Journal of ELECTRICAL ENGINEERING, VOL. 51, NO. 12/s, 2000, 76–80

"SUBMERSIVITY" OF A SOLUTION OF THE EQUATION $z^{(n)}(t) + p_1(t)\, z^{(n-1)}(t) + \cdots + p_n(t)\, z(t) = \alpha \cdot q(t)$

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The main goal is to study certain properties of the solution of the differential equation (2) which are very appropriate for exploring the Property A. One can describe these properties verbally as the ability of the function not to overcome a certain level ε for a certain time interval $[t_0, t_0 + \delta]$.

The function having these properties behaves as follows: from a certain $t > t_0$, it dives under a certain level of ε and keeps being under this level maximally during a time interval δ. For the sake of brevity let us call this behavior of the function *"submersivity"*.

We have found a criterion of "submersivity" as the simplest possible conditions to be imposed on the left side of the equation (2).

Keywords: linear nonhomogeneous equation, estimation of a solution

2000 *Mathematics Subject Classification*: 34C10, 34C15

1 INTRODUCTION

A new idea in the investigation of oscillatory solutions has been brought by Kiguradze in [6]. In that paper the following equation was studied

$$u^{(n)}(t) + u^{(n-2)}(t) = f(t, u'(t), \dots, u^{(n-1)}(t)). \qquad (1)$$

Here $n \geq 3$ and $f\colon [a,\infty) \times R^n \to R$, $a \geq 0$, satisfies local Caratheodory conditions and has the sign property $f(t, x_0, x_1, \dots, x_{n-1}) \operatorname{sign} x_0 \leq 0$. In the case of an oscillatory left-hand side the question of oscillation of solutions of (1) had not been studied before [1]. The results given in it filled this gap to some extent. Consequently the oscillatory cases were studied in a few other papers. The results of this kind were presented only for the third order differential equations. See Došlá [2], Cecchi, Došlá, Marini [1], Greguš, Graef [3], Greguš, Gera, Graef, [4, 5].

"Submersivity" properties can help us to explore the questions of oscillation of solutions in the case of the oscillatory left-hand side operator.

The aim of this contribution is to study the properties of solutions of the n^{th}-order differential equation of the form

$$z^{(n)}(t) + p_1(t)\, z^{(n-1)}(t) + \cdots + p_n(t)\, z(t) = \alpha \cdot q(t) \qquad (2)$$

where $\alpha > 0$ and the functions $p_i(\cdot), q(\cdot) \in C[0,1]$ satisfy conditions

$$0 < q_{\min} \leq q(t) \leq q_{\max}\,, \qquad \forall t \in [0,1]\,, \qquad (3)$$

$$P(t) = \sum_{i=1}^{n} |p_i(t)| \leq P\,, \qquad \forall t \in [0,1]\,, \qquad (4)$$

Let us find a criterion of "submersivity" as the simplest possible conditions to be imposed on the left side of equation (2).

One can describe these properties verbally as the ability of the function not to overcome a certain level ε for a certain time interval $[t_0, t_0 + \delta]$.

The function having these properties behaves as follows: from a certain $t > t_0$, it dives under a certain level of ε and keeps being under this level maximally during a time interval δ. For the sake of brevity let us call this behavior of the function "submersivity".

This property is of major importance for exploring the questions about the oscillating and nonoscillating properties of a solution.

2 MATHEMATICAL PRELIMINARIES

If $\vec{x} = (x_1, x_2, \dots x_n) \in R^n$ then the norm

$$\|\vec{x}\| = \max_{i \in \{1,\dots,n\}} |x_i|\,.$$

Denote further by $\vec{u}(t)$ the vector $\vec{u}(t) = (u(t), u'(t), \dots, u^{(n-1)}(t))$. All the vectors in this paper are considered to be column vectors. Let the norm of $\vec{u}(t)$ be defined by

$$\|\vec{u}\| = \max_t \|\vec{u}(t)\| = \max_t (\max_j |u^{(j)}(t)|)\,.$$

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ISSN 1335-3632

3 "SUBMERSIVITY" OF THE SOLUTION OF THE EQUATION $u^{(n)}(t) = \alpha\, q(t)$.

First, we will study the "submersivity" properties of the equation $u^{(n)}(t) = \alpha\, q(t)$. These properties will be generalized to different types of the equations in the next sections.

Theorem 1. *Let $\alpha > 0$, $0 < q_{\min} \le q_{\max}$ be real constants. Then for each $p \in (0,1)$ there exists a constant ε, $0 < \varepsilon < 1$, such that for each $t_0 \in \mathcal{R}$, $\delta > 0$, and c, $\varepsilon > c > 0$, for all $q \in C[t_0, t_0+\delta]$ satisfying*

$$0 < q_{\min} \le q(t) \le q_{\max}\,, \quad \forall t \in [t_0, t_0+\delta]\,,$$

and for all solutions $u(\cdot) \in C^n[t_0, t_0+\delta]$ of the differential equation

$$u^{(n)}(t) = \alpha\, q(t)\,, \quad \alpha > 0\,,$$

with the property

$$-c \le u(t) \le u(t_0) \;\; \forall t \in [t_0, t_0+\delta]\,, \quad u(t_0) > 0\,,$$

the inequality

$$\mu\left(u^{-1}[-c, u(t_0)\,\varepsilon]\right) \le \delta\, p$$

holds, where μ denotes the Lebesgue measure of sets.

P r o o f . First we will consider the case $t_0 = 0$, $\delta = 1$ and $u(0) = 1$.

Let $0 < \varepsilon \le 1$ be given. Then the set $u^{-1}([0,\varepsilon])$ is closed, and if it is nonempty, then it consists of one-point sets $\{a_j\}$ and of intervals $[t_k, t_{k+1}]$.

Since $u^{(n)}(t) > 0$ in $[0,1]$, there exist at most $n-1$ zeros (counting their multiplicities) of u' in $[0,1]$. $u'(a_j) = 0$ and except the interval $[t_k, 1]$ (if such an interval exists), in each interval $[t_k, t_{k+1}]$ there exists a zero of u'. This implies that there exist at most n intervals $[t_k, t_{k+1}]$ where $0 \le u(t) \le \varepsilon$, $t_k \le t \le t_{k+1}$.

Let their number be m, $0 \le m \le n$. Suppose that

$$\mu\left(u^{-1}[0,\varepsilon]\right) = \tilde{p} > 0\,.$$

Then there exists an interval $[t_k, t_{k+1}]$ with the length d greater than or equal to $\frac{\tilde{p}}{m} \ge \frac{\tilde{p}}{n}$. Write $[t_k, t_{k+1}] = [t_{k_0}, t_{k_0}+d]$. Clearly, there exists a subinterval $[t_{k_n} + t_{k_n} + \frac{d}{4^n}] \subset [t_{k_0}, t_{k_0}+d]$ in which

$$\varepsilon \ge |u(t)| \ge \alpha\, q_{\min} \frac{d^n}{2^{n(n+1)}} \ge \alpha\, q_{\min} \frac{\tilde{p}^n}{n^n\, 2^{n(n+1)}}\,, \qquad t \in [t_{k_n} + t_{k_n} + \frac{d}{4^n}]\,.$$

Hence

$$\varepsilon = \alpha\, q_{\min} \frac{p^n}{n^n\, 2^{n(n+1)}} \implies \mu\left(u^{-1}[0,\varepsilon]\right) \le p\,. \tag{5}$$

In the general case we transform the solution $u(t)$ satisfying all assumptions of the theorem to the solution $u_1(s) = \frac{u(t_0+s\,\delta)}{u(t_0)}$ of the differential equation

$$u_1^{(n)}(s) = \frac{\alpha\,\delta^n}{u(t_0)}\, q(t_0 + s\,\delta) = \frac{\alpha\,\delta^n}{u(t_0)}\, q_1(s)\,, \quad \forall s \in [0,1]\,,$$

and u_1 satisfies the assumptions of our theorem in the case $t_0 = 0$, $\delta = 1$ and $u(0) = 1$. As $-c \le u_1(s) \le \varepsilon$ iff $-c\,u(t_0) \le u(t) \le u(t_0)\,\varepsilon$ at corresponding t, and the transformation $t = h(s) = t_0 + s\,\delta$ where $0 \le s \le 1$ has the property

$$\mu\left(h(A)\right) = \delta \mu(A)\,,$$

for a measurable A, by (5) we come to the implication in the variable t:

$$\varepsilon = \frac{\alpha\,\delta^n}{u(t_0)}\, q_{\min} \frac{p^n}{n^n\, 2^{n(n+1)}} \implies \mu\left(u^{-1}[-c, u(t_0)\,\varepsilon]\right) \le \delta\, p\,,$$

and this gives the statement of our theorem.

4 SOME ESTIMATIONS OF SOLUTIONS OF THE EQUATION $z^{(n)}(t) + p_1(t)\cdot z^{(n-1)}(t) + \dots + p_n(t)\cdot z(t) = q(t)$

Mark $\vec{u}(t) := (u(t), u'(t), \cdots, u^{(n-1)}(t))$ and $\|\vec{u}(t)\| = \max_t(\max_j |u^{(j)}(t)|)$

We will deal with the differential equation

$$z^{(n)}(t) + p_1(t)\cdot z^{(n-1)}(t) + \dots + p_n(t)\cdot z(t) = q(t)\,, \tag{6}$$

where the functions $p_i(\cdot), q(\cdot) \in C[0,1]$ and satisfy conditions

$$|q(t)| \le q_{\max}\,, \qquad \forall t \in [0,1]\,, \tag{7}$$

$$P(t) = \sum_{i=1}^{n} |p_i(t)| \le P\,, \qquad \forall t \in [0,1]\,. \tag{8}$$

Let M be the constant defined by

$$M = \sup\{\|\vec{r}(0)\| : r(\cdot) \in C^n[0,1],\; r^{(n)}(t) = 0 \text{ and } |r(t)| \le 1\,,\; \forall t \in [0,1]\}\,.$$

In the next theorem we will show that $\|\vec{z}(t)\|$, where $z(\cdot) \in C[0,1]$ solves the problem (6), (7) and (8), can be bounded by a constant, which depends only on constants $q_{\max}$, P, and M.

Theorem 2. *If the function $z(\cdot) \in C^n[0,1]$ is a solution of the differential equation* (6) *with the property*

$$0 < z(t) \le 1 \qquad \text{for all } t \in [0,1]\,, \tag{9}$$

where functions $p_i(\cdot), q(\cdot)$ satisfy conditions (7), (8)*. Let the constant P satisfy the inequality*

$$P\, e^{2+P} < \frac{1}{2M}\,,$$

Then there exist constant $z_{\max}$ defined by

$$z_{\max} := e(1+e^{1+P}P)[2M + q_{\max} + e^{1+P} 2M q_{\max}]\,, \tag{10}$$

with the following property:

$$\|\vec{z}(t)\| \le z_{\max} \quad \forall t \in [0,1]\,.$$

P r o o f . The proof will be divided into three parts. **1.** In the first part we find the estimation for $\|\vec{z}(0)\|$. Let us consider the auxiliary differential equation

$$w^{(n)}(t) = 0\,, \quad \vec{w}(0) = \vec{z}(0)\,, \tag{11}$$

whose solution is a function $w \in C^n[0,1]$. Clearly, there exists a positive constant M such that

$$\|\vec{w}(0)\| \le M \cdot \max_{t\in[0,1]} |w(t)|\,.$$

To obtain the estimation of $\|\vec{z}(0)\|$ we use the estimation of $\|\vec{w}(t) - \vec{z}(t)\|$.

If $w(\cdot)$ is a solution of the problem (11), then $w(\cdot)$ is the solution of the differential equation

$$\begin{aligned} &w^{(n)}(t) + p_1(t) w^{(n-1)}(t) + \cdots + p_n(t) w(t) \\ &\qquad = p_1(t) w^{(n-1)}(t) + \cdots + p_n(t) w(t)\,. \end{aligned} \tag{12}$$

The function $z(\cdot)$ is a solution of the equation (6) and $w(\cdot)$ is a solution of the equation (12), hence by (7) and (8) we immediately obtain

$$\begin{aligned} \|\vec{w}(t) - \vec{z}(t)\| &\le e^{\int_0^t (1+P(\tau))\mathrm{d}\tau} \Big[\int_0^t \|\vec{w}(\tau)\| P(\tau) + |q(\tau)| \mathrm{d}\tau \Big] \\ &\le e^{\int_0^t (1+P)\mathrm{d}\tau} \Big[\int_0^t \big(\|\vec{w}(\tau)\| P + q_{\max} \big) \mathrm{d}\tau \Big] \end{aligned}$$

and further

$$\|\vec{w}(t) - \vec{z}(t)\| \le e^{1+P} \Big[P \int_0^1 \|\vec{w}(\tau)\| \mathrm{d}\tau + q_{\max} \Big]\,. \tag{13}$$

And now, in order to obtain the estimation of $\|\vec{w}(t)\|$ we can use the following estimation of $\|\vec{w}(0)\|$. Since $w(\cdot)$ is the solution of (11) on $[0,1]$, we have

$$\|\vec{w}(t)\| \le \|\vec{w}(0)\|\, e^t\,, \quad \forall t \in [0,1]\,. \tag{14}$$

Applying (13) and (14) we find the estimation of $\|\vec{z}(0)\|$. Using the inequalities (9), (13) and (14) we can show that

$$\begin{aligned} \frac{\|\vec{w}(0)\|}{M} &\le \max_{t\in[0,1]} |w(t)| \le \max_{t\in[0,1]} |z(t)| + \|\vec{w}(t) - \vec{z}(t)\| \\ &\le 1 + e^{1+P} \Big[P \int_0^1 \|\vec{w}(\tau)\| \mathrm{d}\tau + q_{\max} \Big] \\ &\le 1 + e^{1+P} \left[P\, \|\vec{w}(0)\| e + q_{\max} \right]\,, \end{aligned}$$

which implies

$$\frac{\|\vec{w}(0)\|}{M} \le 1 + e^{2+P} P \|\vec{w}(0)\| + e^{1+P} q_{\max}\,.$$

By $P \cdot e^{2+P} \le \frac{1}{2M}$, after a simple reduction we get

$$\frac{\|\vec{w}(0)\|}{2M} \le \|\vec{w}(0)\| \Big(\frac{1}{M} - P\, e^{2+P} \Big) \le 1 + e^{1+P} q_{\max}\,.$$

By (11), it is obvious that

$$\|\vec{z}(0)\| = \|\vec{w}(0)\| \le 2M \left(1 + e^{1+P} q_{\max} \right) \tag{15}$$

holds.

2. In this part we will find the estimation of $\|\vec{z}(t) - \vec{v}(t)\|$ on the interval $[0,1]$. (The function $v(t)$ will be defined by (16). Let us consider the auxiliary problem

$$v^{(n)}(t) = q(t)\,, \quad \vec{v}(0) = \vec{z}(0)\,, \tag{16}$$

whose solution is a function $v \in C^{(n)}[0,1]$. The function $v(\cdot)$ also solves the differential equation

$$v^{(n)}(t) + \sum_{i=1}^n p_i(t)\, v^{(n-i)}(t) = \sum_{i=1}^n p_i(t)\, v^{(n-i)}(t) + q(t)\,. \tag{17}$$

As the function $z(\cdot)$ solves the equation (6) and the function $v(\cdot)$ solves the equation (17), we obtain

$$\begin{aligned} \|\vec{z}(t) - \vec{v}(t)\| &\le e^{\int_0^t (1+P(\tau))\mathrm{d}\tau} \int_0^t \|\vec{v}(\tau)\| P(\tau) \mathrm{d}\tau \\ &\le e^{1+P} P \int_0^t \|\vec{v}(\tau)\| \mathrm{d}\tau\,. \end{aligned} \tag{18}$$

Compare the solution of equation (16) with the zero solution of the equation

$$v_0^{(n)}(t) = 0\,, \quad \vec{v}_0(0) = \vec{0}\,. \tag{19}$$

Equation (19) can be put into the form

$$v_0^{(n)}(t) + p_1(t) v_0^{(n-1)}(t) + \cdots + p_n(t) v_0(t) = q(t)\,,$$

where $p_i(t) \equiv 0$, and therefore $P(t) = 0$. Again, we can derive the estimation:

$$\|\vec{v}(t) - \vec{v}_0(t)\| \le e^{\int_0^t \mathrm{d}\tau} \cdot (\|\vec{v}(0)\| + q_{\max}),$$
$$\text{i.e.,} \quad \|\vec{v}(t)\| \le e^t(\|\vec{v}(0)\| + q_{\max}).$$

Using the property $\|\vec{v}(0)\| = \|\vec{z}(0)\|$ and estimation (15) we obtain:

$$\|\vec{v}(t)\| \le e^t \left[2M(1 + q_{\max} \cdot e^{1+P}) + q_{\max}\right]. \tag{20}$$

3. Finally, we estimate $\|\vec{z}(t)\|$ on the interval $[0,1]$. In view of (18) and (20) we have

$$\|\vec{z}(t)\| \le \|\vec{v}(t)\| + \|\vec{z}(t) - \vec{v}(t)\| \le e^t 2M(1 + q_{\max}e^{1+P})$$
$$+ e^t q_{\max} + e^{1+P} P \int_0^t \|\vec{v}(\tau)\| \mathrm{d}\tau \le e^t 2M(1 + q_{\max}e^{1+P})$$
$$+ e^t q_{\max} + e^{1+P} P \int_0^t [2M(1 + q_{\max}e^{1+P})e^{\tau} + q_{\max}e^{\tau}] \mathrm{d}\tau.$$

Since $t \in [0,1]$, it is sufficient to put

$$z_{\max} := (e + (e-1)e^{1+P}P)[2M + q_{\max} + e^{1+P}2Mq_{\max}],$$

which completes the proof.

R e m a r k . If the constant P satisfies the inequality $P \cdot e^{2+P} \le \frac{1}{M+1}$, where M is a positive constant such that $\|\vec{w}(0)\| \le M \cdot \max_{t\in[0,1]} |w(t)|$ then instead of (15) we get

$$\|\vec{w}(0)\| \le M(M+1)[1 + e^{1+P}q_{\max}].$$

Then step by step we come to the inequality

$$z_{\max} = \left(e + (e-1)e^{1+P}P\right)\left[M(M+1) + q_{\max} + e^{1+P}M(M+1)q_{\max}\right].$$

R e m a r k . From the proof of Theorem 2 we can see that this theorem remains to be true when instead of the assumption

$$0 < z(t) \le 1 \quad \text{for all } t \in [0,1].$$

we suppose only that $|z(t)| < 1$ for all $t \in [0,1]$.

Theorem 3. *For each $q_{\max} > 0$ and each $\varepsilon > 0$ the constant $P > 0$ defined by*

$$P = \frac{\varepsilon}{e\, z_{\max}}, \tag{21}$$

where $z_{\max}$ is determined by (10), has the following property:

If the function $z(\cdot) \in C^n[0,1]$ is a solution of (6) such that

$$0 \le z(t) \le 1 \quad \text{for all } t \in [0,1], \tag{22}$$

where the functions $p_i(\cdot), q(\cdot) \in C[0,1]$ and satisfy conditions (7) and (8), then for the solution $u \in C^n[0,1]$ of the differential equation

$$u^{(n)}(t) = q(t), \quad \vec{u}(0) = \vec{z}(0) \tag{23}$$

we have

$$\|\vec{z}(t) - \vec{u}(t)\| \le \varepsilon, \quad \forall t \in [0,1].$$

P r o o f . Let the function $z(\cdot)$ solve the equation (6) and satisfy the assumption (22). We will show that P determined by (21) has the mentioned property. Let us rewrite the equation (6) into the form

$$z^{(n)}(t) = q(t) - p_1(t)z^{(n-1)}(t) - \dots - p_n(t)z(t). \tag{24}$$

Then we can compare the solution of (24) with the solution of (23). Apparently,

$$\|\vec{z}(t) - \vec{u}(t)\| \le e^{\int_0^t 1\mathrm{d}\tau} \int_0^t \left|p_1(\tau)z^{(n-1)}(\tau) - \cdots - p_n(\tau)z(\tau)\right| \mathrm{d}\tau \le e \int_0^t P\|\vec{z}(t)\| \mathrm{d}\tau.$$

Using the definition of P and the fact that the $\|\vec{z}(t)\| \le z_{\max}$ we obtain

$$\|\vec{z}(t) - \vec{u}(t)\| \le ePz_{\max} \le ez_{\max}\frac{\varepsilon}{ez_{\max}} = \varepsilon,$$

which proves the assertion.

5 "SUBMERSIVITY" OF A SOLUTION OF THE EQUATION $z^{(n)}(t) + p_1(t)z^{(n-1)}(t) + \dots + p_n(t)z(t) = \alpha \cdot q(t)$.

In this section, we we will study the "submersivity" properties of the differential equation (2) where $\alpha > 0$ and the functions $p_i(\cdot), q(\cdot) \in C[0,1]$ satisfy conditions

$$0 < q_{\min} \le q(t) \le q_{\max} \qquad \forall t \in [0,1],$$
$$P(t) = \sum_{i=1}^{n} |p_i(t)| \le P \qquad \forall t \in [0,1],$$

Theorem 4. *For arbitrary constants $q_{\min}$ and $q_{\max}$, such that $0 < q_{\min} \le q_{\max}$, and $\alpha_{\max} > 0$, $p \in (0,1)$, there exist $P > 0$ and $\varepsilon \in (0,1)$ with the following property:*

If $z(\cdot) \in C^n[0,1]$ is a solution of (2) such that

$$0 \le z(t) \le 1 \quad \text{for all } t \in [0,1], \quad \text{and } z(0) = 1, \tag{25}$$

where $p_i(\cdot), q(\cdot) \in C[0,1]$ satisfy conditions (3), (4) and $0 < \alpha \le \alpha_{\max}$, then

$$\mu(z^{-1}[0,\varepsilon]) \le p.$$

P r o o f . Denote by $Q_{\max}=\alpha_{\max}q_{\max}$ and choose ε_1 satisfying

$$2\varepsilon_1<\frac{\alpha}{1-\varepsilon_1}q_{\min}\frac{p^n}{n^n2^{n(n+1)}}<1-\varepsilon_1\,. \tag{26}$$

Theorem 3 ensures for constants $Q_{\max}$ and ε_1 chosen by (26) the existence of a constant $P>0$, defined by $P=\frac{\varepsilon_1}{ez_{\max}}$ such that:

If $z(\cdot)$ is the solution of the equation (2) satisfying (25), for which the properties (3), (4) are true, then

$$\|\vec{z}(t)-\vec{u}(t)\|\le\varepsilon_1\quad\forall t\in[0,1]\,,$$

where $u(\cdot)\in C^n[0,1]$ solves the equation $u^{(n)}(t)=\alpha q(t)$, $\vec{u}(0)=\vec{z}(0)$ on the interval $[0,1]$.

Let $u_1\in C[0,1]$ be defined by $u_1(t)=u(t)-\varepsilon_1\ \ \forall t\in[0,1]$. Then $u_1(\cdot)$ satisfies on the interval $[0,1]$ the equation

$$\begin{aligned}u_1^{(n)}(t)&=\alpha q(t)\,,\quad u_1(0)=1-\varepsilon_1\,,\\ u_1^{(i)}(0)&=z^{(i)}(0)\,,\ \text{for } i=1,\dots,n-1\,.\end{aligned}\tag{27}$$

Moreover, the following estimation

$$\begin{aligned}\|\vec{z}(t)-\vec{u}_1(t)\|\le\|\vec{z}(t)-\vec{u}(t)\|&+\|\vec{u}(t)-\vec{u}_1(t)\|\\ &\le\varepsilon_1+\varepsilon_1\le2\varepsilon_1\end{aligned}$$

holds. Since $\min\limits_{t\in[0,1]}z(t)\ge0$ we have

$$-2\varepsilon_1\le u_1(t)\le z(t)\,,\quad\forall t\in[0,1]\,.$$

Since $u_1^{(n)}(t)=\alpha q_{\min}>0$ in $[0,1]$ there exist at most $n-1$ zeros (counting their multiplicities) of u_1' in $[0,1]$. Hence $u_1(t)$ has at most $n-1$ extremal points in $[0,1]$. It follows that there exists a finite number, say l, of intervals such that

$$u_1(t)\le1-\varepsilon_1\,,\quad\forall t\in[t_i,t_{i+1}]\,,\quad i=1,\dots,l-1\,. \tag{28}$$

Note, that due to (27) at least one such interval exists.

Put $M=\{t;\ t\in\bigcup\limits_{i=1}^{l}[t_i,t_{i+1}]\}$, where the intervals $[t_i,t_{i+1}]$ are defined by (28). We thus get

$$-2\varepsilon_1\le u_1(t)\le1-\varepsilon_1=u_1(0)\,,\quad\forall t\in M\,. \tag{29}$$

Due to (29) $u_1(\cdot)$ satisfies on M the assumptions of Theorem 1. Therefore for all $p\in(0,1)$ there exists $\varepsilon_2=\frac{\alpha}{1-\varepsilon_1}q_{\min}\frac{p^n}{n^n2^{n(n+1)}}$ such that

$$\mu\big(u_1^{-1}[-2\varepsilon_1\,,\,u_1(0)\varepsilon_2]\cap M\big)\le p\,.$$

By (26) we have $-2\varepsilon_1\le\varepsilon_2\le1-\varepsilon_1$. From this we obtain

$$\begin{aligned}\mu\big(u_1^{-1}[-2\varepsilon_1\,,\,u_1(0)\varepsilon_2]\big)=\mu\big(u_1^{-1}&[-2\varepsilon_1\,,\,u_1(0)\varepsilon_2]\cap M\big)\\ &+\mu\big(u_1^{-1}[-2\varepsilon_1\,,\,u_1(0)\varepsilon_2]\cap([0,1]-M)\big)\,,\end{aligned}$$

and since $\mu\big(u_1^{-1}[-2\varepsilon_1\,,\,u_1(0)\varepsilon_2]\cap([0,1]-M)\big)=0$ we have

$$\mu\big(u_1^{-1}[-2\varepsilon_1\,,\,u_1(0)\varepsilon_2]\big)=\mu\big(u_1^{-1}[-2\varepsilon_1\,,\,u_1(0)\varepsilon_2]\cap M\big)\le p\,.$$

If we put $\varepsilon:=(1-\varepsilon_1)\varepsilon_2$ then using the previous result we obtain

$$\begin{aligned}\mu\big(z^{-1}[0,\varepsilon]\big)&=\mu\big(z^{-1}[0,(1-\varepsilon_1)\varepsilon_2]\big)\\ &=\mu\big(z^{-1}[0,u_1(0)\varepsilon_2]\big)\le\mu\big(u_1^{-1}[-2\varepsilon_1,u_1(0)\varepsilon_2]\big)\le p\,,\end{aligned}$$

which proves the claim.

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Received 23 June 2000

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