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ESTIMATION OF THE SUPPORT REGION FOR LAPLACIAN SOURCE SCALAR QUANTIZERS

Zoran Peric * — **Jelena Nikolic** * — **Dragoljub Pokrajac** **

The goal of this paper is finding an asymptotic equation which determines the amplitude range of asymptotically optimal scalar quantizers for Laplacian input signals with an unrestricted amplitude range. It is very important to show how the amplitude range depends on the number of quantization levels. The approximation method for determining the amplitude range is suggested. This method can be used for simple and fast estimation of the amplitude range for scalar quantizer, as a result the quantizer is ideally loaded.

K e y w o r d s: Laplacian source, scalar quantizer, support region

1 INTRODUCTION

Considerable attention has been focused on the design of optimal quantizers for sources encountered in image, speech and other compression applications. Sources having exponential and Laplacian probability density functions (pdf) are commonly in use [1], [2] and the methods for designing quantizers for these sources are very similar. The support region is a key issue in quantizer implementation and when a quantizer is applied to a random variable other than the one for which it was optimized. Lloyd-Max style algorithms for designing optimal scalar quantizers [3], [4] begin with an estimate of the support region, the better the estimate, the more rapidly the algorithm converges. In this paper we suggest an approximation method for determining the amplitude range. This method can be used for simple and fast estimation of the support region of optimal scalar quantizers for Laplacian input signals with an unrestricted amplitude range. As a result the quantizer is adapted to amplitude dynamics of input signals, which means it is ideally loaded. After designing optimal scalar quantizers and estimating the support region, using the suggested method, distortion and relative distortion error are calculated.

Optimization of scalar quantizer

We begin by defining the terms and reviewing relevant scalar quantization characteristics. Let an N-level quantizer $Q^{(N)}$ be defined in terms of a set of N positive step sizes $\{\alpha_i\}_{i=1}^N$ and a set of N distances from the representative levels to the nether decision thresholds $\{\delta_i\}_{i=1}^N$. Let $\{t_i\}_{i=1}^N$ be $N+1$ decision thresholds of the quantizer such that:

$$t_0 = -\infty < t_1^{(N)} < \cdots < t_{N-1}^{(N)} < t_N^{(N)} = \infty \qquad (1)$$

and quantizer cells are derived via

$$S_i^{(N)} = \big(t_{i-1}^{(N)}, t_i^{(N)}\big], \quad i = 1, 2, \ldots, N\,. \qquad (2)$$

Cells $S_2^{(N)}, \ldots, S_{N-1}^{(N)}$ will be called the inner cells, while $S_1^{(N)}$ and $S_N^{(N)}$ will be called the outer cells.

Let N output values of the quantizer (representative levels) $\{y_i^{(N)}\}_{i=1}^N$ be given by

$$y_i^{(N)} = t_{i-1}^{(N)} + \delta_i\,, \quad i = 1, 2, \ldots, N\,. \qquad (3)$$

The N-level scalar quantizer is defined as a functional mapping of an input value onto an output representation such that

$$Q^{(N)}(x) = y_i^{(N)}, \quad t_{i-1}^{(N)} < x \le t_i^{(N)}\,. \qquad (4)$$

In this paper, an optimal N-level scalar quantizer, denoted $Q^{*(N)}$, for a source characterized as a continuous random variable X with probability density $p(x)$ is a quantizer that minimizes the mean squared error:

$$D\big(Q^{(N)}\big) = E\big\{\big(X - Q^{(N)}(X)\big)^2\big\}\,, \qquad (5)$$

$$D\big(Q^{(N)}\big) = \sum_{i=1}^{N} \int_{t_{i-1}^{(N)}}^{t_{i1}^{(N)}} \big(x - y_i^{(N)}\big)^2 p(x)\mathrm{d}x\,. \qquad (6)$$

Such a quantizer is optimal in the sense that no other scalar quantizer with N levels can obtain a lower distortion. Let distortion of an optimal N-level scalar quantizer be

$$D^{*(N)} = D\big(Q^{*(N)}\big)\,. \qquad (7)$$

As first discovered by Panter and Dite [5], for large N:

$$D^{*(N)} \cong c_\infty / N^2 \qquad (8)$$

where C_∞ is the Panter-Dite constant

$$C_\infty \triangleq \frac{1}{12}\Big(\int_{-\infty}^{\infty} p^{1/3}(x)\mathrm{d}x\Big)^3. \qquad (9)$$

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Estimation of support region

Construction of optimal scalar quantizer is possible using the analysis given in [6]. The problem of determining the maximal amplitude of the input signal, *eg* the problem of determining the granular region, is very important and was considered in [7] and [8]. Roughly speaking, the support region of the quantizer, also called the granular region, is the interval where quantization errors are small or at least bounded. Correspondingly, errors are large or at least unbounded on the complement of the support region, usually called the overload region. The following equations describe a symmetrical quantizer with a nonuniform amplitude range divided into quantization cells. The total distortion can be found as a sum of granular and overload distortions:

$$D = D_{gran} + D_{ol} \,. \tag{10}$$

Estimates of the support region are useful in specifying the essential limits in the Panter-Dite formula (8):

$$D_{gran} = \frac{1}{12N^2}\Big(\int_{-x_{\max}}^{x_{\max}} P^{\frac{1}{3}}(x)\mathrm{d}x\Big)^3. \tag{11}$$

Let us consider the Laplacian source with memoryless property. Laplacian pdf of the original random variable x can be expressed by

$$p(x) = \frac{e^{-|x|\sqrt{2/\sigma^2}}}{\sqrt{2\sigma^2}}. \tag{12}$$

Without loss of generality we can assume that $\sigma^2 = 1$ and (12) becomes

$$p(x) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|x|}. \tag{13}$$

By substituting (13) in (11), the expression for determining granular distortion is derived as

$$D_{gran} = \frac{9}{2N^2}\Big(1 - e^{-\frac{\sqrt{2}}{3}x_{\max}}\Big)^3. \tag{14}$$

The overload distortion is given by

$$D_{ol} = 2\int_{x_{\max}}^{\infty} \big(x - y^{(N)}\big)^2 p(x)\mathrm{d}x \tag{15}$$

where the symmetry of Laplacian distribution is used. By substituting (13) in (15), the expression for determining overload distortion is derived as

$$D_{ol} = e^{-\sqrt{2}x_{\max}}\big(x_{\max}^2 + \sqrt{2}x_{\max} + 1 - y_N^{(N)}(2x_{\max} + \sqrt{2}) + y_N^{2(N)}\big). \tag{16}$$

In the case of an optimal scalar quantizer with a large number of quantization cells we can assume that representation levels $\{y_i^{(N)}\}_{i=1}^N$ lie in the middle of the interval $S_I^{(N)}$ $i = 1, \dots, N$. Therefore is

$$y_i^{(N)} = \frac{1}{2}\big(t_{i-1}^{(N)} + t_i^{(N)}\big)\,, \tag{17}$$

$$y_N^{(N)} = x_{\max} - \frac{\Delta_N}{2}\,. \tag{18}$$

Δ_N denotes the width of Nth interval. A nonuniform scalar quantizer for $\Delta_N = 1$ is designed in [6]. Using the assumption that representation levels are centred in appropriate cells we can conclude that $\Delta_N = 2$. Using this fact and equations (16) and (18), final expression for overload distortion is derived as

$$D_{ol} = e^{-\sqrt{2}x_{\max}}(2 + \sqrt{2})\,. \tag{19}$$

By using (14) and (19), the expression for total distortion is obtained as

$$D = \frac{9}{2N^2}\Big(1 - e^{-\frac{\sqrt{2}}{3}x_{\max}} + e^{-\sqrt{2}x_{\max}}(2 + \sqrt{2})\,. \tag{20}$$

For large values of $x_{\max}$ expression (14) can be approximated as

$$D_{gran} = \frac{9}{2N^2}\Big(1 - 3e^{-\frac{\sqrt{2}}{3}x_{\max}}\Big). \tag{21}$$

Granular distortion of an N-level quantizer can be decomposed into an inner distortion and an outer distortion. Inner distortion is a sum of distortion contribution of the inner cells $S_2^{(N)}, \dots, S_{N-1}^{(N)}$. Correspondingly, outer distortion is a sum of distortion contribution of the outer cells $S_1^{(N)}$ and $S_N^{(N)}$:

$$D_{gran} = 2D_{gran}(N) + \sum_{i=2}^{N-1} D_{gran}(i) \tag{22}$$

where

$$2D_{gran}(N) = 2\int_{x_{\max}-\Delta_N}^{x_{\max}} \big(x - y_N^{(N)}\big)^2 p(x)\mathrm{d}x\,, \tag{23}$$

$$\sum_{i=2}^{N-1} D_{gran}(i) = \frac{1}{12(N-2)^2}\Big(2\int_0^{x_{\max}-\Delta_N} p^{\frac{1}{3}}(x)\mathrm{d}x\,. \tag{24}$$

For a large number of quantization cells N, pdf can be considered as constant within each of quantization cells and approximated with appropriate values in representation points

$$2D_{gran}(N) \approx 2p\big(y_N^{(N)}\big)\int_{x_{\max}-\Delta_N}^{x_{\max}} \big(x - y_N^{(N)}\big)^2\mathrm{d}x\,. \tag{25}$$

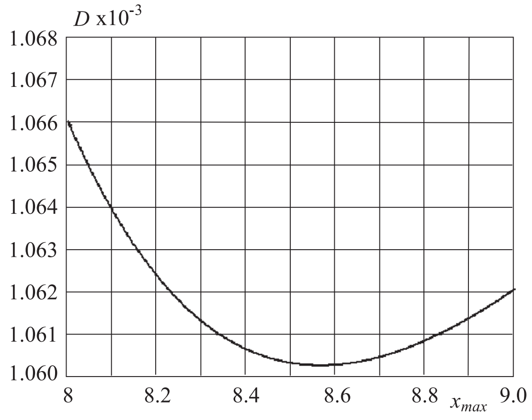

Fig. 1. Dependence of distortion on maximal input signal amplitude for nonuniform scalar quantizer with $N = 64$.

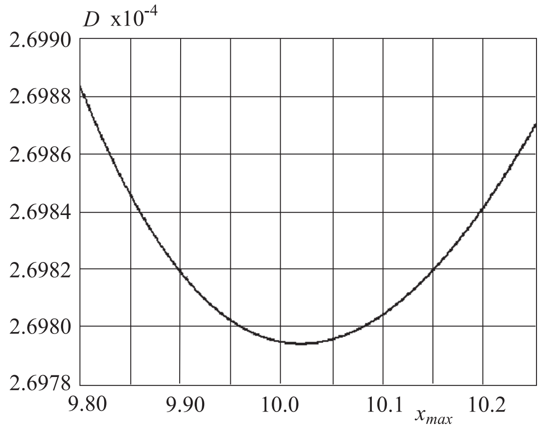

Fig. 2. Dependence of distortion on maximal input signal amplitude for nonuniform scalar quantizer with $N = 128$.

By solving integrals (24) and (25) it is possible to determine the following expressions for granular distortion:

$$2D_{gran}(N) \approx \frac{2\sqrt{2}}{3} e^{\sqrt{2}} e^{-\sqrt{2}x_{\max}}, \tag{26}$$

$$\sum_{i=2}^{N-1} D_{gran}(i) = \frac{9}{2(N-2)^2}\Big(1 - e^{-\frac{\sqrt{2}}{3}(x_{\max}-\Delta_N)}\Big)^3, \tag{27}$$

$$\sum_{i=2}^{N-1} D_{gran}(i) \approx \frac{9}{2(N-2)^2}\Big(1 - 3e^{-\frac{\sqrt{2}}{3}(x_{\max}-\Delta_N)}\Big). \tag{28}$$

Using (21), (22), (26) and (28) the following equation is obtained:

$$\frac{2\sqrt{2}}{3} e^{\sqrt{2}} e^{-\sqrt{2}x_{\max}} + \frac{27}{2}\Big(\frac{1}{N^2} - \frac{e^{\frac{2\sqrt{2}}{3}}}{(N-2)^2}\Big) e^{-\frac{\sqrt{2}}{3}x_{\max}} + \frac{9}{2}\Big(\frac{1}{(N-2)^2} - \frac{1}{N^2}\Big) = 0\,. \tag{29}$$

By solving (29) and using approximation for a large N

$$\frac{1}{N^2} \approx \frac{1}{(N-2)^2} \tag{30}$$

it is possible to approximately determine how the optimal amplitude range depends on the number of quantization levels:

$$x^{apr}_{\max opt} = \frac{3}{\sqrt{2}} \ln \frac{N}{1.0849}\,. \tag{31}$$

The optimal amplitude range, *eg* the accurate value for $x_{\max opt}$, can be evaluated by minimization of distortion. Therefore, differentiating Eq. (20) and equating to zero leads to

$$\frac{\partial D}{\partial x_{\max}} = \frac{9}{\sqrt{2}N^2} e^{-\frac{\sqrt{2}}{3}x_{\max}} \Big(1 - e^{-\frac{\sqrt{2}}{3}x_{\max}}\Big)^2 - 2(1+\sqrt{2}) e^{-\sqrt{2}x_{\max}} = 0\,, \tag{32}$$

$$x_{\max opt} = \frac{3}{\sqrt{2}} \ln \frac{N^2 - 1.318}{1.148N - 1.318}\,. \tag{33}$$

2 NUMERICAL RESULTS

To easily find nearly exact solutions, we suggest an approximation method for determining the support region of asymptotically optimal scalar quantizers for Laplacian input signals with an unrestricted amplitude range. From Eq. (31) it is easy to calculate the approximate value for x^{apr}_{maxopt} and from Eq. (33) the accurate value for $x_{\max opt}$. In Table 1 are given the values of the maximal input signal amplitude, calculated by using the suggested approximation method and by using the differentiation method that minimizes the distortion when the numbers of quantization cells are $N = 64$, $N = 128$, $N = 256$ and $N = 512$. In Table 2 are given numerical values of distortion calculated by using the suggested approximation method and using the differentiation method that minimizes distortion for different numbers of quantization cells. The dependence of distortion on the maximal input signal amplitude for nonuniform scalar quantizer with $N = 64$, $N = 128$, $N = 256$ and $N = 512$ for Laplacian source is shown in Figs. 1 to 4. From these figures $x_{\max opt}$ that corresponds to the minimum of distortion can be ordered. From Figs. 1 to 4 and Tables 1 and 2 it is possible to notice that the appropriate values of $x^{apr}_{\max opt}$ and $x_{\max opt}$, and also the appropriate values of D^{apr} and D are almost identical. The relative distortion error calculated using proposed approximation method is given by

$$\delta = \Big|\frac{D^{apr} - D}{D}\Big| \tag{34}$$

or expressed in [db]

$$SNRO - SNRO^{apr} = 10 \log \frac{\sigma^2}{D}\frac{D^{apr}}{\sigma^2}\,, \tag{35}$$

$$SNRQ - SNRQ^{apr} = 10 \log \frac{D^{apr}}{D}\,. \tag{36}$$

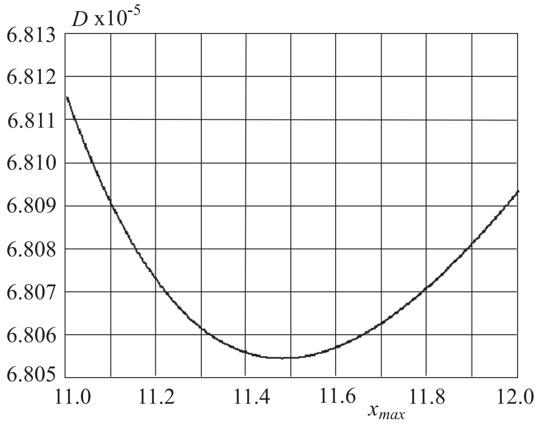

Fig. 3. Dependence of distortion on maximal input signal amplitude for nonuniform scalar quantizer with $N = 256$.

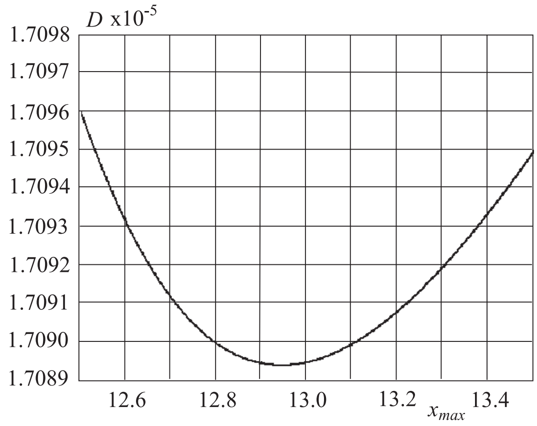

Fig. 4. Dependence of distortion on maximal input signal amplitude for nonuniform scalar quantizer with $N = 512$.

Table 1. Parallel comparison of support region calculated using the approximation method with an appropriate support region calculated using the differentiation method that minimizes distortion, for optimal scalar quantizers.

$\log_2 N$	N	$x^{apr}_{\max opt}$	$x_{\max opt}$
6	64	8.6494	8.5672
7	128	10.1198	10.0188
8	256	11.5902	11.4798
9	512	13.0606	12.9454

Table 2. Parallel comparison of distortion calculated using the approximation method, with appropriate distortion calculated by the differentiation method that minimizes distortion, for optimal scalar quantizers.

$\log_2 N$	N	D^{apr}	D
6	64	0.0011	0.0011
7	128	2.6975×10^{-4}	2.698×10^{-4}
8	256	6.8051×10^{-5}	6.8053×10^{-5}
9	512	1.7089×10^{-5}	1.7089×10^{-5}

Table 3. Numerical values of δ and $SNRQ - SNRQ^{apr}$ for optimal scalar quantizers for different numbers of quantization cells N.

$\log_2 N$	N	δ	$SNRQ - SNRQ^{apr}$(db)
6	64	0	0
7	128	1.8532×10^{-4}	-8.0491×10^{-4}
8	256	2.9388×10^{-5}	-1.2763×10^{-4}
9	512	0	0

It is possible to notice from Table 3 that relative distortion errors caused by using the suggested approximation method are very small, approximately about 10^{-5}. Hereby, the validity of this method is proved again.

3 CONCLUSION

Method presented in this paper enables estimation of the support region of asymptotically optimal scalar quantizers for Laplacian input signals with an unrestricted amplitude range. It is very important to point out that we derived in closed form the equation which provides easily finding the values of maximum amplitude *eg*the support region of quantizer. Due to its simplicity the proposed method is useful when designing scalar quantizers. Particularly, it provides fast and efficient design of scalar quantizers that are used for source coding of images [9] and speech [2]. Furthermore, numerical results demonstrate that by using the proposed method relative distortion error is approximately about 10^{-5}. Hereby, the proposed method validity is confirmed.

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