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PSEUDO–EFFECT ALGEBRAS

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As a non-commutative generalization of effect algebras, we introduce pseudo-effect algebras. We list some of their basic properties, and we introduce a property of Riesz type in a similar manner as known for *po-groups*. We then show that any Riesz pseudo-effect algebra is representable by a unit interval of some *po-group*.

K e y w o r d s: pseudo-effect algebra, effect algebra, *po-groups*, interval effect algebra

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0 INTRODUCTION

In the field of quantum structures, various algebras originating from Hilbert space have been studied, so as to give way to a better understanding of the quantum-mechanical formalism. The probably most prominent one is the orthomodular lattice of closed subspaces of the standard Hilbert space.

In recent times, the attention moved to another part of the Hilbert space; instead of the closed subspaces, which correspond to the projection operators, all the positive operators lying below the identity, called (quantum) effects, have been taken into consideration. To model this set, several axiom systems were proposed, among which we find the one corresponding to effect algebras [6].

In this paper, we shall generalize the latter structure. We basically keep all axioms of effect algebras, but we drop commutativity.

We are working towards a structure theory of the new kind of algebra. Since not much is known here even for effect algebras, we assume a certain further property comparable to the Riesz Decomposition Property as known for *po-groups*. It is then in fact possible to show that our algebras are representable by intervals of — not necessarily abelian — *po-groups*.

Most of the proofs are not shown in this paper; they may be found in [4, 5].

1 PSEUDO–EFFECT ALGEBRAS

We introduce the notion of a pseudo-effect algebra. The axioms are similar to those of effect algebras; but the axiom of commutativity is dropped.

Definition 1.1. A structure $(E;+,0,1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* if, for all $a,b,c \in E$, the following holds.

(E1) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$.

(E2) There is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$.

(E3) If $a+b$ exists, there are elements $d,e \in E$ such that $a+b = d+a = b+e$.

(E4) If $1+a$ or $a+1$ exists, then $a=0$.

In view of (E2), we may define the two unary operations $^\sim$ and $^-$ by requiring for any $a \in E$

(EC) $\qquad a + a^\sim = a^- + a = 1$.

It is clear that effect algebras are exactly those pseudo-effect algebras $(E;+,0,1)$ in which the following holds for any $a,b \in E$:

(A) $a+b$ exists if and only if $b+a$ exists, in which case $a+b = b+a$.

In the sequel, by any sentence to hold we mean: All terms that occur in it are defined, and it holds. Furthermore, we will denote finite sums usually without brackets, which is justified by axiom (E1).

Lemma 1.2. *Let $(E;+,0,1)$ be a pseudo-effect algebra. For all $a,b,c \in E$ we have the following.*

(i) $a+0 = 0+a = a$.

(ii) $a+b=0$ *implies* $a=b=0$.

(iii) $0^\sim = 0^- = 1$, $1^\sim = 1^- = 0$.

(iv) $a^{\sim -} = a^{-\sim} = a$.

(v) $a+b = a+c$ *implies* $b=c$*; and* $b+a = c+a$ *implies* $b=c$.

(vi) $a+b=c$ *iff* $a = (b+c^\sim)^-$ *iff* $b = (c^- + a)^\sim$.

In the usual way, pseudo-effect algebras may be partially ordered.

Definition 1.3. Let $(E;+,0,1)$ be a pseudo-effect algebra. We define for $a,b \in E$

$$a \le b \quad \text{iff} \quad a+c=b \text{ for some } c \in E\,.$$

Lemma 1.4. *Let $(E;+,0,1)$ be a pseudo-effect algebra. The following holds in E for all $a,a_1,b,b_1,c \in E$.*

(i) *$\le$ is a partial order on E. E is, by $\le$, naturally ordered, i.e. $a \le b$ iff $a+c=b$ for some $c \in E$ iff $d+a=b$ for some $d \in E$.*

(ii) $a \le b$ *iff* $b^- \le a^-$ *iff* $b^\sim \le a^\sim$.

(iii) $a+b$ *exists iff* $a \le b^-$ *iff* $b \le a^\sim$.

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(iv) *If $b+c$ exists, then $a \le b$ if and only if $a+c$ exists and $a+c \le b+c$. If $c+b$ exists, then $a \le b$ if and only if $c+a$ exists and $c+a \le c+b$.*

2 INTERVAL PSEUDO–EFFECT ALGEBRAS

We are interested in pseudo-effect algebras that arise from intervals in partially ordered groups in the following manner.

Definition 2.1. Let $(G;+,\le)$ be a *po-group* and u a positive element of G. We denote by (G,u) the structure $(G;+,\le,u)$, i.e. we add the element u as a constant. (G,u) is called a *unital po-group* if u is a strong unit of G.

We call the set

$$\Gamma(G,u) \overset{\text{def}}{=} \{g \in G^+ : g \le u\}$$

the *unit interval* of (G,u). We denote by $(\Gamma(G,u);+,0,u)$ the structure consisting of the unit interval of (G,u), the partial binary operation $+$ that is the restriction of the group addition to those pairs of elements of $\Gamma(G,u)$ whose sum lies again in $\Gamma(G,u)$, the neutral element of G, 0, and the positive element u.

As it is easily checked, $(\Gamma(G,u);+,0,u)$ is a pseudo-effect algebra. For $g \in \Gamma(G,u)$ we have here $g^{\sim} = -g+u$ and $g^{-} = u-g$. Furthermore, it is clear that the order defined for $(\Gamma(G,u);+,0,u)$ coincides with the order of the *po-group* G restricted to $\Gamma(G,u)$.

Definition 2.2. A pseudo-effect algebra $(E;+,0,1)$ is called an *interval pseudoeffect algebra* if there is a unital *po-group* (G,u) such that
$(E;+,0,1)$ and $(\Gamma(G,u);+,0,u)$ are isomorphic.

An example of a non-commutative *po-group* leading to a non-commutative pseudo-effect algebra is the following [3, Example 4.1].

Example 2.3. Let $G = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$; define for every two elements of G

$$(a_1,b_1,c_1)+(a_2,b_2,c_2) \overset{\text{def}}{=} \begin{cases} (a_1+a_2, b_1+b_2, c_1+c_2) & \text{if } a_2 \text{ is even,} \\ (a_1+a_2, b_2+c_1, b_1+c_2) & \text{if } a_2 \text{ is odd;} \end{cases}$$

and define $(a_1,b_1,c_1) \le (a_2,b_2,c_2)$ to hold if $a_1 < a_2$ or $a_1 = a_2$, $b_1 \le b_2$ and $c_1 \le c_2$. Then $(G;+,\le)$ is an ℓ-group, and

$$\Gamma(G,(1,0,0)) = \{(0,b,c) : b,c \ge 0\} \cup \{(1,b,c) : b,c \le 0\}$$

becomes a pseudo-effect algebra with the sum and constants defined according to Definition 2.1. Both structures are non-commutative, since e.g. $(0,1,2)+(1,-2,-2) = (1,0,-1)$, but $(1,-2,-2)+(0,1,2) = (1,-1,0)$.

3 RIESZ PROPERTIES FOR PSEUDO–EFFECT ALGEBRAS AND *po*–GROUPS

For the purpose of a structure theory, we now introduce a Riesz-like property for pseudo-effect algebras.

Definition 3.1. Let $(E;+,0,1)$ be a pseudo-effect algebra.

(a) For $a,b \in E$, we write $a \,\mathbf{com}\, b$ to mean that for all $a_1 \le a$ and $b_1 \le b$, a_1 and b_1 commute.

(b) We say that E fulfils the *Commutational Riesz Decomposition Property*, (RDP_c) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1+a_2 = b_1+b_2$ there are $d_1,d_2,d_3,d_4 \in E$ such that

(i) $d_1+d_2 = a_1$, $d_3+d_4 = a_2$,
$d_1+d_3 = b_1$, $d_2+d_4 = b_2$,

(ii) $d_2 \,\mathbf{com}\, d_3$.

A pseudo-effect algebra fulfilling (RDP_c) will be called a *Riesz pseudo-effect algebra* in the sequel.

R e m a r k 3.2. Clearly, the **com**-relation is symmetric. Furthermore, it is easy to see that when assuming (RDP_c), it is also additive: For any elements a,b,c of a Riesz pseudo-effect algebra such that $a+b$ exist, from $a \,\mathbf{com}\, c$ and $b \,\mathbf{com}\, c$ it follows $a+b \,\mathbf{com}\, c$. (RDP_c) is equivalent to a stronger version of Riesz property, involving any finite number of elements rather than just pairs of two.

Lemma 3.3. *Let $(E;+,0,1)$ be a Riesz pseudo-effect algebra. Let*

$$a_1+\cdots+a_m = b_1+\cdots+b_n\,,$$

where $m,n \ge 1$. Then there are $d_{11},\dots,d_{1n},\dots,$ $d_{m1},\dots,d_{mn} \in E$ such that

(i) *$d_{i1}+\cdots+d_{in} = a_i$ for $i=1,\dots,m$ and*
$d_{1j}+\cdots+d_{mj} = b_j$ for $j=1,\dots,n$,

(ii) *for $1 \le i < m$, $1 \le j < n$ we have*
$d_{i+1,j}+\cdots+d_{mj} \,\mathbf{com}\, d_{i,j+1}+\cdots+d_{in}$.

P r o o f . Similar to the proof of [7, Theorem V.1].

In exact analogy to the case of pseudo-effect algebras, we define the Commutational Riesz Decomposition Property also for *po-groups*. The Riesz Decomposition Property of *po-groups*, as it is known from literature, is usually defined similarly, but requires just condition (i) of Definition 3.4 (b) to hold.

Definition 3.4. Let $(G;+,\le)$ be a directed *po-group*.

(a) For $a,b \ge 0$, we write $a \,\mathbf{com}\, b$ to mean that for all a_1,b_1 such that $0 \le a_1 \le a$ and $0 \le b_1 \le b$, a_1 and b_1 commute.

(b) We say that G fulfils the *Commutational Riesz Decomposition Property*, (RDP_c) for short, if for any $a_1,a_2,b_1,b_2 \ge 0$ such that $a_1+a_2 = b_1+b_2$ there are $d_1,d_2,d_3,d_4 \ge 0$ such that

(i) $d_1+d_2 = a_1$, $d_3+d_4 = a_2$,
$d_1+d_3 = b_1$, $d_2+d_4 = b_2$,

(ii) $d_2 \,\mathbf{com}\, d_3$.

Proposition 3.5. *Any ℓ-group fulfils (RDP_c).*

P r o o f . Given elements $a_1,a_2,b_1,b_2 \ge 0$ of an ℓ-group such that $a_1+a_2 = b_1+b_2$, there are $d_1,d_2,d_3,d_4 \ge 0$ such that (i) $d_1+d_2 = a_1$, $d_3+d_4 = a_2$, $d_1+d_3 = b_1$, $d_2+d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$ by [7, Theorem V.1]. Two elements of ℓ-groups whose infimum is 0 commute by [2, XIII, §3, Eq. (13)]. It follows that (RDP_c) holds.

4 REPRESENTATION OF PSEUDO–EFFECT ALGEBRAS BY UNIT–INTERVALS OF GROUPS

We present in this section our main result; we will show that any Riesz pseudo-effect algebra is an interval pseudo-effect algebra.

We make use of the word technique, as introduced by Baer [1] and Wyler [9], which was also successfully applied to effect algebras with a Riesz-like Property [8]. In a first step, we embed a given pseudo-effect algebra into a semigroup. The semigroup will then, in a second step, be extended to a *po-group*.

Definition 4.1. Let $(E;+,0,1)$ be a pseudo-effect algebra.

(i) A sequence $A = (a_1,\dots,a_n)$ of finite, but non-zero, length with entries from E is called a *word* in E. We denote by $\mathcal{W}(E)$ the set of all words; that is $\mathcal{W}(E) \stackrel{\text{def}}{=} \{(a_1,\dots,a_n)\colon a_1,\dots,a_n \in E,\ n \geq 1\}$. We define an *addition* in $\mathcal{W}(E)$ as the concatenation; that is $+\colon \mathcal{W}(E)\times\mathcal{W}(E) \to \mathcal{W}(E)$, $\big((a_1,\dots,a_m),(b_1,\dots,b_n)\big) \mapsto (a_1,\dots,a_m,b_1,\dots b_n)$.

(ii) We call two words A and B of E *directly similar*, in symbols $A \sim B$, if one of it has the form $(a_1,\dots,a_n)$, $n \geq 2$, and the other one has the form $(a_1,\dots,a_p + a_{p+1},\dots,a_n)$, $1 \leq p < n$.

We call two words A and B *similar*, in symbols $A \simeq B$, if there are words $A_0,\dots,A_k$, $k \geq 0$, such that $A = A_0 \sim A_1 \sim \cdots \sim A_k = B$. In such a case we say that A and B are *connected by a chain of length* k.

We set for $a_1,\dots,a_n \in E$, $n \geq 1$, $[a_1,\dots,a_n] \stackrel{\text{def}}{=} \{A \in \mathcal{W}(E)\colon A \simeq (a_1,\dots,a_n)\}$, and we put $\mathcal{C}(E) \stackrel{\text{def}}{=} \{[a_1,\dots,a_n]\colon a_1,\dots,a_n \in E, n \geq 1\}$.

Lemma 4.2. *Let $(E;+,0,1)$ be a Riesz pseudo-effect algebra.*

Then similarity in $\mathcal{W}(E)$ is an equivalence relation compatible with $+$. $+$ being the induced relation, $(\mathcal{C}(E);+)$ is a semigroup in which the following holds for any $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c} \in \mathcal{C}(E)$.

(i) *$[0]$ is a neutral element.*

(ii) *From $\mathfrak{a}+\mathfrak{b} = [0]$ it follows $\mathfrak{a} = \mathfrak{b} = [0]$.*

(iii) *From $\mathfrak{a}+\mathfrak{b} = \mathfrak{a}+\mathfrak{c}$ it follows $\mathfrak{b} = \mathfrak{c}$; from $\mathfrak{b}+\mathfrak{a} = \mathfrak{c}+\mathfrak{a}$ it follows $\mathfrak{b} = \mathfrak{c}$.*

(iv) *There is a $\mathfrak{d} \in \mathcal{C}(E)$ such that $\mathfrak{a}+\mathfrak{b} = \mathfrak{d}+\mathfrak{a}$, and there is an $\mathfrak{e} \in \mathcal{C}(E)$ such that $\mathfrak{a}+\mathfrak{b} = \mathfrak{b}+\mathfrak{e}$.*

Sketch of proof. Evidently, $\mathcal{C}(E)$ is well-defined as a semigroup. (i) is also obvious.

(ii) For any word $\mathfrak{x} = [x_1,\dots,x_n] \in \mathcal{C}(E)$, it follows from $\mathfrak{x} = [0]$ by induction on the length of the chain by which $(x_1,\dots,x_n)$ is connected with (0), that $x_1+\cdots+x_n = 0$, whence $x_1 = \cdots = x_n = 0$. So from $\mathfrak{a}+\mathfrak{b} = [0]$ it follows $\mathfrak{a} = \mathfrak{b} = [0]$.

(iii) We may assume $\mathfrak{a} = [a]$, $a \in E$. We apply Lemma 3.3, extended to a case that two equal words are given rather than two equal sums of elements of E, to $[a]+\mathfrak{b} = [a]+\mathfrak{c}$ or $\mathfrak{b}+[a] = \mathfrak{c}+[a]$, respectively. Taking into account the **com**-relations provided by Lemma 3.3, we see that $\mathfrak{b} = \mathfrak{c}$.

(iv) We may assume $\mathfrak{a} = [a]$ and $\mathfrak{b} = [b]$, $a, b \in E$. By (RDP_c) there are $d_1,\dots,d_4 \in E$ such that $d_1 + d_2 = a$, $d_3 + d_4 = a^{\sim}$, $d_1 + d_3 = b^{-}$, $d_2 + d_4 = b$. By (E3) there are elements $d_2', d_4' \in E$ such that $(a,b) = (d_1 + d_2, d_2 + d_4) \sim (d_2' + d_1, d_2, d_4) \simeq (d_2', a + d_4) = (d_2', d_4' + a) \sim (d_2', d_4') + (a)$. This shows one half; the other one is seen analogously.

By [7 Theorem II.4], the conditions (i) to (iv) of this Lemma are necessary and sufficient conditions for a semigroup being the positive cone of some *po-group*. So we arrive at our main theorem:

Theorem 4.3. *Let $(E;+,0,1)$ be a Riesz pseudo-effect algebra. Then we have $\mathcal{C}(E) = \mathcal{G}(E)^+$ for some po-group $\mathcal{G}(E)$. The map $\iota_E\colon E \to \mathcal{G}(E)\, a \mapsto [a]$ determines an isomorphism between $(E;+,0,1)$ and $\big(\Gamma(\mathcal{G}(E),[1]);\ +,[0],[1]\big)$. $[1]$ is a strong unit of $\mathcal{G}(E)$.*

In particular, E is an interval pseudo-effect algebra.

We finally note that the mapping Δ, that associates with every pseudo-effect algebra E its representing unital *po-group* $(\mathcal{G}(E),[1])$, and the mapping Γ, that associates with any unital *po-group* (G,u) the interval pseudo-effect algebra $\Gamma(G,u)$, define a categorical equivalence between pseudo-effect algebras and unital *po-groups* that both fulfil (RDP_c).

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