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THE McKAY–MILLER–ŠIRÁŇ GRAPHS AND LIFTS OF AUTOMORPHISMS

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McKay, Miller, Širáň [2] discovered a family of vertex-transitive graphs G_d of degree $d = (3q-1)/2$ for prime powers $q \equiv 1 \pmod 4$, diameter 2, and order $\frac{8}{9}(d+\frac{1}{2})^2$. The graphs G_d are the currently largest known vertex-transitive graphs of diameter 2 and given degree d. They were originally constructed as lifts of the graphs $K^*_{q,q}$ obtained from the complete bipartite graphs $K_{q,q}$ by attaching $\frac{q-1}{4}$ loops to each vertex. Recently, the author simplified the construction of the graphs G_d by showing that they can be obtained as regular covers of the graphs D^*_q, the q-dipoles with $\frac{q-1}{4}$ loops at each of the two vertices.

In this contribution we show that certain automorphisms of the graphs G_d are not lifts of the automorphisms of D^*_q.

K e y w o r d s: regular coverings of graphs; voltage assignments; lifts of automorphisms.

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INTRODUCTION

The McKay-Miller-Širáň graphs [2] are the currently largest known vertex-transitive graphs of diameter 2 and degree d for infinitely many values of d. They were found as regular covering spaces of certain complete bipartite graphs with loops. One way of proving vertex transitivity of the graphs of McKay, Miller and Širáň is to lift suitable automorphisms of the complete bipartite graphs.

Observing that the special complete bipartite graphs regularly cover certain two vertex graphs and using a necessary and sufficient condition for regularity of a composition of two regular coverings [3], a simplified construction of the McKay-Miller-Širáň graphs was presented in [4]. The aim of this note is to show that, paradoxically, vertex-transitivity of these graphs cannot be proved by lifting automorphisms of the dipoles.

BACKGROUND

For an undirected graph G let $D(G)$ be the set of all darts of G. (A dart is an edge with a chosen direction.) If Γ is a group, a mapping $\alpha\colon D(G) \to \Gamma$ is a *voltage assignment* on G if $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for each dart $e \in D(G)$. The *lift* of G with respect to α, denoted by G^α, is a graph with vertex set $V(G^\alpha) = V(G) \times \Gamma$ and dart set $D(G^\alpha) = D(G) \times \Gamma$. A dart e_g in G^α emanates from u_g and terminates at v_h if and only if e is a dart in G from u to v, and $h = g\alpha(e)$. The reverse of the dart e_g is the dart $(e^{-1})_{g\alpha(e)}$. This pair of darts form an undirected edge in G^α, and so the lift is undirected. The *natural projection* $\pi\colon G^\alpha \to G$ such that $\pi(u_g) = u$ and $\pi(e_g) = e$ for each $u \in V(G)$, $e \in D(G)$, and $g \in \Gamma$ is a graph homomorphism. This homomorphism is called a *covering projection*.

A sequence $W = e_1 e_2 \dots e_k$ of darts e_i of G is a *walk* in G of length k if the terminal vertex of e_{i-1} is the same as the initial vertex of e_i, $2 \le i \le k$. If α is a voltage assignment on G, then the product $\alpha(W) = \alpha(e_1)\alpha(e_2)\dots\alpha(e_m)$ is the *net voltage* on W.

Let A be an automorphism of G and let $\pi\colon G^\alpha \to G$ be a covering projection. An automorphism A' of G^α is a *lift* of A if $A' \circ \pi = \pi \circ A$. It was proved in [1] that an automorphism A of G lifts to an automorphism A' of G^α if and only if for each closed walk W in G,

$$\alpha(W) = 1_\Gamma \Leftrightarrow \alpha(AW) = 1_\Gamma\,. \tag{1}$$

Lifts of automorphisms are also useful for determining regularity of a composition of two coverings. Let β be a voltage assignment on the lift G^α in a group Γ'. Assuming that both G' and $(G^\alpha)^\beta$ are connected, we know by [3] that the composition $\pi_\beta \circ \pi_\alpha$ is regular if and only if for every covering transformation A of π_α (that is, an automorphisms of G such that $\pi(A(e)) = \pi(e)$ for each dart $e \in E(G)$) and for each closed walk W in the graph G^α we have

$$\beta(W) = 1_{\Gamma'} \Leftrightarrow \beta(AW) = 1_{\Gamma'}\,. \tag{2}$$

With help of voltage assignments, McKay, Miller and Širáň described in [2] the currently largest graphs G_d of diameter 2 and given degree d, for $d = \frac{3q-1}{2}$ where q is a prime power congruent with $1 \pmod 4$. The graph G_d is a lift of the base graph $K^*_{q,q}$, the complete bipartite

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graph with $2q$ vertices and $\frac{q-1}{4}$ loops at each vertex. The voltage assignment on $K^*_{q,q}$ in the additive group $GF(q)$ is as follows. If u_g and v_h, $g,h \in GF(q)$ are vertices in the two parts of $K^*_{q,q}$, then the dart e from u_g to v_h has voltage $g \cdot h$. Let ξ be a primitive element of $GF(q)$ and $X = \{1, \xi^2, \xi^4, \dots, \xi^{2l-2}\}$, $Y = \{\xi, \xi^3, \dots, \xi^{2l-1}\}$, where $l = \frac{q-1}{4}$. To the loops at each vertex u_g (v_h) we bijectively assign voltages from X (Y), respectively.

In [4] the above construction was simplified in the following way. The graph $K^*_{q,q}$ regularly covers D^*_q, a dipole with $\frac{q-1}{4}$ loops at both vertices. By the necessary and sufficient condition of [3], $G_d \to K^*_{q,q} \to D^*_q$ is a regular covering. The corresponding voltage assignment α is in the additive group $GF(q) \times GF(q)$. To describe it, let u, v be the two vertices of D^*_q. Let e_g, $g \in GF(q)$ denote the darts from u to v and let a_i, b_j, $i,j \in \{1,2,\dots \frac{q-1}{4}\}$, be the loops at u, v, respectively. Then we define $\alpha(e_g) = (g, g^2)$, $\alpha(a_i) = (0, \xi^{2i})$, $\alpha(b_j) = (0, \xi^{2j-1})$. It can be shown that the lift $(D^*_q)^\alpha$ is isomorphic to the McKay-Miller-Širáň graphs G_d.

THE RESULT

In [2] it was proved that the graphs G_d are vertex transitive; as we mentioned before, this can be done by lifting suitable automorphisms of $K^*_{q,q}$ to G_d. It is therefore of interest to see if automorphisms of the dipole D^*_q lift to G_d in a similar manner. Very surprisingly, the answer is negative.

Theorem 1. *Let A be an automorphism of the graph D^*_q such that $A(u) = v$. Then A does not lift onto $(D^*_q)^\alpha$.*

Proof. Assume that A is an automorphisms of D^*_q which lifts onto $(D^*_q)^\alpha$, such that $A(u) = v$.

We distinguish two cases, depending on whether the element $2 \in GF(q)$ is an even or an odd power of the primitive element ξ.

(a) $2 \in X \cup (-X) = \{\xi^2, \xi^4, \dots, \xi^{4l-2}, \xi^{4l} = 1\}$. Let $g \in GF(q)$ be such that $-2g^2 \in X$; for example, as $\xi^{2l} = -1$, we may take for g either the element 1 or ξ^l, according as -2 or 2 is in X. Let U be a closed walk in D^*_q of the form $U = e_g e_0^{-1} e_{-g} e_0^{-1} a_i$ where the index i, $1 \le i \le l$ is chosen in such a way that $\xi^{2i} = -2g^2$. The choice of U guarantees that $\alpha(U) = (0,0)$, and by the assumed existence of a lift and condition (1) we have $\alpha(AU) = (0,0)$ as well. To evaluate $\alpha(AU)$, let $A(e_0) = e_h^{-1}$, $A(e_g) = e_r^{-1}$, $A(e_{-g}) = e_s^{-1}$ for suitable $h, r, s \in GF(q)$, and let $A(a_i) = b_j^\varepsilon$ for some j, $1 \le j \le l$, and $\varepsilon \in \{-1, +1\}$. Then $AU = e_r^{-1} e_h e_s^{-1} e_h b_j^\varepsilon$. Observe that $\alpha(b_j^\varepsilon) = (0,t)$ for some $t \in Y \cup (-Y) = \{\xi, \xi^3, \dots, \xi^{4l-1}\}$. We therefore have

$$0 = \alpha(AU) = 2(h, h^2) - (r, r^2) - (s, s^2) + (0, t).$$

This gives the two equations

$$r + s = 2h \quad \text{and} \quad r^2 + s^2 = 2h^2 + t.$$

Raising the first equation to the power of two and subtracting the result from the second equation multiplied by two, we obtain

$$(r-s)^2 = 2t.$$

As both $(r-s)^2$ and 2 belong to $X \cup (-X)$, it follows that t must be an even power of 2, which contradicts the fact that $t \in Y \cup (-Y)$.

(b) $2 \in Y \cup (-Y)$. The procedure is similar to the preceding one, except that now U will have the form $U = e_g e_0^{-1} e_{-g} b_j e_0^{-1}$ where b_j is a directed loop at v such that $\alpha(b_j) = -2g^2 = \xi^{2j-1}$ for a suitable j, $1 \le j \le l$, and $g \in GF(q)$; again, one of $g = 1$ or $g = \xi^l$ will do it. This time we will have $A(b_j) = a_i^\varepsilon$ with $\alpha(a_i^\varepsilon) = (0,t)$ for some $t \in X \cup (-X)$. Keeping the remaining notation as above, we will again arrive at the equation $(r-s)^2 = 2t$, and as now 2 is an odd power of ξ, we must have the same for t, contrary to $t \in X \cup (-X)$.

The identity automorphism of D^*_q lifts to covering transformations which act on the lift $(D^*_q)^\alpha$ with two orbits. By [2] we know, however, that there exist automorphisms of the lift which interchange the two orbits; our result shows on the other hand that these automorphisms cannot be obtained as lifts. A deeper understanding of the ways such automorphisms can arise (and their full characterization) remains open.

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