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GERSGORIN–STYLE ESTIMATION OF THE SPECTRAL RADIUS AND OF THE SMALLEST, IN THE MODULUS, EIGENVALUE

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This paper is a continuation of the work "Optimal Gersgorin-Style Estimation of Extremal Singular Values" (by C.R. Johnson, T. Szulc and D. Wojtera-Tyrakowska [4]).

The purpose of this paper is to describe the best possible estimates for both the spectral radius and, in nonsingular case, the smallest (in the modulus) eigenvalue based on Gersgorin-type information.

K e y w o r d s: spectral radius, eigenvalue, bound for extremal (in the moduli) eigenvalues, equimodular, equiradial

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1 INTRODUCTION

By $M_n(\mathbb{C})$ we will mean the set of all n-by-n complex matrices. For $A = (a_{ij}) \in M_n(\mathbb{C})$ we define the matrices:

- $|A| = (|a_{ij}|)$
- $D(A) = \mathrm{diag}(a_{11}, \dots, a_{nn})$
- the *comparison matrix* of A by
$$M(A) = \begin{cases} |a_{ij}| & \text{if } i = j\,, \\ -|a_{ij}| & \text{otherwise.} \end{cases}$$

Let $R_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|$ be the sum of the moduli of the off-diagonal entries. If $A = (a_{ij}) \in M_n(\mathbb{C})$, then A is called *strictly diagonally dominant* if $|a_{ii}| > R_i(A)$, $i = 1, \dots, n$. In addition, recall that *a nonsingular M-matrix* is one that is equal to its comparison matrix and has an entry-wise nonnegative inverse, and an *H-matrix* is one whose comparison matrix is a nonsingular M-matrix.

For $A \in M_n(\mathbb{C})$ we define two classes of matrices:

- the class $\Omega(A)$ of matrices *equimodular* to A by
$$\Omega(A) = \{B \in M_n(\mathbb{C}) : |B| = |A|\}\,,$$
- the class $\Lambda(A)$ of matrices *equiradial* to A by
$$\Lambda(A) = \{B \in M_n(\mathbb{C}) \colon |D(B)| = |D(A)| \text{ and } R_i(B) = R_i(A), i = 1, \dots, n\}\,.$$

By $\rho(A)$ we will denote the *spectral radius* of $A \in M_n(\mathbb{C})$.

It is well known that the spectral radius plays an important role in solving linear stationary iterations, whereas the smallest (in the modulus) eigenvalue can be used in studying spectral properties of inverses of matrices. It is common for applications to require an upper bound for the spectral radius and a lower bound for the smallest (in the modulus) eigenvalue. One of the possibilities to get such bounds is to use the Gersgorin theorem (see [2]). That implies the following Gersgorin-style questions about the spectral radius and the smallest (in the modulus) eigenvalue.

Question 1. *What are*
$$\max_{B\in\Omega(A)} \{\rho(B)\} \text{ and } \max_{B\in\Lambda(A)} \{\rho(B)\}\,?$$

Question 2. *If A is an H-matrix, what is* $\min_{B\in\Omega(A)} \{|\lambda_{min}(B)|\}$, *where* $|\lambda_{min}(B)| = \min_{1\le i\le n} \{|\lambda_i(B)|\}$ *?*

The assumption that A is an H-matrix assures the minimum over $\Omega(A)$ to be non-zero.

Question 3. *If A is strictly diagonally dominant, what is*
$$\min_{B\in\Lambda(A)} \{|\lambda_{min}(B)|\}\,?$$

2 EIGENVALUES (IN THE MODULI) EXTREMIZERS IN $\Omega(\mathbf{A})$

Theorem 1. *The matrix $|A|$ is a spectral radius maximizer in $\Omega(A)$.*

P r o o f . Let B be any member of $\Omega(A)$. Then it is well known that
$$\rho(B) \le \rho(|B|)\,, \tag{1}$$
from which, by the definition of $\Omega(A)$, we get
$$\rho(B) \le \rho(|B|) = \rho(|A|)\,.$$

So, the assertion follows.

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Theorem 2. *If A is an H-matrix then $M(A)$ is a $|\lambda_{min}|$-minimizing matrix in $\Omega(A)$.*

Proof. Let B be any member of $\Omega(A)$. First we will prove that if B is an H-matrix, then B is invertible. Since $M(B)$ is a nonsingular M-matrix, it can be expressed as

$$M(B) = sI - X\,,$$

where $X \geq 0$ and s is a positive real number such that $s > \rho(X)$. Let $U \in M_n(\mathbb{C})$ be a matrix, the entries of which have modulus one and let $D = I \circ U$ be a diagonal matrix with $|D| = I$, ($\circ$ denotes the Hadamard product). Matrix B can be written in the form

$$\begin{aligned} B = M(B) \circ U &= (sI - X) \circ U \\ &= sI \circ U - X \circ U = sD - X \circ U \\ &= sD\big(I - (sD)^{-1}(X \circ U)\big)\,. \end{aligned}$$

Now we will show that the matrix $I - (sD)^{-1}(X \circ U)$ is a nonsingular matrix. We have

$$\begin{aligned} \rho((sD)^{-1}(X \circ U)) &\leq \rho\big(|(sD)^{-1}(X \circ U)|\big) \\ &\leq \rho\big(|(sD)^{-1}|\,|X \circ U|\big) = \rho\Big(\frac{1}{s}X\Big) < 1\,, \end{aligned}$$

which implies that $I - (sD)^{-1}(X \circ U)$ is nonsingular. Since $s > 0$ and D is a nonsingular matrix, $B = sD\big(I - (sD)^{-1}(X \circ U)\big)$ is invertible [4]

Another argumentation of this fact can be as follows. By the definition of H-matrix, $M(B)$ is a nonsingular M-matrix. So, by Theorem 2.5.3 from [3] there is a positive diagonal matrix E such that $EM(B)E^{-1}$ is strictly diagonally dominant. Hence, by the definition of $M(B)$, EBE^{-1} is strictly diagonally dominant, and B is invertible.

Therefore as B is an H-matrix, there exists B^{-1} and we have

$$|\lambda_{min}(B)| = \frac{1}{\rho(B^{-1})}\,. \tag{2}$$

Since for any $X \in M_n(\mathbb{C})$ we have

$$\rho(X) \leq \rho(|X|)$$

hence

$$\rho(B^{-1}) \leq \rho\big(|B^{-1}|\big)\,.$$

From this inequality and (2) we obtain

$$|\lambda_{min}(B)| \geq \frac{1}{\rho(|B^{-1}|)}\,. \tag{3}$$

As B is an H-matrix, therefore by Ostrowski's result [5], we have

$$|B^{-1}| \leq (M(B))^{-1}\,.$$

Hence, as $\rho(\cdot)$ is a nondecreasing function of the entries of a nonnegative matrix,

$$\rho(|B^{-1}|) \leq \rho((M(B))^{-1}) \tag{4}$$

from which, by (3) we get

$$|\lambda_{min}(B)| \geq \frac{1}{\rho((M(B))^{-1})} = \big|\lambda_{min}\big(M(B)\big)\big|\,. \tag{5}$$

As $M(B)$ is a nonsingular M-matrix, we have

$$\big|\lambda_{min}\big(M(B)\big)\big| = \lambda_{min}\big(M(B)\big)\,,$$

therefore (5) becomes

$$|\lambda_{min}(B)| \geq \lambda_{min}\big(M(B)\big)$$

and by the definition of $\Omega(A)$, we get

$$|\lambda_{min}(B)| \geq \lambda_{min}\big(M(A)\big)\,.$$

3 EIGENVALUES (IN THE MODULI) EXTREMIZERS IN $\Lambda(\mathbf{A})$

Theorem 3. *Let $s, t \in \{1, 2, \ldots, n\}$, $s \neq t$, be such that*

$$\begin{aligned} &\frac{1}{2}[|a_{ss}| + |a_{tt}| + \sqrt{(|a_{ss}| - |a_{tt}|)^2 + 4R_s(A)R_t(A)}] \\ &= \max_{\substack{1 \leq i,j \leq n \\ i \neq j}} \Big\{\frac{1}{2}\Big[|a_{ii}| + |a_{jj}| \\ &\qquad + \sqrt{(|a_{ii}| - |a_{jj}|)^2 + 4R_i(A)R_j(A)}\Big]\Big\}. \end{aligned}$$

Then any $B = (b_{ij}) \in \Lambda(A)$, where $b_{ss} = |a_{ss}|$, $b_{tt} = |a_{tt}|$, $b_{st} = R_s(A)$, $b_{ts} = R_t(A)$ and $b_{sj} = b_{tj} = 0$ for $j \in \{1, 2, \ldots, n\} \setminus \{s, t\}$, is a spectral radius maximizer in $\Lambda(A)$.

Proof. Let $C = (c_{ij}) \in \Lambda(A)$ be any member of $\Lambda(A)$. Then, by Brauer's bound [1] and by the definition of s and t, we have

$$\begin{aligned} \rho(C) &\leq \max_{\substack{1 \leq i,j \leq n \\ i \neq j}} \Big\{\frac{1}{2}\Big[|c_{ii}| + |c_{jj}| \\ &\qquad + \sqrt{(|c_{ii}| - |c_{jj}|)^2 + 4R_i(C)R_j(C)}\Big]\Big\} \\ &= \frac{1}{2}\Big[|a_{ss}| + |a_{tt}| + \sqrt{(|a_{ss}| - |a_{tt}|)^2 + 4R_s(A)R_t(A)}\Big]. \end{aligned}$$

Observe that

$$\frac{1}{2}\Big[|a_{ss}| + |a_{tt}| + \sqrt{(|a_{ss}| - |a_{tt}|)^2 + 4R_s(A)R_t(A)}\Big] = \rho(\widetilde{C})\,,$$

where, assuming without losing generality that $s < t$,

$$\widetilde{C} = \begin{bmatrix} |a_{ss}| & R_s(A) \\ R_t(A) & |a_{tt}| \end{bmatrix}.$$

So, again by Brauer's bound [1] and by the definition of s and t we have

$$\rho(C) \leq \rho(B)\,,$$

and the assertion follows.

Example. Let

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 1 & 4 & 1 \\ -3 & 1 & 7 \end{bmatrix}.$$

Then, following Theorem 3, the candidates for a ρ-maximizer in $\Lambda(A)$ are the matrices

$$A_1 = \begin{bmatrix} 5 & 0 & 4 \\ 1 & 4 & 1 \\ 4 & 0 & 7 \end{bmatrix}, \; A_2 = \begin{bmatrix} 5 & 4 & 0 \\ 2 & 4 & 0 \\ 3 & 1 & 7 \end{bmatrix}, \; A_3 = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 2 \\ 0 & 4 & 7 \end{bmatrix}.$$

Calculation yields $\rho(A) = 8.5953$, $\rho(|A|) = 9.1926$, $\rho(A_1) = \mathbf{10.1231}$, $\rho(A_2) = 7.3723$, $\rho(A_3) = 8.7016$. So, in this case a ρ-maximizing matrix is

$$A_1 = \begin{bmatrix} 5 & 0 & 4 \\ 1 & 4 & 1 \\ 4 & 0 & 7 \end{bmatrix},$$

Theorem 4. *Let A be strictly diagonally dominant and let $p, k \in \{1, 2, \ldots, n\}$, $p \neq k$, be such that*

$$\frac{1}{2}\Big[|a_{pp}| + |a_{kk}| - \sqrt{(|a_{pp}| - |a_{kk}|)^2 + 4R_p(A)R_k(A)}\Big]$$
$$= \min_{\substack{1 \le i,j \le n \\ i \neq j}} \left\{ \frac{1}{2}\Big[|a_{ii}| + |a_{jj}| - \sqrt{(|a_{ii}| - |a_{jj}|)^2 + 4R_i(A)R_j(A)}\Big] \right\}.$$

Then any $C = (c_{ij}) \in \Lambda(A)$, where $c_{pp} = |a_{pp}|$, $c_{kk} = |a_{kk}|$, $c_{pk} = R_p(A)$, $c_{kp} = R_k(A)$ and $c_{pj} = c_{kj} = 0$ for $j \in \{1, 2, \ldots, n\} \setminus \{p, k\}$, is a $|\lambda_{min}|$-minimizer in $\Lambda(A)$.

P r o o f . Since A is a strictly diagonally dominant matrix, $|\lambda_{min}(A)|$ is non-zero. Let $B = (b_{ij})$ be any element of $\Lambda(A)$. Notice that B is strictly diagonally dominant, hence, by Brauer's theorem [1] and by definition of p and k, we have

$$|\lambda_{min}(B)| \ge \min_{\substack{1 \le i,j \le n \\ i \neq j}} \left\{ \frac{1}{2}\Big[|b_{ii}| + |b_{jj}| - \sqrt{(|b_{ii}| - |b_{jj}|)^2 + 4R_i(B)R_j(B)}\Big] \right\}$$
$$= \frac{1}{2}\Big[|a_{pp}| + |a_{kk}| - \sqrt{(|a_{pp}| - |a_{kk}|)^2 + 4R_p(A)R_k(A)}\Big].$$

Without losing generality we assume that $p < k$, and observe that

$$\frac{1}{2}\Big[|a_{pp}| + |a_{kk}| - \sqrt{(|a_{pp}| - |a_{kk}|)^2 + 4R_p(A)R_k(A)}\Big] = |\lambda_{min}(\widetilde{B})|,$$

where

$$\widetilde{B} = \begin{bmatrix} |a_{pp}| & R_p(A) \\ R_k(A) & |a_{kk}| \end{bmatrix}.$$

Using again Brauer's theorem [1] and concerning definition of p and k we obtain

$$|\lambda_{min}(B)| \ge |\lambda_{min}(C)|,$$

which proves the assertion.

Example. Let

$$A = \begin{bmatrix} 7 & 2 & 3 \\ -1 & 5 & -2 \\ -1 & -\frac{1}{2} & 2 \end{bmatrix}.$$

Then, following Theorem 4, the candidates for a $|\lambda_{min}(A)|$-minimizer in $\Lambda(A)$ are the matrices

$$A_1 = \begin{bmatrix} 7 & 5 & 0 \\ 3 & 5 & 0 \\ \frac{1}{4} & \frac{5}{4} & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 7 & 0 & 5 \\ 2 & 5 & 1 \\ \frac{3}{2} & 0 & 2 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 7 & 5 & 0 \\ 0 & 5 & 3 \\ 0 & \frac{3}{2} & 2 \end{bmatrix}.$$

Calculation yields $|\lambda_{min}(A)| = 2.6763$, $|\lambda_{min}(M(A))| = 0.9475$, $|\lambda_{min}(A_1)| = 2$, $|\lambda_{min}(A_2)| = \mathbf{0.7919}$, $|\lambda_{min}(A_3)| = 0.9019$. So, in this case a $|\lambda_{min}(A)|$-minimizing matrix is

$$A_2 = \begin{bmatrix} 7 & 0 & 5 \\ 2 & 5 & 1 \\ \frac{3}{2} & 0 & 2 \end{bmatrix}.$$

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