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DERIVATIVE OF FUZZY REAL FUNCTIONS

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The notions of derivative and integral are examples of the most common in real functions analysis. They are also amongst the most important notions of mathematical analysis. Since 1965 we are witnesses of massive reasearch in the fuzzy sets theory including fuzzy real functions theory. A simple approach on diferentiation of these fuzzy real functions based on α-level functions is studied. Some properties of this derivative are shown, equivalents of standard theorems such as Rolle theorem are proven. A similar fuzzy integral is also defined and its connection to a fuzzy derivative is studied.

K e y w o r d s: fuzzy number, fuzzy derivative, fuzzy integral

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1 INTRODUCTION

Each set can be described in terms of a membership function μ. The membership function is a mapping that assigns a membership degree to each element of the domain set. For crisp (non-fuzzy) sets this membership degree is either 1 (the element belongs to the set) or 0 (the element does not belong to the set). Fuzzy set $A \subset X$ is usually identified with its membership function $\mu_A : X \to [0,1]$, where the set A is assumed to be a subset of a real numbers domain R.

The α-cuts or α-level sets are important for working with fuzzy sets.

Definition 1. Let $\alpha \in]0,1]$. Then α-cut of fuzzy set A is set A_α,

$$A_\alpha = \{x \in R; \mu_A(x) \geq \alpha\}.$$

For $\alpha = 0$ the α-cut is set

$$A_0 = cl\{x \in R; \mu_A(x) > 0\}\,.$$

A fuzzy number is a fuzzy set which fulfils following conditions:

(1) There exists a unique $y \in R$, such that $\mu_A(y) = 1$,
(2) A is bounded: there exists interval $[y_1, y_2]$ such that $A_0 \subseteq [y_1, y_2]$,
(3) A is convex $\forall y, \forall z > y, \forall x \in [y,z], \mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\}$.

The set of all real fuzzy numbers is denoted by $F(R)$. Sometimes the term fuzzy interval is used for fuzzy set that does not fulfil the uniqueness in the first condition or is not bounded.

Definition 2. Fuzzy real function is a mapping:

$$f : R \to F(R).$$

As the α-level sets belong to the fuzzy set, there are α-level functions that belong to fuzzy function.

Definition 3. Let f be a fuzzy real function. Then we denote:

(1) $f_1(x) = \{y; \mu_{f(x)}(y) = 1\}$,
(2) $f_0(x) = \sup\{y; \mu_{f(x)}(y) > 0\}$,
(3) $f_{-0}(x) = \inf\{y; \mu_{f(x)}(y) > 0\}$,
(4) for $\alpha \in]0,1]$, $f_\alpha(x) = \sup\{y; \mu_{f(x)}(y) \geq \alpha\}$,
(5) for $\alpha \in [-1,0[$, $f_\alpha(x) = \inf\{y; \mu_{f(x)}(y) \geq \alpha\}$.

2 DERIVATIVE OF FUZZY REAL FUNCTION

There are several attitudes in differentiating fuzzy functions. Kalina in [1] showed three of them and also Dubios and Prade in [2,3,4] introduced a differential calculus. We try to use level functions for the definition of the fuzzy derivative.

Definition 4. Fuzzy function f is differentiable on $]a,b[$, whenever all of its α-levels are differentiable on $]a,b[$.

Definition 5. $(f'(x))_\alpha$ denotes the derivative of fuzzy function f on the level α at the point x.

$$(f'(x))_\alpha = [I_\alpha; S_\alpha],$$

where

$$I_\alpha = \inf\{f'_\beta(x); |\beta| \geq \alpha\},$$

$$S_\alpha = \sup\{f'_\beta(x); |\beta| \geq \alpha\},$$

for $\alpha \in]0;1]$ and $|\beta| \in]0;1]$.

$$I_0 = \inf_{0<\alpha\leq 1} I_\alpha, \text{ a } S_0 = \sup_{0<\alpha\leq 1} S_\alpha.$$

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Definition 6. Derivative of a fuzzy function f at point $x \in]a,b[$ is a fuzzy set $f'(x)$ defined as follows:

$$\mu_{f'(x)}(z) = \sup\{\alpha; z \in [I_\alpha, S_\alpha]\}$$

where $[I_\alpha; S_\alpha]$ are derivatives on level α of the fuzzy function f at the point x, assuming $\sup \emptyset = 0$.

Remark. Of course it may happen that $I_\alpha = -\infty$, or $S_\alpha = \infty$.

The basic property of a derivative is its linearity. For fuzzy functions such a property cannot hold generally. Only weak (partial) linearity property holds.

Proposition 1. *(Weak linearity of fuzzy derivative) Let f a g be differentiable fuzzy functions. Then:*

$$f'(a) + g'(a) \supseteq (f+g)'(a).$$

Proof. From the definition of derivatives at level α it is easy to see that $I^f_\alpha + I^g_\alpha \le I^{f+g}_\alpha$ and $S^f_\alpha + S^g_\alpha \ge S^{f+g}_\alpha$. From this and from the properties of addition of fuzzy numbers (Zadeh's extension principle using minimum T-norm), it is clear that α-level derivatives of the fuzzy function $f+g$ are subsets of α-cuts of $f'+g'$. And therefore the proposition holds.

Using this fuzzy derivative, it is easy to rewrite the Rolle theorem for fuzzy real functions.

Theorem (Rolle). *Let f and g be differentiable fuzzy functions on $[a,b]$ and $f'(a) = f'(b)$. Then $\forall \alpha \in [0,1]$ there exists $c \in]a,b[$ such that $[f'(c)]_\alpha \ni 0$.*

Proof. The proof is elementary.

3 INTEGRAL OF A FUZZY REAL FUNCTION

It is easy to define the (Riemann) integral of a fuzzy real function in a similar way as to define the derivative. We call a fuzzy real function integrable if all its level functions are integrable.

Definition 7. $\left(\int_I f(x)\right)_\alpha$ denotes the integral of a fuzzy real function f on the level α on the interval I.

$$\left(\int_I f(x)\right)_\alpha = [I_\alpha; S_\alpha],$$

where

$$I_\alpha = \inf\left\{\int_I f_\beta; |\beta| \ge \alpha\right\}$$

$$S_\alpha = \sup\left\{\int_I f_\beta; |\beta| \ge \alpha\right\}$$

$$\alpha \in]0;1] \text{ a } |\beta| \in]0;1].$$

Definition 8. The integral of fuzzy function f on some interval I is the fuzzy set $\int_I f(x)$ defined as follows:

$$\mu_{\int_I f(x)}(z) = \sup\{\alpha; z \in [I_\alpha, S_\alpha]\}$$

where $[I_\alpha; S_\alpha]$ are integrals on level α of fuzzy function f on interval I, assuming $\sup \emptyset = 0$.

Remark. It is easy to see that integrals on the level α are the sets $\left[\int_I f_{-\alpha}(x), \int_I f_\alpha(x)\right]$. And also these intervals are the α – *cuts* of the integral of the fuzzy real function. That is why equations $I^f_\alpha + I^g_\alpha = I^{f+g}_\alpha$ and $S^f_\alpha + S^g_\alpha = S^{f+g}_\alpha$ hold and therefore this integral has the linearity property.

Definition 9. The antiderivative of a fuzzy function f is denoted by F, and defined as follows:

$$F(x) = \int_a^x f \text{ , where } a \in I.$$

Proposition 3.

$$(F')_\alpha = f_\alpha, \text{ where } 1 \ge |\alpha| \ge 0.$$

If all α-cuts of values of fuzzy function f are closed intervals, the equation $F' = f$ holds.

Proof. It is enough to mention that α-levels of the antiderivative are the antiderivatives of the corresponding α-levels and then

$$I_\alpha = \inf\{[F_\beta]'(x); |\beta| \ge \alpha\}$$
$$= \inf\left\{\left[\int_a^x f_\beta\right]'; |\beta| \ge \alpha\right\} = \inf\{f_\beta(x); |\beta| \ge \alpha\}.$$

And because $I_\alpha = \inf\{f_\beta(x); |\beta| \ge \alpha\} = f_\alpha(x) = \inf[f(x)]_\alpha$, the first part of the proposition holds.

The second part is a consequence of the properties of the α-cuts of F', which are closed intervals.

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