ON SOME VARIETIES OF WEAKLY ASSOCIATIVE LATTICE RINGS

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The notion of a weakly associative lattice ring (\textit{wal-ring}) is a generalization of that of a lattice ordered ring in which the identities of associativity of the lattice operations join and meet are replaced by the identities of weak associativity. In the paper the classes of \textit{wal-rings} representable as subdirect products of \textit{to-rings} and \textit{so-rings} (both being non-transitive generalizations of the class of \textit{f-rings}) are characterized.

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1 BASIC NOTIONS

A \textit{semiorder} of a non-void set \(A\) is any reflexive and antisymmetric binary relation on \(A\).

A \textit{weakly associative lattice} (\textit{wa-lattice}) (see [2], [10]) is an algebra \(A = (A, \land, \lor)\) with two binary operations satisfying the identities

(I) \(a \lor a = a; \quad a \land a = a.\)

(C) \(a \lor b = b \lor a; \quad a \land b = b \land a.\)

(Abs) \(a \lor (a \land b) = a; \quad a \land (a \lor b) = a.\)

(WA) \(((a \lor c) \lor (b \land c)) \lor c = c; ((a \lor c) \land (b \land c)) \land c = c.\)

We can define a binary relation \(\preceq\) on \(A\) as follows:
\[
a, b \in A; \quad a \preceq b \iff_a a \land b = a.
\]

This relation is a semiorder and every two-element subset \(\{a, b\} \subseteq A\) has the join \(\sup\{a, b\} = a \lor b\) and the meet \(\inf\{a, b\} = a \land b\) in \(A\), i.e., \(a \preceq \sup\{a, b\} \leq \sup\{a, b\}\) and simultaneously, \(\forall u \in A; \ (a \preceq u \land b \preceq u) \Longrightarrow \sup\{a, b\} \leq u\) (dually). Moreover, each such binary relation defines on \(A\) a structure of a \textit{wa-lattice}.

A special case of a \textit{wa-lattice} is a tournament. A semiordered set \((A, \preceq)\) is said to be a \textit{tournament (totally semiordered set)} if any elements \(a, b \in A\) are comparable, i.e.,
\[
\forall a, b \in A; \ a \preceq b \text{ or } b \preceq a.
\]

\textbf{Definition.} A system \(R = (R, +, \cdot, \preceq)\) is called a \textit{semiordered ring (so-ring)} if

(R1) \((R, +, \cdot)\) is an (associative) ring;

(R2) \((R, \preceq)\) is a semiordered set;

(R3) \(\forall a, b, c \in R; \ a \preceq b \Longrightarrow a + c \preceq b + c;\)

(R4) \(\forall a, b, c \in R; \ 0 \preceq c, \ a \preceq b \Longrightarrow ac \preceq bc\) and \(ca \preceq cb\).

If \((R, \preceq)\) is a \textit{wa-lattice}, then we say that \(R = (R, +, \cdot, \preceq)\) is a \textit{weakly associative lattice ring (wal-ring)}.

If \((R, \preceq)\) is a lattice, then \(R = (R, +, \cdot, \preceq)\) is said to be a \textit{lattice ordered ring (l-ring)}. If for \textit{wal-ring} \(R\) the corresponding \textit{wa-lattice} \((R, \preceq)\) is a tournament, then \(R\) is called a \textit{totally semiordered ring (to-ring)}.

(For some properties of \textit{so-groups} and \textit{wal-groups} see [4] and [5], for those of \textit{l-rings} see [1].)

\textbf{Definition.} Let \(R\) be an \textit{so-ring}. Denote \(R^+ = \{x \in R; \ 0 \preceq x\}\), \(R^+\) will be called the \textit{positive cone of} \(R\).

\textbf{Proposition 1.} a) Let \(R = (R, +, \cdot, \preceq)\) be an \textit{so-ring}.

The positive cone \(R^+\) has the following properties

(1) \(R^+ \cap -R^+ = \{0\},\)

(2) \(R^+ \cdot R^+ \subseteq R^+\).

b) If \((R, +, \cdot)\) is a ring, \(P\) a subset with \(0\) in \(R, P \subseteq R\) satisfies (1) and (2), then \(R = (R, +, \cdot, \preceq),\) where \(a \preceq b\) iff \(b - a \in P\) for all \(a, b \in R\), is an \textit{so-ring} and \(R^+ = P\).

In contrast to lattice ordered rings (\textit{l-rings}), there are many non-trivial finite \textit{so-rings} and \textit{wal-rings}. The following example serves as the simplest case of a \textit{to-ring}.

\textbf{Example 2.} Let us consider the ring \(Z_3 = \{0, 1, 2\}\) with the addition and multiplication mod 3. We denote \(R = (Z_3, +, \cdot), R^+ = \{0, 1\}\). It is clear that \(Z_3^+\) is the positive cone of a total semiorder of the ring \(Z_3\).

\textbf{Example 3.} The ring \(R = (Z, +, \cdot)\) with the positive cone \(R^+ = \{0, 1, 2, 4, 6, \ldots\}\) is a \textit{wal-ring}, not a \textit{to-ring}.

If \(x \in R\), then it holds:

1) \(x \in R^+ \Longrightarrow x \lor 0 = x;\)

2) \(-x \in R^+ \Longrightarrow x \lor 0 = 0;\)

3) \(x \notin R^+, -x \notin R^+ \Longrightarrow x \lor 0 = \max\{x, 0\} + 1,\) where \(\max\{x, 0\}\) is meant in the natural ordering of \(Z\).

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Let us consider a family \( \{ R_i; i \in I \} \) of semiorordered rings. The direct product of so-rings, denoted by \( R = \prod_{i \in I} R_i \), is the direct product of the rings \( R_i \) in which a relation \( \leq \) is defined: If \( a = (a_i)_{i \in I} \) and \( b = (b_i)_{i \in I} \), then
\[
a \leq b \iff a_i \leq b_i \quad \text{for all } i \in I.
\]
This relation is a semiorder. If we suppose every \( R_i \) to be a wal-ring, then \( R \) is the wal-ring and
\[
a \vee b = (a_i \vee b_i)_{i \in I}, \quad a \wedge b = (a_i \wedge b_i)_{i \in I}.
\]
Let \( (R, +, \cdot, \leq) \) and \( (R', +', \cdot', \leq') \) be wal-rings. A mapping \( h: R \to R' \) will be called a wal-homomorphism if simultaneously \( h \) is a ring homomorphism and \( (R, +, \cdot) \to (R', +', \cdot') \) and a wal-lattice homomorphism.

Let \( R = (R, +, \cdot, \leq) \) be an so-ring, \( \theta \neq A \subseteq R \). Then we say that \( A \) is a convex subset of \( R \) if \( a \leq x, x \leq b \) imply \( x \in A \) for all \( a, b \in A, x \in R \).

**Definition.** Let \( R = (R, +, \cdot, \leq) \) be a wal-ring and \( I \) an ideal of \( R \). If a convex ideal \( I \) is a wa-sublattice of \( (R, \leq) \) and satisfies the condition:
\[
(I_{\text{wal}}) \forall a, b \in I, x, y \in R \text{ such that } x \leq a, y \leq b \text{ there exists } c \in I \text{ such that } x \vee y \leq c,
\]
then \( I \) is called a wal-ideal of \( R \).

**Proposition 4** [9, Theorem 1.4.2]. Let \( R = (R, +, \cdot, \leq) \) be a wal-ring. A subset \( L \subseteq R \) is a wal-ideal if and only if \( L \) is the kernel of a wal-homomorphism.

### 2 THE LATTICE OF WAL–IDEALS

Let \( (R, +, \cdot, \leq) \) be a wal-ring. The set of all wal-ideals of the ring \( (R, +, \cdot, \leq) \) is denoted by \( \mathcal{I}(R) \). Further we denote the set of all wal-ideals of the additive wal-group \( (R, +, \leq) \) by \( \mathcal{L}(R) \). The set of all wal-ideals of a wal-group \( (R, +, \leq) \) coincides with the subgroup of the additive group \( (R, +) \) generated by these ideals as subgroups. (See [5, 7].)

We will denote the subgroup of the additive group \( R \) generated by a system \( \{ A_i, i \in J \} \) of subgroups of \( R \) by \( \bigcup_{i \in J} A_i \).

**Proposition 5.** If \( R \) is a wal-ring, then \( \mathcal{I}(R) \) is a complete sublattice of the lattice \( \mathcal{L}(R) \) of wal-ideals of the additive wal-group \( (R, +) \).

**Theorem 6.** The class of all wal-rings is a variety of algebras of type \( \langle +, 0, - , \cdot , \vee , \wedge \rangle \) of signature \( \{ 2, 0, 1, 2, 2, 2 \} \).

**Proof.** It is sufficient to show that the condition (R4) in the definition of a wal-ring can be replaced by some identities. Indeed the condition \( 0 \leq c, a \leq b \implies ac \leq be \) and \( ca \leq cb \) is equivalent to the following two identities:
\[
(a \vee b)(c \vee 0) \geq a(c \vee 0) \vee b(c \vee 0),
\]
\[
(c \vee 0)(a \vee b) \geq (c \vee 0)a \vee (c \vee 0)b.
\]
Let the condition (R4) hold. Since \( a \vee b \geq a, a \vee b \geq b \), \( 0 \leq c \vee 0 \leq c \), according to (R4), we get \( (a \vee b)c' \geq ac' \) and \( (a \vee b)c' \geq bc' \) and hence \( (a \vee b)c' \geq ac' \vee bc' \) and \( (a \vee b)c' \geq ac' \vee bc' \). Similarly the other identity can be proved.

Conversely, let the identities be fulfilled and \( 0 \leq c, a \leq b \). Then \( c \vee 0 = c, a \vee b = b \). We have \( bc \geq ac \vee bc \). the proof for \( ca \leq cb \) is similar.

Wal-rings are \( \Omega \)-groups in the sense of Kurosch (see [3]). The kernels of homomorphisms of an \( \Omega \)-group are precisely all its ideals. Hence a wal-ideal of a wal-ring is also an ideal in the sense of an ideal of an \( \Omega \)-group. Hence by [3, III.2.5], a partition to blocks of any wal-ring \( R \) defines a congruence on \( R \) if and only if it is the partition by some wal-ideal in \( R \).

Now we can show that the lattice \( \mathcal{I}(R) \) is distributive. For this we will use the known properties of varieties of algebras. Let us recall that a variety of algebras is called arithmetical if it is both congruence-distributive and congruence-permutable.

Every wal-ring is an expansion of a wal-group. By [7, Theorem 3], the variety of all wal-groups is arithmetical. Therefore the following proposition holds.

**Proposition 7.** The variety of all wal-rings is arithmetical.

**Corollary 8.** The lattice of wal-ideals of any wal-ring is distributive.

Let \( R \) be a wal-ring and \( I \in \mathcal{I}(R) \). Consider the following conditions for \( I \).
1. If \( a, b \in R \) and \( a \leq b \), then \( a \in I \) or \( b \in I \).
2. If \( a, b \in R \) and \( a \wedge b = 0 \), then \( a \in I \) or \( b \in I \).
3. \( R/I \) is a totally semi-ordered set.
4. \( \{ a \in \mathcal{I}(R); I \subseteq A \} \) is a linearly ordered set.
5. If \( A, B \in \mathcal{I}(R) \) and \( A \cap B = I \), then \( A = I \) or \( B = I \).
6. If \( A, B \in \mathcal{I}(R) \) and \( A \cap B \subseteq I \), then \( A \subseteq I \) or \( B \subseteq I \).

**Proposition 9.** If \( I \) is a wal-ideal of a wal-ring \( R \), then
\[
(1) \iff (2) \iff (3) \iff (4) \iff (5) \iff (6).
\]

**Definition.** A wal-ideal of a wal-ring \( R \) satisfying conditions (1), (2) and (3) will be called a straightening ideal of \( R \).

If a wal-ideal \( I \) of a wal-ring \( R \) satisfies conditions (5) and (6), then \( I \) is said to be an irreducible ideal of \( R \).
Definition. A wal-ideal $I$ of a wal-ring $R$ is called semi-maximal if there exists an element $a \in R$ such that $I$ is a maximal wal-ideal of $R$ with respect to the property “not containing $a$”.

Proposition 10. A wal-ideal $I \in \mathcal{I}(R)$ is semimaximal if and only if it is infinitely irreducible, i.e., if $I = \bigcap_{\alpha \in \Gamma} J_{\alpha}, \ J_{\alpha} \in \mathcal{I}(R)$ implies the existence of an $\alpha_0 \in \Gamma$ such that $I = J_{\alpha_0}$.

Proof. Let $I$ be a semimaximal wal-ideal of $R$ with respect to the property “not containing $a$”. Let $I = \bigcap_{\alpha \in \Gamma} J_{\alpha}, \ J_{\alpha} \in \mathcal{I}(R)$. Then there exists $\alpha$ such that $a \notin J_{\alpha}$. But $I$ is maximal with this property, hence $I = J_{\alpha}$.

Conversely, let $I$ be infinitely irreducible and $I^*$ the intersection of all wal-ideals containing $I$ as a proper set $I \subset I^*$. Then there exists $\alpha \in I^* \setminus I$. If $I \subset J$, then $a \in J$, that means $I$ is maximal with respect to the property “not containing $a$”, i.e., $I$ is semimaximal.

Let us denote by $V(a)$ the set of all semimaximal wal-ideals, maximal with respect to the property “not containing $a$”.

Proposition 11. If $I \in \mathcal{I}(R)$ and $a \in R \setminus I$, then there exists $H \in V(a)$ such that $I \subseteq H$.

3 VARIETY OF REPRESENTABLE WAL–RINGS

Let us recall [1] that an $l$-ring $R$ is called a ring of functions (f-ring) if $R$ is isomorphic to a subdirect product of linearly ordered rings (o-rings).

Definition. If $R$ is a wal-ring, then $R$ is called representable if it is isomorphic to a subdirect product of to-rings.

Proposition 12. A wal-ring is representable if and only if the intersection of all its straightening wal-ideals is equal to $\{0\}$.

Proof. Let $R$ be a representable wal-ring. Then there exists a family of surjective wal-homomorphisms $p_i: R \rightarrow R_i, \ i \in I$ such that every $R_i$ is totally semiordered and $\bigcap_{i \in I} \ker p_i = \{0\}$. Hence $R/\ker p_i (i \in I)$ is totally semiordered, which holds if and only if $\ker p_i (i \in I)$ is a straightening ideal.

The converse implication is obvious.

Proposition 13. If every semimaximal wal-ideal of a wal-ring $R$ is straightening then $R$ is representable.

Proof. By [9, Corollary 2.2.6], the intersection of all semimaximal wal-ideals of a wal-ring is equal to $\{0\}$.

Theorem 14. The class $\mathcal{R}\mathcal{R}_{\text{wal}}$ of all representable wal-rings is a variety of wal-rings.

Proof. By Birkhoff’s theorem, a nonempty class of algebras of a given type is a variety if it is closed under direct products, subalgebras and homomorphic images.

a) Obviously, the direct product of representable wal-rings is a representable wal-ring, too.

b) Let $R \in \mathcal{R}\mathcal{R}_{\text{wal}}$ and $S$ be a wal-subring of $R$. Let $K_\beta$ be a straightening wal-ideal of $R$. Let $\mathcal{S}_\beta = S \cap K_\beta$. It is obvious that $\mathcal{S}_\beta$ is an ideal of the ring $S$ which is a wa-sublattice of the wa-lattice $S$. Let $a, b \in \mathcal{S}_\beta, \ x, y \in S, \ a \leq x, \ x \leq b$. Then $a \in \mathcal{S}_\beta, \ x \in \mathcal{S}_\beta$, hence $\mathcal{S}_\beta$ is convex.

Let $a, b, c \in S_\beta, \ x, y \in S, \ x \leq a, \ y \leq b$. Then $(x \vee y) \cap \mathcal{S} = (x \cap \mathcal{S}) \setminus S_\beta$ and so $S_\beta$ is a wal-ideal of $S$.

Let $x, y \in S, \ x \wedge y = 0$. Then $x \in K_\beta$ or $y \in K_\beta$, hence $x \in S_\beta$ or $y \in S_\beta$. That means $S_\beta$ is straightening.

Now, let $\{K_\beta; \ \beta \in \Delta\}$ be the system of all straightening wal-ideals of $R$. Then $\bigcap_{\beta \in \Delta} \mathcal{S}_\beta = \bigcap_{\beta \in \Delta} (S \cap K_\beta) \subseteq \bigcap_{\beta \in \Delta} K_\beta = \{0\}$, and so, by Proposition 12, $S$ is a representable wal-ring.

c) Let $R, R'$ be wal-rings and $f$ be a surjective wal-homomorphism of $R$ onto $R'$. Since wal-rings are $\Omega$-groups in the sense of Kurosch, by [3, III.2.13], if $J$ is a wal-ideal of $R$ and $J' = f(J)$ then $J'$ is a wal-ideal of $R'$.

Suppose $J$ is straightening. Consider $x + J' + J' \in R'/J'$. Let $x, y \in R$, $f(x) = x',\ f(y) = y'$. We can assume that $x + J \leq y + J$. Then there exists $a \in J$ such that $x + a \leq y$, and from this $x' + f(a) \leq y'$. We have $x' + J' \leq y' + J'$ because $f(a) \in J'$. Therefore $J'$ is straightening.

Let $R$ be representable and let $\{J_\alpha; \ \alpha \in \Gamma\}$ be the system of all straightening wal-ideals of $R$. If there exists $\beta \in \Gamma$ such that $f(J_\beta) = \{0\}$, then $\{0\}$ is a straigntening wal-ideal of $R'$ and from this $R'$ is a to-ring and so representable.

Let $J'_\alpha = f(J_\alpha) \neq \{0\}$ for each $\alpha \in \Gamma$. $f$ induces a bijection preserving inversions of the set of all wal-ideals of $R$ which are not contained in $J_\alpha$ onto the set of all wal-ideals of $R'$. At the same time the wa-lattices $R/J_\alpha$ and $R'/J'_\alpha$ are isomorphic, hence $f$ induces also a bijection of the set of all straightening wal-ideals of $R$ onto the set of all straightening wal-ideals of $R'$. Let $J' = \bigcap_{\alpha \in \Gamma} J'_\alpha \neq \{0\}$. Then $J = f^{-1}(J')$ is a wal-ideal of $R$ which is contained in all straightening wal-ideals of $R$, hence $J = \{0\}$, a contradiction. Therefore $J' = \{0\}$, that means $R'$ is representable.

Evidently, o-rings are special cases of to-rings, thus f-rings are special cases of representable wal-rings and they form a subvariety of the variety $\mathcal{R}\mathcal{R}_{\text{wal}}$. 
4 THE VARIETY OF AO–REPRESENTABLE WAL–RINGS

We could see that representable wal-rings are nontransitive generalization of f-rings and, in addition, an l-ring is an f-ring if and only if it is a representable wal-ring. Nevertheless, the class $\mathcal{RR}_{\text{wal}}$ of all representable wal-rings remains to be a rather large extension of the class $\mathcal{RR}_f$ of all f-rings because the notion of a to-ring is a considerable generalization of that of an o-ring. Therefore, in this part we will deal with subdirect products of to-rings with total semirorders very close to linear orders.

A tournament $(T, \leq)$ is said to be circular if (a) there exist $a, b, c \in T$ such that $a < b < c < a$, and (b) whenever $x, y, z \in T$ satisfy $x < y < z < x$, then there exists no $w \in T$ such that $w \notin \{x, y, z\}$ or $w > \{x, y, z\}$.

Definition. A to-group $G$ is called circular if the tournament $(G, \leq)$ is circular. A to-ring $R$ is called circular if the tournament $(R, \leq)$ is circular.

Definition. A to-group $G$ is called an almost o-group (ao-group) if $G$ is either an o-group or a circular to-group. A to-ring $R$ is called an almost o-ring (ao-ring) if $R$ is either an o-ring or a circular to-ring.

The circular to-groups and ao-groups have been introduced and studied in [6] and [8].

Proposition 15. Let $R$ be a to-ring. Then $R$ is an ao-ring if and only if $R^+$ is a linearly ordered set.

Proof. Let $R$ be a circular to-ring, $a, b, c \in R^+ \setminus \{0\}$, $a < b < c$. Consider $a > c$. Then $a < b < c < a$ and $0 \notin \{a, b, c\}$, a contradiction. Thus $a < c$, therefore the restriction of $< t R^+$ is transitive.

Conversely, let $R^+$ be a linearly ordered set and let $R$ be not a linearly ordered ring. Then there exist $a, b, c, d \in R$ such that $a < b < c < a$ and, for example, $d < \{a, b, c\}$. Then $-d + a < -d + b < -d + c < -d + a$, and $0 \notin \{-d + a, -d + b, -d + c\}$. Hence $R^+$ is not linearly ordered, a contradiction. Similarly for $d > \{a, b, c\}$. It follows that $R$ is circular.

Definition. A wal-ideal $I$ of a wal-ring $R$ is called an ao-straightening wal-ideal of $R$ if $R/I$ is an ao-ring.

Definition. A wal-ring $R$ is called ao-representable if it is isomorphic to a subdirect product of ao-rings.

Obviously, every ao-straightening wal-ideal is also straightening and every ao-representable wal-ring is also representable.

Proposition 16. A wal-ring is ao-representable if and only if the intersection of all its ao-straightening wal-ideals is equal to $\{0\}$.

Proof. The proof is similar to that of Proposition 12.

Proposition 17. The class $\mathcal{AoRR}_{\text{wal}}$ of all ao-representable wal-rings is a variety of wal-rings.

Proof. Similarly as in Theorem 14, we will use Birkhoff’s characterization of a variety as a class of algebras of a given type closed under direct products, subalgebras and homomorphic images.

References


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