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Adaptive observer design for a class of nonlinear fractional-order Lipschitz systems with unknown time-varying parameters

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The confluence of nonlinearity, unavailable states, and unknown time-varying parameters poses profound estimation challenges in fractional-order dynamical systems. This paper presents a novel adaptive observer design for nonlinear fractional-order Lipschitz systems with unknown, slowly time-varying parameters. Drawing on recent advancements in fractional-order calculus, a rigorous stability analysis is conducted, deriving the updating law and formulating the observer's viability and stability conditions in terms of linear matrix inequalities (LMIs) and linear matrix equalities (LMEs). The proposed observer ensures the stability of both state observation and parameter estimation errors, along with the asymptotic convergence of the observation error norm square mean value to zero. Empirical results from a case study on a fractional-order financial system validate the efficacy of the proposed observer, thereby advancing the field of states and parameters estimation theory for non-integer order nonlinear systems.

Keywords: fractional-order observer, nonlinear system, Lipschitz continuity, fractional-order adaptive control, parameter identification

1 Introduction

Recently, considerable attention has been focused on the estimation of unmeasurable states in nonlinear dynamical systems within various engineering disciplines, particularly in the systems and control fields [1], where state measurements are indispensable. Despite extensive research efforts devoted to this area, a generalized observer design method for nonlinear systems remains an unresolved challenge. The existing synthesis techniques in the literature can be categorized into two primary approaches. The model-based approaches integrate the system's dynamics into the observer design, including two main techniques: One technique involves transforming the observation error dynamics to achieve linearity, thereby enabling the use of conventional linear observation methods [2]. However, this approach is limited by the inherent complexity of establishing such a transformation. whereas the second technique directly leverages the system's dynamics [3]. The data-driven approaches rely solely on input-output measurements, without considering the system's internal dynamics [4].

Over the past three decades, fractional calculus has gained substantial prominence across a wide range of scientific disciplines. Researchers have increasingly explored the application of fractional-order theory in many scientific areas such as dielectric polarization [5], electromagnetism [7], biology [8], economics [9], and encryption [10]. While fractional-order theory has proven valuable in these fields, its application in systems and control has garnered particular interest due to the distinctive advantages it offers. The memory effect inherent in fractional differential equations enables more precise modeling and control of complex dynamical systems [11, 12]. Compared to traditional integer-order methods, fractional-order approaches often lead to improved accuracy, robustness, and stability, making them especially effective for addressing the challenges of real-world systems [13, 14].

Observers for fractional-order nonlinear systems, developed using the model-based approach that incorporates system dynamics, include extensions of linear observers, such as the extended Kalman filter [15] and high-gain observer [16], as well as specific nonlinear observers, such as super-twisting sliding mode observers [17]. However, in practical applications, systems are often subject to unknown time-varying inputs or parameters, which introduce additional complexities. Accurately estimating these uncertainties or determining their upper bounds is critical for ensuring the observer's effective design. To address this, observers are typically augmented with adaptation or data-driven approximators [18, 19]. While neural networks and fuzzy logic approximators are powerful tools, they pose significant computational and complexity challenges. Adaptation algorithms, in contrast, offer a concise solution, providing efficiency, simplicity, and real-time adaptability – essential for control and observation systems design. In spite of these advantages, fractional-order adaptive observers and controllers design remains underexplored, primarily due to the lack of an extension of Barbalat's lemma for fractional orders within the interval (0,1) or

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any alternative theoretical framework to address this gap. This limitation largely hinders the development of adaptive high-order sliding mode observers or adaptive backstepping mode controllers for fractional-order systems. On the other hand, since there are no signals convergence conditions in their design, only extended linear observers can currently be rendered adaptive. Nevertheless, their applicability is confined to specific classes of nonlinear systems, constrained by small Lipschitz constants for Lipschitz systems [20] and by the significant conservatism inherent in the design conditions for one-sided Lipschitz systems [21]. In response, a growing body of research has focused on developing observers for conventional and fractional-order nonlinear systems with relatively large Lipschitz constants, either by establishing new sufficient conditions [22] or by adopting generalized Lipschitz continuity as in [23]. However, comparatively little attention has been directed toward adaptive observer design for systems subject unknown parameters or inputs [24, 25].

Motivated by the aforementioned discussion and inspired by [22], the primary contribution of this study lies in:

- This research introduces a novel, model-based, adaptive fractional-order observer, based on the foundational principles outlined in [22]. The proposed methodology is specifically designed for simultaneously estimating the system's states and unknown, slowly time-varying parameters in fractional-order Lipschitz nonlinear dynamical systems, enabling enhanced precision in parameter identification.
- A detailed stability analysis of the proposed observer is conducted using recent advances in fractional-order calculus. The stability of both the observation and parameter estimation errors is established. Furthermore, the asymptotic convergence of the observation error norm square mean value is proved.
- The numerical validation of our proposed adaptive observer's effectiveness through the observation of the full state vector and the estimation of an unknown parameter in a fractional-order financial system.

The subsequent sections of this manuscript are structured in the following manner: In section 2, we provide preliminaries and relevant results related to fractional-order calculus. Next, we delve into a comprehensive discussion of the main problem in section 3. In section 4 the essential foundational results upon which this study builds are presented, providing the necessary theoretical basis for the proposed observer design. Following this, section 5 presents our main findings, including a detailed stability analysis of the proposed adaptive observer. Section 6 provides simulation results for a fractional order financial system states and unknown parameter estimation to illustrate the efficiency of the proposed design strategy. Finally, our conclusions are announced in Section 7.

2 Preliminaries

Definition 1. [26] The fractional integration, according to Riemann-Liouville, of a continuous function $f(t)$ is provided by:

$$ {}_{t_0}^{RL}I_t^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau \tag{1} $$

where $\Gamma(.)$ is the gamma function defined as $\Gamma(\omega) = \int_0^{\infty} s^{\omega-1} e^{-s} ds$. Here, $\beta \in \mathbb{R}^{+}$and t_0 is the initial time.

Definition 2. [26] The continuous function $f(t)$ fractional derivation in the Riemann-Liouville concept is defined as follows:

$$ {}_{t_0}^{RL}D_t^{\beta} f(t) = \frac{\mathrm{d}^{\beta} f(t)}{\mathrm{d}t^{\beta}} = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau \tag{2} $$

where $n \in \mathbb{N}$, satisfying $n-1 < \beta < n$.

Definition 3. [26] The Caputo's β^{th} order fractional derivative of a continuous function $f(t)$ is defined by

$$ {}_{t_0}^{C}D_t^{\beta} f(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} \mathrm{d}\tau & n-1 < \beta < n \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) & \beta = n \end{cases} \tag{3} $$

here n denotes the next integer greater than β.

Lemma 1. [27] Consider $\psi(.)\colon \mathbb{R}^{+} \to \mathbb{R}^{n}$ as a derivable and continuous functions vector. For any specified time instant $t \geq t_0$, the following inequality holds:

$$ \frac{1}{2} {}^{C}D_t^{\beta} \psi^2(t) \leq \psi(t) {}^{C}D_t^{\beta} \psi(t), \quad \forall \beta \in (0,1] \tag{4} $$

Lemma 2. [28] Consider $\varphi(t) \in \mathbb{R}$ defined on the right real axis as a bounded nonnegative function, if

$$ \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-\tau)^{\beta-1} \varphi(\tau) d\tau < C \tag{5} $$

holds for some $\beta \in (0,1]$ with $t \geq t_0$ and $C \in (0,\infty)$, then

$$\lim_{t\to+\infty}\left[t^{\beta-\epsilon}\frac{\int_{t_0}^{t}\phi(\tau)d\tau}{t}\right]=0,\forall\epsilon>0 \quad (6)$$

Throughout the rest of this work, the symbol D^{α} refers to the Caputo fractional-order derivative.

3 Problem statement

In this paper we aim to provide an efficient methodology for designing observers for fractional order nonlinear systems described by

$$\begin{aligned} D^{\alpha}x &= Ax+\Phi(x,u)+B\psi(x,u)\theta \\ y &= Cx \end{aligned} \quad (7)$$

where $\alpha\in(0,1)$ denotes the fractional order, $x\in\mathbb{R}^n$ and $\theta\in\mathbb{R}^{\nu}$represent the state and unknown parameters vectors to be estimated. The variables $u\in\mathbb{R}^p, y\in\mathbb{R}^q$ denote the input and output vectors, respectively. $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}, C\in\mathbb{R}^{q\times n}$ are appropriate matrices. The nonlinear functions $\Phi:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}^n, \psi:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}^{m\times\nu}$ are assumed to satisfy the Lipschitz condition for any admissible control law u, with $\gamma_{\Phi},\gamma_{\psi}$ as the Lipschitz constants, respectively, i.e.,

$$\|\Phi(x,u)-\Phi(\bar{x},u)\|\le\gamma_{\Phi}\|x-\bar{x}\|,\ \forall x,\bar{x}\in\mathbb{R}^n \quad (8)$$

$$\|\psi(x,u)-\psi(\bar{x},u)\|\le\gamma_{\psi}\|x-\bar{x}\|,\ \ \forall x,\bar{x}\in\mathbb{R}^n \quad (9)$$

In this study, we tackle the critical challenge of adaptive observer design for the fractional order Lipschitz nonlinear dynamical systems Specifically, for the system defined by (7), we assume that only the input u and output y are available for measurement. By exploiting these signals, we aim to accurately reconstruct the full state vector, $x=(x_1,\dots,x_n)^T$, and the unknown parameters, $\theta=col(\theta_1(t),\dots,\theta_m(t))$, simultaneously, through our developed observer which is structured as follows:

$$\begin{aligned} &D^{\alpha}\hat{x}-A\hat{x}+\Phi(\hat{x},u)+B\psi(\hat{x},u)\hat{\theta}-L(C\hat{x}-y)\\ &\hat{y}(t)=C\hat{x}(t)\\ &D^{\alpha}\theta=H(y,\hat{x}) \end{aligned} \quad (10\ a,b,c)$$

Detailed design strategy for the observer gain L and the dynamics of the updating laws, denoted as $H(y,\hat{x})$, will be elaborated in the subsequent section. First, we need to write down the observer error dynamical system.

Let $\tilde{x}(t)\triangleq\hat{x}(t)-x(t)$ and $\tilde{\theta}(t)\triangleq\hat{\theta}(t)-\theta(t)$, then

$$D^{\alpha}\tilde{x}(t)=(A-LC)\tilde{x}+\hat{\Phi}-\Phi+B\big(\hat{\psi}\hat{\theta}-\psi\theta\big) \quad (11)$$

given that $\Phi\triangleq\Phi(x,u), \hat{\Phi}\triangleq\Phi(\hat{x},u), \psi\triangleq\psi(x,u)$, and $\hat{\psi}\triangleq\psi(\hat{x},u)$.

4 Background results

In the literature, several techniques for observer design have been proposed for the class of nonlinear dynamical systems that can be modeled by:

$$\begin{aligned} \dot{x}(t) &= Ax(t)+\Phi(x,u) \\ y(t) &= Cx(t) \end{aligned} \quad (12)$$

where $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$ represent the state and output vectors, respectively, while $u\in\mathbb{R}^p$ denotes the admissible control input. The system's model matrices $A\in\mathbb{R}^{n\times n}, C\in\mathbb{R}^{m\times n}$ along with the nonlinearity function vector $\Phi(x,u)$, are assumed to be known.

The observers to which this study pertains to generalize are the Leuenberger-like observers, defined by

$$\dot{\hat{x}}=A\hat{x}+\Phi(\hat{x},u)+L(C\hat{x}-y)$$

Thus, the corresponding error dynamical system is given by

$$\dot{\tilde{x}}=(A-LC)\tilde{x}+\tilde{\Phi}$$

with $\tilde{\Phi}=\hat{\Phi}-\Phi$, the observer gain vector design is typically based on a stability analysis using the Lyapunov theorem. In this context, the commonly considered Lyapunov function candidate is given by

$$V=\tilde{x}^TP\tilde{x}$$

with P being a symmetric definite positive matrix, and hence its time derivative is

$$\dot{V}=\tilde{x}^T[(A-LC)^TP+P(A-LC)]\tilde{x}+2\tilde{x}^T\tilde{\Phi}$$

The standard and dominant approach is that proposed by Thau [29]. This later involves selecting the observer gain L such that

$$(A-LC)^TP+P(A-L)=-Q$$

where $Q\in\mathbb{R}^{n\times n}$ is a positive definite matrix. To guarantee the asymptotic convergence of $\hat{x}$ toward x, the system nonlinearities must satisfy

$$2\tilde{x}^T\tilde{\Phi}<\tilde{x}^TQ\tilde{x} \quad (13)$$

This latter condition ensures that the linear part of the observation error dynamics dominates the nonlinearities. To facilitate this, the Lipschitz continuity property is typically invoked, as all nonlinearities satisfy the Lipschitz continuity condition, at least locally. One of the notable outcomes in this topic is discussed in [29], under the following assumption:

Assumption 1. [30] The pair (A, C) is observable and there exists a positive constant η such that

$$\eta P + A^T P + PA - C^T C = 0 \tag{14}$$

with P being a symmetric definite positive matrix. The observer gain L is selected as $L = \sigma P^{-1} C^T$, with $\sigma \geq 1$.

The asymptotic convergence of the corresponding observation error to the origin is guaranteed by the following proposition, under the conditions outlined below.

Proposition 1. [28] If Assumption 1 holds, with Φ satisfying the condition specified in (8), and if

$$\gamma_\phi < \frac{\lambda_{\min}(\eta P + (2\sigma - 1)C^T C}{2\lambda_{\max}(P)}, \tag{15}$$

then the dynamical system

$$\dot{\hat{x}} = A\hat{x} + \Phi(\hat{x}, u) - \sigma P^{-1} C^T (C\hat{x} - y) \tag{16}$$

is an observer for the system (12).

Remark 1. A notable strength of this observer in comparison to the other existing techniques lies in the fact that the upper bound of the Lipschitz constant γ_Φ is dependent on the parameter η. Specifically, one can identify a value η that maximizes the righthand side of inequality (17). Additionally, the term $\lambda_{\min}(\eta P + (2\sigma - 1)C^T C)$ can be made arbitrarily large by selecting a sufficiently large $\sigma \geq 1$. These provides more flexibility in observer design for nonlinear system with relatively large Lipschitz constants.

Recently, this observation technique has been extended in [22] to be applicable to the class of fractional-order systems, modeled by:

$$\begin{aligned} D^\alpha x &= Ax + \Phi(x, u) \\ y &= Cx \end{aligned} \tag{17}$$

where $\alpha \in (0,1)$ is the system's fractional order.

The proposed fractional-order observer is structured as follows:

$$\begin{aligned} &D^\alpha \hat{x} - A\hat{x} + \Phi(\hat{x}, u) - \sigma P^{-1} C^T (C\hat{x} - y) \\ &y = C\hat{x} \end{aligned} \tag{18}$$

The observation error dynamical system is globally Mittag-Leffler stable under specific sufficient conditions.

Theorem 1. If Assumption 1 holds and Φ meets the Lipschitz condition (8), with the satisfaction of the condition outlined in (15), then the dynamical system defined in (18) serves as a global Mittag-Leffler observer for the class of systems considered in (17).

Regrettably, for fractional-order nonlinear systems subject to unknown slowly time-varying parameters, as described in (7), the implementation of this fractional-order observer is deemed impractical. Therefore, our primary objective is to extend this approach into an adaptive observer to address the problem outlined in the previous section.

5 Main results

In this section, we detail our adaptive observer design methodology, starting with foundational assumptions crucial for ensuring stability, as analyzed next.

Assumption 2. For any symmetric positive definite matrix *P*, there exists a positive scalar $\boldsymbol{\eta}$ such that:

$$\eta P + A^T P + PA - C^T C < 0 \tag{19}$$

$$B^T P C^\perp = 0 \tag{20}$$

given that the system (A, C) is observable and $C^\perp$ represents the orthogonal projection onto null (C).

Assumption 3. The system's unknown, slowly time-varying parameters are bounded by θ_M, i.e.

$$|\theta(t)| \leq \theta_M, \forall t \geq t_0$$

Based on the above assumptions, we suggest the following observation scheme:

$$\begin{aligned} D^\alpha \hat{x}(t) &= A\hat{x} + \Phi(\hat{x}(t), u) + B\psi(\hat{x}, u)\hat{\theta} - \\ &\quad \sigma P^{-1} C^T (C\hat{x} - y) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{21}$$

for $t \geq t_0$ and $\sigma \geq 1$. Augmented by the following updating law

$$D^\alpha \hat{\theta} = -\mu\, \psi^T(\hat{x}, u) B^T P \tilde{x}(t) \tag{22}$$

The following theorem summarizes our findings and establishes the sufficient conditions under which both the observation and unknown parameter estimation errors remain stable.

Theorem 2. Assuming all the trajectories of system (7) remain bounded, the observer maintains its structure as defined in (21), with the unknown parameters updating law (22). Given Assumptions 2, 3 hold, and if

$$\gamma_\Phi + \gamma_\psi \theta_M \parallel B \parallel < \frac{1}{2} \frac{\lambda_{\min}(\eta P + (2\sigma - 1)C^T C)}{\lambda_{\max}(P)} \quad (23)$$

it follows that

- $\tilde{x}(t), \tilde{\theta}(t)$ are bounded $\forall t \geq t_0$,
- the mean value of $\parallel \tilde{x}(t) \parallel^2$ converge to zero as t approaches the infinity.

Proof. Let us assume that $\tilde{\theta}$ is differentiable. Then, by referring to Lemma 1, we derive the following inequality:

$$D_t^\alpha \left[\tilde{x}^T(t) P \tilde{x}(t) + \frac{1}{\mu}\left\|\tilde{\theta}(t)\right\|^2\right] \leq$$
$$2\tilde{x}^T P (D_t^\alpha \tilde{x}) + \frac{2}{\mu}\tilde{\theta}^T D_t^\alpha \tilde{\theta}, \qquad \forall t \geq t_0 \quad (24)$$

Substituting the observer error dynamics in the right-hand side of inequality (24), one can get:

$$D_t^\alpha \left[\tilde{x}^T(t) P \tilde{x}(t) + \frac{1}{\mu}\left\|\tilde{\theta}(t)\right\|^2\right] \leq$$
$$\left[\tilde{x}^T (A - \sigma P^{-1} C^T C)^T + \tilde{\Phi}^T\right]^T P \tilde{x} +$$
$$\tilde{x}^T P\left[(A - \sigma P^{-1} C^T C)\tilde{x} + \tilde{\Phi}\right] + 2\tilde{x}^T P B\left(\hat{\psi}\hat{\theta} - \psi\theta\right) +$$
$$\frac{2}{\mu}\tilde{\theta}^T D_t^\alpha \hat{\theta} \leq$$
$$\tilde{x}^T[(A - \sigma P^{-1} C^T C)^T P + P(A - \sigma P^{-1} C^T C)]\tilde{x} +$$
$$2\tilde{x}^T P \tilde{\Phi} + 2\tilde{x}^T P B\left(\hat{\psi} - \psi\right)\theta + 2\tilde{x}^T P B\, \hat{\psi}\left(\hat{\theta} - \theta\right) +$$
$$\frac{2}{\mu}\tilde{\theta}^T D_t^\alpha \hat{\theta} \leq \tilde{x}^T (A^T P - 2\sigma C^T C + PA)\tilde{x} + 2\tilde{x}^T P \tilde{\Phi} +$$
$$2\tilde{x}^T P B \tilde{\psi}\theta 2\tilde{x}^T P B\, \hat{\psi}\tilde{\theta} + \frac{2}{\mu}\tilde{\theta}^T D_t^\alpha \hat{\theta} \quad (25)$$

Building on Assumptions 2, 3, and integrating the updating law dynamics (22), we can formally derive the following outcome

$$D_t^\alpha \left[\tilde{x}^T(t) P \tilde{x}(t) + \frac{1}{\mu}\left\|\tilde{\theta}(t)\right\|^2\right] \leq$$
$$-\eta\tilde{x}^T P \tilde{x} + (1 - 2\sigma)\tilde{x}^T C^T C \tilde{x} +$$
$$2\,\|\tilde{x}\|\,\|P\|\,\left\|\tilde{\Phi}\right\| \,+\, 2\,\|\tilde{x}\|\,\|P\|\,\|B\|\,\left\|\tilde{\psi}\right\|\,\|\theta\|$$
$$\leq\, -\lambda_{\min}(\eta P + (2\sigma - 1)C^T C)\,\|\tilde{x}\|^2 +$$
$$2\gamma_\Phi \lambda_{\max}(P)\,\|\tilde{x}\|^2 + 2\gamma_\psi \theta_M\,\|B\|\,\lambda_{\max}(P)\,\|\tilde{x}\|^2 \quad (26)$$

Per condition in (23), we obtain the ensuing outcome

$$D_t^\alpha \left[\tilde{x}^T(t) P \tilde{x}(t) + \frac{1}{\mu}\left\|\tilde{\theta}(t)\right\|^2\right] \leq -m\,\|\tilde{x}\|^2 \quad (27)$$

with

$$m = \lambda_{\min}(\eta P + (2\sigma - 1)C^T C) -$$
$$2\left(\gamma_\Phi + \gamma_\phi \theta_M \|B\|\,\right)\lambda_{\max}(P) > 0$$

The fractional-order integral of this last inequality gives

$$\left[\tilde{x}^T(t) P \tilde{x}(t) + \frac{1}{\mu}\left\|\tilde{\theta}(t)\right\|^2 - \tilde{x}^T(t_0) P \tilde{x}(t_0) -\right.$$
$$\left.\frac{1}{\mu}\left\|\tilde{\theta}(t_0)\right\|^2\right] \leq -\mu I^\alpha \|\tilde{x}(t)\|^2 \quad (28)$$

Since$I^\alpha \parallel \tilde{x}(t) \parallel^2 \geq 0, \forall t \geq t_0$, with P being a symmetric positive definite matrix and $\mu > 0$, it consequently follows that

$$\tilde{x}^T(t) P \tilde{x}(t) + \frac{1}{\mu}\left\|\tilde{\theta}(t)\right\|^2 \leq$$
$$\tilde{x}^T(t_0) P \tilde{x}(t_0) + \frac{1}{\mu}\left\|\tilde{\theta}(t_0)\right\|^2 \quad (29)$$

With finite initial conditions $\tilde{x}(t_0)$ and $\tilde{\theta}(t_0)$, inequality (29) implies that $\tilde{x}(t), \tilde{\theta}(t))$ are uniformly bounded $\forall t \geq t_0$.

Given the inequality (28) and the bounded nature of $\tilde{x}(t), \tilde{\theta}$, it can be concluded that:

$$I^\alpha \|\tilde{x}(t)\|^2 < M, \quad M \in \mathrm{R} \quad (30)$$

Thus, by applying Lemma 2, we obtain:

$$\lim_{t \to +\infty}\left[t^{\alpha - \epsilon}\, \frac{\int_{L_0}^{t} \|\tilde{x}(\tau)\|^2 d\tau}{t}\right] = 0, V_\epsilon > 0 \quad (31)$$

The result derived in (31) is equivalent to the assertion that the mean value of $\|\tilde{x}(t)\|^2$ asymptotically converges to the origin.

6 Simulation results

In this section, we conduct a numerical analysis to assess the performance of our proposed adaptive observation approach for state estimation and parameter identification in a fractional-order financial system [31], defined by

$$\begin{aligned} D^\alpha x_1 &= -s x_1 + x_3 + \theta x_1 x_2 \\ D^\alpha x_2 &= -c x_2 + 1 - x_1^2 \\ D^\alpha x_3 &= -x_1 - e x_3 \end{aligned} \quad (32)$$

where the state variables x_1, x_2, and x_3 represent the interest rate, investment demand, and price index, respectively. The associated system parameters are defined as follows: s corresponds to the total savings, c represents the cost per unit of investment, and e quantifies the elasticity of demand within commercial markets.

The scenario under consideration assumes that the parameter θ is unknown, slowly time-varying, and takes the form of a sinusoidal wave, mathematically represented as follows:

$$\theta(t) = z_0 + z_1 \sin(2\pi f t) \quad (33)$$

The system parameters are configured as $a = 0.3$, $b = 2, c = 4$, and $\alpha = 0.85$. The sinusoidal wave signal parameters are set to $z_0 = 1.3$, $z_1 = 0.3, f = \frac{1}{40}$. The initial conditions are specified as $x_0 = [-0.8, -2, 1]$. It is important to note that, in addition to the unknown parameter θ, the price index (x_3) is not directly measurable. The system's nonlinear function vectors are $\Phi = [0, 1 - x_1^2, 0]^T$ and $\psi = [x_1 x_2, 0, 0]^T$. As the Lipschitz constants cannot be directly inferred, we proceed with their numerical estimation by evaluating the norms of the corresponding Jacobian matrices

$$\frac{\partial\Phi(\mathrm{x},\mathrm{u})}{\partial \mathrm{x}} = \begin{bmatrix} 0 & 0 & 0 \\ -2x_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \frac{\partial\psi(\mathrm{x},\mathrm{u})}{\partial \mathrm{x}} = \begin{bmatrix} x_2 & x_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the obtained results are illustrated in Figs. 1 and 2.

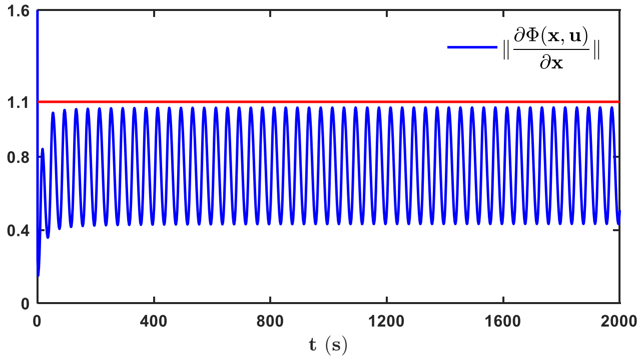

Fig. 1. Time evolution of the Jacobian matrix norm of the nonlinear function Φ

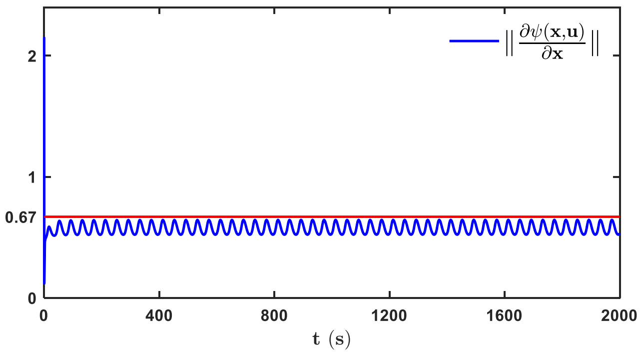

Fig. 2. Time evolution of the Jacobian matrix norm of the nonlinear function ψ

From these, it is clear that the appropriate choices for the Lipschitz constants are $\gamma_\phi = 1.1$ and $\gamma_\psi = 0.67$. The observer is synthesized following the methodology presented in Section 5, with the parameters set as $\eta = 7.05$ and $\sigma = 1$. The linear matrix inequality (LMI) in (21) and the linear matrix equality (LME) in (22) are solved using the YALMIP Toolbox, yielding the following results:

$$P = \begin{bmatrix} 0.1154 & 0 & 0 \\ 0 & 0.2400 & 0 \\ 0 & 0 & 0.2152 \end{bmatrix}$$

$$L = \begin{bmatrix} 8.6621 & 0 \\ 0 & 4.1662 \\ 0 & 0 \end{bmatrix}$$

Consequently,

$$\gamma_\Phi + \gamma_\psi 0_M \|B\| =$$

$$2.44 < \frac{1}{2}\frac{\lambda_{\min}(\eta P + (2\sigma - 1)C^T C)}{\lambda_{\max}(P)} = 3.138$$

Thus, we ensure the satisfaction of condition (23). The observer state vector $\hat{x}(t)$ and the estimated unknown parameter $\hat{\theta}(t)$ are initialized to zero. The simulation results shown in Fig. 3, which compare the observed states with the actual ones, clearly demonstrate that the proposed observer accurately reconstructs the full state vector of the financial system, even in the presence of an unknown, slowly time-varying parameter. This is further validated by the asymptotic convergence of the observation errors, as shown in Fig. 4. Regarding the parameter estimation, the proposed observer provides instantaneously precise estimations as illustrated in Figs. 5 and 6. These results confirm the observer's efficiency in addressing challenges related to both state and parameter estimation in fractional-order Lipschitz nonlinear systems.

7 Conclusion

In this work, we have introduced an adaptive fractional-order observation technique for the reconstruction of unmeasurable states and the estimation of slowly time-varying unknown parameters in continuous-time fractional-order nonlinear dynamical systems that satisfy the Lipschitz condition. The key contribution of this research lies in the extension of an existing observer, enabling it to handle systems with unknown slowly time-varying parameter. The existence and stability conditions for the adaptive observer are formulated through a Linear Matrix Inequality (LMI) and a Linear Matrix Equality (LME). Based on the principle of fractional calculus, we demonstrate the boundedness of both the observation and the parameter estimation errors. Additionally, we have proven the asymptotic convergence of the observation error norm square mean value to the origin.

This technique provides a reliable framework for a wide range of applications in fields such as control systems for fault detection, financial and biological systems for parameter identification, where fractional-order models are frequently employed for their ability to capture complex dynamics.

Despite these significant contributions, several limitations of the current observation technique warrant further investigation. Notably, the robustness of the observer against external disturbances, such as noise and unmodeled dynamics, needs to be explored in more depth.

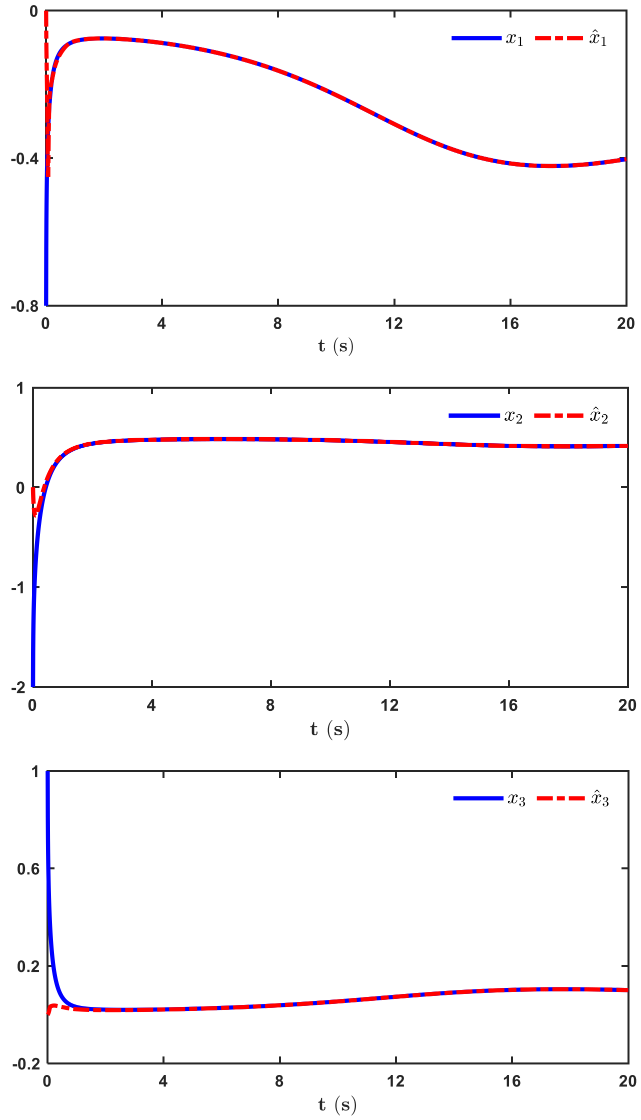

Fig. 3. Time response of actual and observed state trajectories

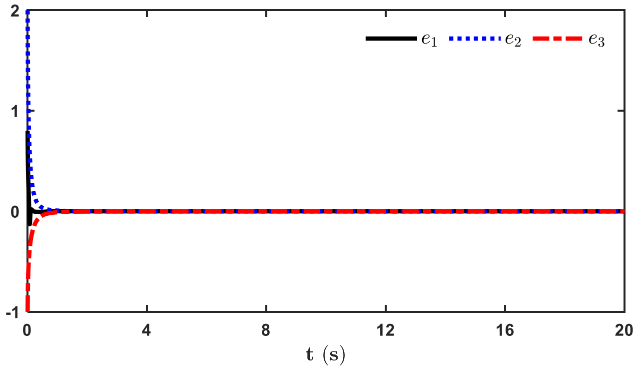

Fig. 4. Evolution of the observation error over time

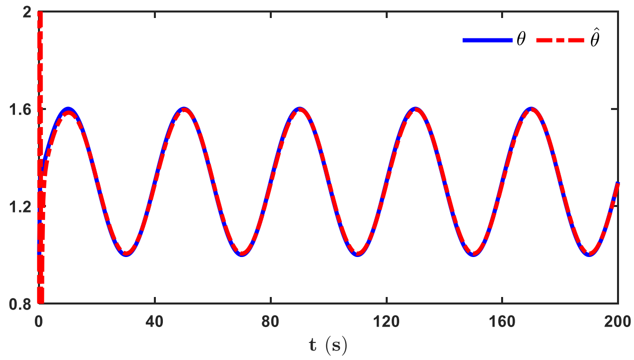

Fig. 5. Temporal evolution of the unknown parameter and its adaptive estimate

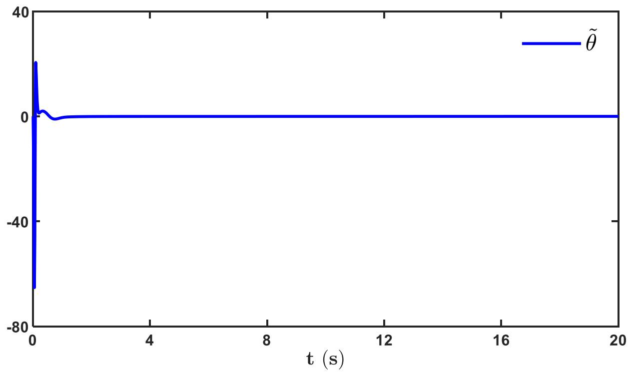

Fig. 6. Time evolution of the estimation error for the unknown parameter

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Statement and declaration

Conflict of interest: The authors declare that there is no conflict of interest.

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