Hamiltonian of the classical projection is given

$$h_{\mathrm{cp}}^\lambda\big(P_x^\lambda\big) = h_{\mathrm{cp}}^\lambda(x) = \mathrm{Tr}\big(P_x^\lambda H_{\mathrm{q}}^\lambda\big)\,.$$

As the orbits $P(O_\phi^\lambda)$ can be identified with $\mathbb{R}^{2n}$, the classical projection can be considered to be a classical Hamiltonian system on the phase space $\mathbb{R}^{2n}$.

The next material is taken from [2]. Let G_{WH} be the Weyl Heisenberg group parametrized by $q = (q_1,\dots,q_n) \in \mathbb{R}^n$, $p = (p_1,\dots,p_n) \in \mathbb{R}^n$, $s \in \mathbb{R}$ with group multiplication

$$g(q,p,s)g(q',p',s') = g\Big(q+q', p+p', s+s'+\frac{1}{2}(q'\cdot p - p'\cdot q)\Big)$$

where $q{\cdot}p$ denotes the scalar product of q,p. If $Lie(G_{WH})$ denotes the Lie algebra of G_{WH} and $Lie(G_{WH})^*$ is its dual, then the elements $F \in Lie(G_{WH})^*$ can be parametrized by parameters $q_0, p_0 \in \mathbb{R}^n, s_0 \in \mathbb{R}$: $F = F(q_0,p_0,s_0)$. For $s_0 \neq 0$ the coadjoint action of G_{WH} in the space $Lie(G_{WH})^*$ in this parametrization has the simple form

$$\mathrm{Ad}^*(g(q,p,s))F(q_0,p_0,s_0) = F(q+q_0, p+p_0, s_0)\,. \tag{6}$$

The group G_{WH} is a central extension of the additive group $\mathbb{R}^{2n}$. The abovementioned projective representation U_x of $\mathbb{R}^{2n}$ can be extended to a unitary representation $U(G_{WH})$ of G_{WH} in this way: if $x = (q,p)$ then

$$U(g(q,p,s)) = \exp\Big(\frac{is}{\hbar}\Big)U_x\,.$$

Given a unitary representation U of a Lie Group G, let us define the corresponding momentum mapping

$$\mathbb{F}\colon P(\mathcal{H}) \to Lie(G)^*\,,$$

$$\mathbb{F}(P_\psi) = F_\psi \in Lie(G)^*\,,\quad F_\psi(\xi) = \mathrm{Tr}(P_\psi X_\xi)$$

where $\xi \in Lie(G)$ and X_ξ is given by

$$U(\exp t\xi) = \exp(-itX_\xi)\,.$$

If $G = G_{WH}$ then we can choose a basis $\{\xi_i; i = 0,1,\dots,2n\}$ in $Lie(G_{WH})$ such that $X_{\xi_i} = X_i$ for $i = 1,\dots,2n$, $X_{\xi_0} = X_0 = \hbar I$.

3 EVOLUTION OF MEAN VALUES OF QUANTUM POSITION AND MOMENTUM

In [3], an approach to the classical limit of QM is given which relates the time evolution of the mean values of position and momentum of a quantum system to the time evolution of its classical limit via the transition to the limit $\hbar \to 0$. For each value of $\hbar$, there is given a quantum Hamiltonian $H_\hbar$, operators of position and momentum $q_\hbar, p_\hbar$ and an initial state of the system. This initial state is for each $\hbar$ chosen conveniently on the orbit of Glauber coherent states. Then it is shown that for $\hbar \to 0$ the time evolution of the mean values of quantum position and momentum converges to the time evolution of the classical limit.

We shall proceed in a similar way but only for quadratic Hamiltonians. Let $\lambda > 0$ be given and H^λ be given by (2). As in [3], a crucial role will be played by a convenient choice of the initial state for given λ. In [3] Glauber coherent states were used; we shall use the generalized (Perelomov) coherent states (see, e.g. [5]). Let $P_\phi \in P(\mathcal{H})$ be such that $\mathrm{Tr}(P_\phi X_i) = 0$ for all $i = 1, \dots, 2n$. For each $\lambda > 0$ we choose the initial state on the orbit $P(O_\phi^\lambda)$:

$$P^\lambda_{x(0)} = U^\lambda_{x(0)} P_\phi (U^\lambda_{x(0)})^{-1} .$$

Here $x(0) \in \mathbb{R}^{2n}$ parametrizes the initial state. It will be fixed and λ will converge to 0.

For these initial states we shall investigate the time evolution of mean values of observables X_i^λ (which play the role of position and momentum). For a given value $\lambda > 0$ and Hamiltonian H^λ let us denote the quantal time evolution by

$$\varphi^{H^\lambda}_{t,\lambda}(P^\lambda_{x(0)}) = \exp\Big(\frac{-it}{\lambda^2\hbar}H^\lambda\Big) P^\lambda_{x(0)} \exp\Big(\frac{it}{\lambda^2\hbar}H^\lambda\Big).$$

The mean value of the observable X_i^λ in this evolved state will be $\mathrm{Tr}(\varphi^{H^\lambda}_{t,\lambda}(P^\lambda_{x(0)})X_i^\lambda)$. The result is that these values are for all $\lambda > 0$ identical to their classical counterparts $x_i(t)$.

Theorem 1. *Let $x(0), x(t)$ be the classical states defined by* (4), (5). *Let $\lambda > 0$ be arbitrary. Then*

$$\mathrm{Tr}\big(\varphi^{H^\lambda}_{t,\lambda}(P^\lambda_{x(0)})X_i^\lambda\big) = x_i(t) , \quad i = 1, \dots, 2n .$$

P r o o f . Let us first note that

$$\exp\Big(\frac{it}{\lambda^2\hbar}H^\lambda\Big) = \exp\Big(\frac{it}{\lambda^2\hbar}\lambda^2 H\Big) = \exp\Big(\frac{it}{\hbar}H\Big),$$

so

$$\varphi^{H^\lambda}_{t,\lambda} \equiv \varphi^H_{t,1}.$$

For the initial state $P^\lambda_{x(0)}$ the mean value of the observable X_i is

$$\mathrm{Tr}\big(P^\lambda_{x(0)}X_i\big) = \mathrm{Tr}\big(P_{\frac{x(0)}{\lambda}}X_i\big) = \frac{x_i(0)}{\lambda} .$$

Now we can use the well-known Ehrenfest theorem [4] for the quadratic Hamiltonian $H = H^1$ and the fact that for a quadratic Hamiltonian the right-hand side of Hamilton equations consists of linear functions. We obtain

$$\mathrm{Tr}\big(\varphi^{H^\lambda}_{t,\lambda}(P^\lambda_{x(0)})X_i\big) = \mathrm{Tr}\Big(\varphi^H_{t,1}\big(P_{\frac{x(0)}{\lambda}}\big)X_i\Big) = \frac{x_i(t)}{\lambda} .$$

So we have

$$\mathrm{Tr}\big(\varphi^{H^\lambda}_{t,\lambda}(P^\lambda_{x(0)})X_i^\lambda\big) = \lambda\,\mathrm{Tr}\big(\varphi^{H^\lambda}_{t,\lambda}(P^\lambda_{x(0)})X_i\big) = \lambda\frac{x_i(t)}{\lambda} = x_i(t)$$

which completes the proof.

Corollary. *It is obvious that*

$$\lim_{\lambda\to 0}\mathrm{Tr}\big(\varphi^{H^\lambda}_{t,\lambda}(P^\lambda_{x(0)})X_i^\lambda\big) = x_i(t) .$$

4 CLASSICAL PROJECTIONS

Another approach to the classical limit is by using the classical projections [1], [6], [7]. If H^λ_{q} is a oneparameter family of quantum Hamiltonians, the classical projection for a given value of λ is given by the Hamiltonian

$$h^\lambda_{\mathrm{cp}}(x) = \mathrm{Tr}(P^\lambda_x H^\lambda_{\mathrm{q}}) .$$

if h_{cl} is the corresponding classical limit, then it can be shown

$$h^\lambda_{\mathrm{cp}}(x) \xrightarrow[\lambda\to 0]{} h_{\mathrm{cl}}(x) .$$

In [6], an attempt is made to extend the convergence also to the dynamics. Unfortunately, there is a mistake in the proof of Theorem 2 in [6] but it seems that Theorem 3 in [6] will hold with some additional assumptions. The convergence of dynamics means that if for each $\lambda > 0$ and corresponding classical projection we have the same initial condition as for the classical limit then the time evolution of classical projection will converge to the time evolution of classical limit as $\lambda \to 0$.

We shall again consider only quadratic Hamiltonians given by (2). Given $\lambda > 0$ then the Hamiltonian of the classical projection is

$$h^\lambda_{\mathrm{cp}}(x) = \mathrm{Tr}(P^\lambda_x H^\lambda).$$

We will prove a

Lemma. *For each $\lambda > 0$*

$$(U^\lambda_x)^{-1}X_i^\lambda U^\lambda_x = X_i^\lambda + x_i I , \quad i = 1, \dots, 2n .$$

P r o o f . We have (see [1])

$$(U_x)^{-1}X_i U_x = X_i + x_i I .$$

Then

$$(U^\lambda_x)^{-1}X_i^\lambda U^\lambda_x = \big(U_{\frac{x}{\lambda}}\big)^{-1}\lambda X_i U_{\frac{x}{\lambda}} = \lambda\big(U_{\frac{x}{\lambda}}\big)^{-1}X_i U_{\frac{x}{\lambda}} = \lambda\Big(X_i + \frac{x_i}{\lambda}I\Big) = \lambda X_i + x_i I = X_i^\lambda + x_i I .$$

Then we have

Theorem 2. *Let h be given by (3), H^λ by (2), $\lambda > 0$. Then there exist a constant C_λ such that*

$$h^\lambda_{\mathrm{cp}}(x) = \mathrm{Tr}\big(P^\lambda_x H^\lambda\big) = h(x) + C_\lambda$$

and

$$\lim_{\lambda\to 0} C_\lambda = 0.$$

Hence

$$\lim_{\lambda\to 0} h^\lambda_{\mathrm{cp}}(x) = h(x).$$

P r o o f . It will be sufficient to prove the statement for $H^\lambda = X_i^\lambda X_j^\lambda$. We have (here the projector P_ϕ is the one specified in Section 2)

$$\begin{aligned} h^\lambda_{\rm cp}(x) &= \mathrm{Tr}\big(P^\lambda_x H^\lambda\big) = \mathrm{Tr}\big(U^\lambda_x P_\phi (U^\lambda_x)^{-1} X^\lambda_i X^\lambda_j\big) \\ &= \mathrm{Tr}\big(P_\phi (U^\lambda_x)^{-1} X^\lambda_i X^\lambda_j U^\lambda_x\big) \\ &= \mathrm{Tr}\big(P_\phi (U^\lambda_x)^{-1} X^\lambda_i U^\lambda_x (U^\lambda_x)^{-1} X^\lambda_j U^\lambda_x\big) \\ &= \mathrm{Tr}\big(P_\phi (X^\lambda_i + x_i I)(X^\lambda_j + x_j I)\big) \\ &= \mathrm{Tr}\big(P_\phi (\lambda X_i + x_i I)(\lambda X_j + x_j I)\big) \\ &= \lambda^2\, \mathrm{Tr}\big(P_\phi X_i X_j\big) + \lambda x_i\, \mathrm{Tr}\big(P_\phi X_j\big) + \lambda x_j\, \mathrm{Tr}\big(P_\phi X_i\big) \\ &\quad + x_i x_j\, \mathrm{Tr}\big(P_\phi\big) = x_i x_j + \lambda^2\, \mathrm{Tr}\big(P_\phi X_i X_j\big)\,. \end{aligned}$$

We can put $C_\lambda = \lambda^2\, \mathrm{Tr}\big(P_\phi X_i X_j\big)$.

As classical Hamiltonians differing only by a constant generate the same dynamics, the proof of the next Theorem is trivial. Let $x(0)$ and $x(t)$ be given by (4), (5); let $x^\lambda_{\rm cp}(0)$ be the initial state for the classical projection given by Hamiltonian $h^\lambda_{\rm cp}(x) = \mathrm{Tr}\big(P^\lambda_x H^\lambda\big)$ and $x^\lambda_{\rm cp}(t)$ the evolved state at time t.

Theorem 3. *Let* $\lambda > 0$ *be given and*

$$x(0) = x^\lambda_{\rm cp}(0)\,.$$

Then for arbitrary t

$$x(t) = x^\lambda_{\rm cp}(t)\,.$$

Corollary. *The limit transition* $\lim_{\lambda\to 0} x^\lambda_{\rm cp}(t) = x(t)$ *is satisfied trivially.*

5 NONLINEAR EVOLUTION

We shall look for the solutions of the following nonlinear Schroedinger equation [2, Section 3.5]:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \sum_{j=1}^{2n} \frac{\partial}{\partial F_j} h\big(\mathrm{Tr}(P_{\psi(t)}X_1), \ldots, \mathrm{Tr}(P_{\psi(t)}X_{2n})\big) X_j \psi(t) \quad (7)$$

where the state $\psi(0) \in \mathcal{H}$ at time $t = 0$ is an analytic vector [2] for the representation $U(G_{WH})$ given in Section 2, h is given by (3) and the quantities $\mathrm{Tr}(P_{\psi(t)}X_j)$ are inserted for the components F_j into

$$\frac{\partial h(F_1, \ldots, F_{2n})}{\partial F_j}, \quad j = 1, \ldots, 2n\,.$$

Let

$$\mathrm{Tr}\big(P_{\psi(0)}X_j\big) = x_j(0)\,, \quad j = 1, \ldots, 2n\,,$$

so $P_{\psi(0)} = P^1_{x(0)}$ in the notation of Section 2. (There exists a corresponding projector P_ϕ according to Section 2.) We obtain the following

Theorem 4. *The solution* $\psi(t)$ *of the equation* (7) *is such that the projector* $P_{\psi(t)} \in P(O^1_{\psi(0)})$ *is the unique one having the property*

$$\mathrm{Tr}(P_{\psi(t)}X_j) = x_j(t)\,, \quad j = 1, \ldots, 2n$$

where $x_j(t)$ *is determined by* (5).

P r o o f . According to [2, Theorem 3.5.1] in the corresponding notation it is sufficient to find the cocycle element $g_h(t, F(0)) \in G_{WH}$ where

$$F(0) = \mathbb{F}(P_{\psi(0)}) = (x_1(0), \ldots, x_{2n}(0), s_0)\,,$$

and $0 \neq s_0 \in R$ is a constant. Let φ^h_t denote the classical flow on $Lie(G_{WH})^*$ with Hamiltonian h. The abovementioned cocycle element is defined by the relation

$$\varphi^h_t F(0) = \mathrm{Ad}^*(g_h(t, F(0))F(0)\,.$$

We have

$$\varphi^h_t F(0) = (x_1(t), \ldots, x_{2n}(t), s_0)$$

where $x_j(t)$ be determined by (5) as φ^h_t is the usual Hamiltonian flow on the corresponding coadjoint orbit. According to (6) we obtain

$$g_h(t, F(0)) = g(q(t) - q(0), p(t) - p(0), s)$$

for some $s \in \mathbb{R}$, so according to [2, Theorem 3.5.1]

$$\psi(t) = U(g(q(t) - q(0), p(t) - p(0), s))\psi(0).$$

Since

$$\psi(0) = U(g(q(0), p(0), s'))\phi$$

where $s' \in \mathbb{R}$ and the vector ϕ determined in Section 2 has the property that $\mathrm{Tr}\big(P_\phi X_j\big) = 0$ for $j = 1, \ldots, 2n$, we obtain

$$\psi(t) = U(g(q(t), p(t), s''))\phi$$

where $s'' \in \mathbb{R}$ and finally

$$\mathrm{Tr}\big(P_{\psi(t)}X_j\big) = x_j(t)\,, \quad j = 1, \ldots, 2n\,.$$

R e m a r k . From Theorem 4 it follows that the nonlinear evolution considered can be formally identified in a natural sense with the classical evolution.

R e m a r k . If the orbit $O^1_{\psi(0)}$ contains an eigenvector of H then according to [1] the quantum evolution given by the Hamiltonian H lies in this orbit. If we use the Ehrenfest theorem, we obtain that this quantum evolution is identical with the nonlinear evolution considered here.

6 EVOLUTION PROJECTED BY MOMENTUM MAPPING

If we have a quantum Hamiltonian H (not necessarily quadratic), it generates a time evolution in $P(\mathcal{H})$. The space $P(\mathcal{H})$ is projected into $Lie(G_{WH})^*$ by the momentum mapping $\mathbb{F}$. More precisely, it is projected onto a coadjoint orbit of G_{WH} in $Lie(G_{WH})^*$. There arises a natural question: if the quantum flow after the projection by $\mathbb{F}$ gives a well defined "flow" in $Lie(G_{WH})^*$. More precisely: if $\mathbb{F}(P_\psi) = \mathbb{F}(P_\eta)$, does it follow that $\mathbb{F}(\varphi^H_{t,1}(P_\psi)) = \mathbb{F}(\varphi^H_{t,1}(P_\eta))$?

It can be shown that if H is quadratic then the answer is positive.

Theorem 5. *Let H be given by (1). Let $\psi, \eta \in \mathcal{H}$ be such that $\mathbb{F}(P_\psi) = \mathbb{F}(P_\eta)$. Then for arbitrary time t*

$$\mathbb{F}(\varphi^H_{t,1}(P_\psi)) = \mathbb{F}(\varphi^H_{t,1}(P_\eta))\,.$$

P r o o f . If we denote

$$\mathbb{F}(P_\psi) = \mathbb{F}(P_\eta) = F(0) = (F_1(0), \dots, F_{2n}(0), s_0)$$

where $0 \neq s_0 \in \mathbb{R}$, then

$$F_i(0) = \mathrm{Tr}(P_\psi X_i) = \mathrm{Tr}(P_\eta X_i) = x_i(0), \quad i = 1, \dots, 2n.$$

Let

$$\mathbb{F}(\varphi^H_{t,1}(P_\psi)) = F^\psi(t) = (F^\psi_1(t), \dots, F^\psi_{2n}(t), s_0)\,,$$

$$\mathbb{F}(\varphi^H_{t,1}(P_\eta)) = F^\eta(t) = (F^\eta_1(t), \dots, F^\eta_{2n}(t), s_0)\,.$$

Then a consequence of Ehrenfest theorem and of the fact that H is quadratic is that

$$F^\psi_i(t) = \mathrm{Tr}(\varphi^H_{t,1}(P_\psi) X_i) = x_i(t)\,, \quad i = 1, \dots, 2n\,,$$

$$F^\eta_i(t) = \mathrm{Tr}(\varphi^H_{t,1}(P_\eta) X_i) = x_i(t), \quad i = 1, \dots, 2n\,.$$

R e m a r k . From the proof it is clear that the "projected" evolution considered in the given coordinates of $Lie(G_{WH})^*$ can be formally identified with the corresponding classical evolution.

7 CONCLUSION

It was shown that for quadratic Hamiltonians various constructions of time evolutions can be in a natural way regarded as being identical. There arises a natural question if there exist some deeper reasons for this fact.

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