NUMERICAL ANALYSIS OF ELECTROMAGNETIC AXISYMMETRIC PROBLEMS USING ELEMENT FREE GALERKIN METHOD

Fatima-Zohra Louaï — Nasreddine Naït-Saïd — Said Drid *

The element free Galerkin method (EFGM) is a method for solving partial differential equations with moving least squares interpolants. This method, based as the finite element method on an integral formulation, requires only a set of nodes distributed on the analysis area for the construction of the shape functions, no element connectivity is needed. The objective of this paper is to present the application of EFGM to axisymmetric electromagnetic problems which describe wide types of induction heating device. A non-uniform nodal distribution algorithm is developed. Numerical examples show the effect of the nodal distribution (uniform or non-uniform distribution) on the solution. The results compared to the finite element method (FEM) show the accuracy and the power of the developed algorithm.

K e y w o r d s: element free Galerkin method (EFGM), moving least square (MLS) method, weight function, influence domain, non-uniform nodal distribution

1 INTRODUCTION

The FEM for the solution of complex problems in science and engineering is well established. It is a robust and thoroughly developed technique, but it is not without shortcomings. Generating adequate discretization meshes is a difficult task, in particular for complex three-dimensional domains or for modelling large deformation processes, where considerable loss in accuracy arises when the elements in the mesh become extremely skewed or compressed [1, 2].

Recently, interest has grown in a new class of methods called meshless methods. In these methods, only a set of nodes and boundary description are needed to formulate the discrete equations. The EFGM which is regarded as a meshless method employs the MLS approximation to approximate the unknown function. The MLS method was introduced and studied first by Lancaster *et al* [3], in curve and surface fitting. Essential idea of this method is that a global approximation can be achieved by going through a moving process. This approximation is constructed from three components: a weight function associated to each node, a polynomial basis and a set of coefficients which depend on the position. The first to use MLS interpolants in conjunction with a Galerkin method were Nayrols *et al* [1]. They applied their method to linear problems of heat conduction. Also Belytschko *et al* [2,4] applied MLS interpolants to solve problems of elasticity and crack propagation. Afterwards this method was found very promising in electromagnetic problems [5–8]. The major disadvantage of the EFGM is that, since the MLS interpolants do not pass through the data because the interpolation functions are not equal to unity at the nodes, it is complicating the imposition of essential boundary conditions. To overcome this difficulty, several solutions are presented in literature: the use of Lagrange multipliers [2], the modified variational principle [4] or coupling of the EFGM with the FEM on the boundary [6].

This paper presents an application of the EFGM to axisymmetric electromagnetic problems with the study of the sensitivity of the approximation solution to different types of nodal distribution (uniform or non-uniform distribution). It shows that the use of non-uniform nodal distribution can increase the accuracy with decreasing the computation time.

The outline of this paper is as follows: in Section 2, description of the MLS method principle is given. In Section 3, the EFGM is applied to formulate axisymmetric electromagnetic problems. Numerical examples which show improvements in accuracy and computation time due to the non-uniform nodal distribution are given in Section 4, followed by conclusions in Section 5.

2 MLS APPROXIMATION

The MLS method is an approach adopted to develop the approximations of the meshless methods. Consider the Cartesian coordinates system. For approximation of the unknown function $u(\mathbf{x})$ the EFG method utilizes the MLS method in which the interpolant function $u^h(\mathbf{x})$ of the function $u(\mathbf{x})$ is given by [2]

$$u^h(\mathbf{x}) = \sum_{j}^{m} p_j(\mathbf{x}) a_j(\mathbf{x}) = P^T(\mathbf{x}) a(\mathbf{x}) \,, \tag{1}$$

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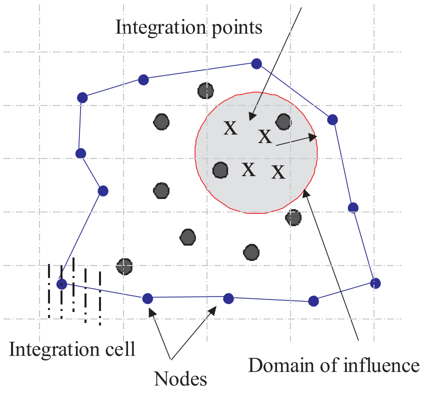

Fig. 1. Geometrical model of EFG method.

where $P^T(\mathbf{x})$ is a complete polynomial basis of order m in space coordinates $\mathbf{x}^T = [x, y]$. Common linear and quadratic bases in $1D$ and $2D$ space are

- Linear bases:

$$\begin{aligned} p^T(\mathbf{x}) &= [1, x] && \text{in } 1D\,, \\ p^T(\mathbf{x}) &= [1, x, y] && \text{in } 2D\,. \end{aligned} \tag{2}$$

- Quadratic bases:

$$\begin{aligned} p^T(\mathbf{x}) &= [1, x, x^2] && \text{in } 1D\,, \\ p^T(\mathbf{x}) &= [1, x, y, x^2, xy, y^2] && \text{in } 2D\,. \end{aligned} \tag{3}$$

The $a(\mathbf{x})$ components are unknown parameters which are functions of position $\mathbf{x}$

$$a^T(\mathbf{x}) = [a_0(\mathbf{x}), a_1(\mathbf{x}), a_2(\mathbf{x}), \dots, a_m(\mathbf{x})]\,. \tag{4}$$

Thus, to determine these coefficients, it is necessary to minimize the following weighted quadratic form:

$$J = \sum_{I=1}^{n} w(\mathbf{x} - \mathbf{x}_I)\left[u_L^h(\mathbf{x}_I, \mathbf{x}) - u_I\right]^2, \tag{5}$$

where $u_L^h(\mathbf{x}_I, \mathbf{x})$ is the local approximation at point $\mathbf{x}$, defined by

$$u_L^h(\mathbf{x}_I, \mathbf{x}) = \sum_{j}^{m} p_j(\mathbf{x}_I) a_j(\mathbf{x}) = P^T(\mathbf{x}_I) a(\mathbf{x})\,. \tag{6}$$

Then

$$J = \sum_{I=1}^{n} w(\mathbf{x} - \mathbf{x}_I)\left[P^T(\mathbf{x}_I) a(\mathbf{x}) - u_I\right]^2, \tag{7}$$

where n is the number of nodes in the neighborhood of $\mathbf{x}$ for which the weight function $w(\mathbf{x} - \mathbf{x}_I) \neq 0$ and u_I are the nodal parameters of $u(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}_I$. This neighborhood of $\mathbf{x}$ is called the influence domain of $\mathbf{x}$.

Minimization of J in Eq. (7) with respect to $a(\mathbf{x})$ leads to the following linear relation between $a(\mathbf{x})$ and $\mathbf{u}$:

$$\mathbf{A}(\mathbf{x}) a(\mathbf{x}) = \mathbf{B}(\mathbf{x}) \mathbf{u}\,, \tag{8}$$

where

$$\mathbf{u}^T = [u_1, u_2, u_3, \dots, u_n]\,. \tag{9}$$

Then

$$a(x) = \mathbf{A}^{-1}(x) \mathbf{B}(x) u\,, \tag{10}$$

where

$$\mathbf{A}(\mathbf{x}) = \sum_{I=}^{n} w(\mathbf{x} - \mathbf{x}_I) P(\mathbf{x}_I) P^T(\mathbf{x}_I) \tag{11}$$

and
$$\begin{aligned} \mathbf{B}(\mathbf{x}) = \big[& w(\mathbf{x} - \mathbf{x}_1) P(\mathbf{x}_1), \\ & w(\mathbf{x} - \mathbf{x}_2) P(\mathbf{x}_2), \dots, w(\mathbf{x} - \mathbf{x}_n) P(\mathbf{x}_n)\big]\,. \end{aligned} \tag{12}$$

By substituting (10) into (1), the MLS approximants can be defined as

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \phi(\mathbf{x}) u_I = \phi(\mathbf{x}) \mathbf{u}\,, \tag{13}$$

where the shape function ϕ_I is defined by

$$\phi_I(\mathbf{x}) = \sum_{j}^{m} p_j(\mathbf{x}) \big(\mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x})\big)_{jI}\,. \tag{14}$$

In matrix form we have

$$\phi_I(\mathbf{x}) = P^T \big(\mathbf{A}^{-1} \mathbf{B}\big)_I\,. \tag{15}$$

The weight function is defined with a compact support, often called the domain of influence (Fig. 1). This domain can be circular, elliptic or rectangular [9], its width is defined by

$$\mathrm{d}_{mI} = \mathrm{d}_{\max} C_I\,, \tag{16}$$

where $\mathrm{d}_{\max}$ is a scaling factor which is chosen such that matrix $\mathbf{A}$ is non-singular [7] and C_I is the difference between node x_I and its nearest neighbour when the nodal distribution is uniform. If the nodes are non-uniformly distributed, C_I is taken as follows [2]:

$$C_I = \max_{J \in S_J} \|x_J - x_I\|\,, \tag{17}$$

where S_J is the minimum set of neighbouring points of x_I which construct a polygon surrounding point x_I.

Cubic spline, exponential and Gaussian functions are some of the commonly used weights [8]. The weight function used in this paper is the cubic spline function:

$$w(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & \text{for } r \leq \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & \text{for } \frac{1}{2} < r \leq 1 \\ 0 & \text{for } r > 1 \end{cases} \tag{18}$$

where $r = \mathrm{d}_I / \mathrm{d}_{mI}$ is the normalized distance with $d_I = \|x - x_I\|$.

If the influence domain is rectangular, the weight function is derived as a tensor product which is described in [9].

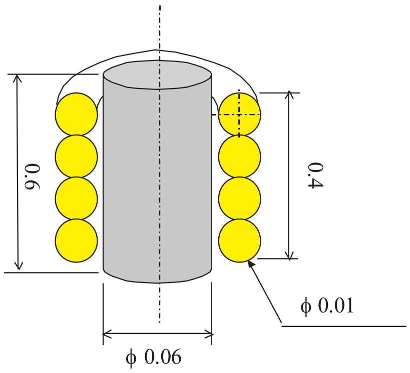

Fig. 2. The heating device under study.

3 VARIATIONAL FORMULATION OF AN AXISYMMETRIC PROBLEM

The axisymmetric problem describing most induction heating devices is obtained from the electromagnetic equation in terms of the magnetic vector potential $\vec{A}$:

$$\overrightarrow{curl}\big(v\overrightarrow{curl}\vec{A}\big)+\sigma\frac{\partial\vec{A}}{\partial t}=\vec{J}\,, \tag{19}$$

where v, σ, $\vec{J}$ are, respectively, the magnetic reluctivity, the electric conductivity and the source current density. In the axisymmetric case the electric current density $\vec{J}$ has only the φ-component which is independent of φ, so the resulting magnetic vector potential $\vec{A}$ has only the φ-component. Using 2D cylindrical r, z coordinates; equation (19) is developed as

$$-\frac{\partial}{\partial r}\Big(\frac{1}{r\mu_r}\frac{\partial(rA_\varphi)}{\partial r}\Big)-\frac{\partial}{\partial z}\Big(\frac{1}{\mu_r}\frac{\partial A_\varphi)}{\partial z}\Big)+j\omega\sigma\mu_0A_\varphi = \mu_0J_\varphi\,, \tag{20}$$

where μ_r is the relative magnetic permeability and μ_0 is the permeability of free space. In harmonic analysis, the time derivative $(\partial\vec{A}/\partial t)$ is replaced by the complex operation $j\omega\vec{A}$ $(j^2=-1)$, ω is the angular frequency. Equation (20) can also be written as

$$-\frac{\partial}{\partial r}\Big(\frac{1}{r\mu_r}\frac{\partial(rA_\varphi)}{\partial r}\Big)-\frac{\partial}{\partial z}\Big(\frac{1}{r\mu_r}\frac{\partial(rA_\varphi)}{\partial z}\Big)+j\frac{\omega\sigma\mu_0}{r}(rA_\varphi) = \mu_0J_\varphi\,. \tag{21}$$

If we introduce the transformation

$$A=rA_\varphi\,, \tag{22}$$

equation (21) becomes

$$-\frac{\partial}{\partial r}\Big(\frac{1}{r\mu_r}\frac{\partial A}{\partial r}\Big)-\frac{\partial}{\partial z}\Big(\frac{1}{r\mu_r}\frac{\partial A}{\partial z}\Big)+j\frac{\omega\sigma\mu_0}{r}A=\mu_0J_\varphi\,. \tag{23}$$

Let Ω be the study domain enclosed by $\Gamma=\Gamma u\cup\Gamma_q$ with boundary conditions

$$A=\overline{A}\quad \text{on } \Gamma_u\,, \tag{24a}$$

$$\frac{1}{r}\frac{\partial A}{\partial n}=\overline{q}\quad \text{on } \Gamma_q\,. \tag{24b}$$

$\overline{A}$ and $\overline{q}$ are the prescribed unknown potential and normal magnetic flux, respectively, on the essential boundary Γ_u and on the flux boundary Γ_q and n is the unit outward normal direction to the boundary Γ_q.

The functional corresponding to the axisymmetric equation (23) with boundary conditions (24) is given by

$$I=\frac{1}{2}\int_\Omega\Big(\frac{1}{r\mu_r}\big(\frac{\partial A}{\partial r}\big)^2+\frac{1}{r\mu_r}\big(\frac{\partial A}{\partial z}\big)+j\frac{\omega\sigma\mu_0}{r}A^2- 2\mu_0J_\varphi A\Big)\mathrm{d}r\mathrm{d}z+\int_{\Gamma_q}\frac{\partial A}{\partial n}A\mathrm{d}s+\int_{\Gamma_u}\lambda(A-\overline{A})\mathrm{d}s\,, \tag{25}$$

where λ is the Lagrange multiplier used to impose the essential boundary conditions. It can be expressed by

$$\lambda(s)=\sum_i N_i(s)\lambda_i\,, \tag{26}$$

where $N_i(s)$ is a Lagrange interpolant and s is the arc length along the boundary Γ_u. The necessary condition for (25) to reach its minimum yields [3]

$$\delta_AI(A,\lambda)=\int_\omega\Big[(\delta A)_r\frac{A_r}{r\mu_r}+(\delta A)_z\frac{A_z}{r\mu_r}+(\delta A)\frac{j\omega\sigma}{r}A -\mu_0J(\delta A)\Big]\mathrm{d}r\mathrm{d}z-\int_{\Gamma_q}\overline{q}\delta A\mathrm{d}s+\int_{\Gamma_u}\lambda\delta A\mathrm{d}s=0\,, \tag{27}$$

$$\delta_\lambda I(A,\lambda)=\int_{\Gamma_u}\delta\lambda(A,\lambda)=0\,. \tag{28}$$

δA and $\delta\lambda$ are test functions, $(\delta A)_r$ and $(\delta A)_z$ (A_r and A_z) are the derivatives of δA (A) in terms, respectively, of r and z.

The MLS approximation of A, as described above can be defined as

$$A(r,z)=\sum_{I=1}^{n}\phi_I(r,z)A_I=\phi(r,z)A\,. \tag{29}$$

Substituting (29) and (26) into (27) and (28), letting $\delta A=\phi_I(X)$, $\delta\lambda=N_I(s)$ and then integrating, one obtains

$$\begin{bmatrix}\mathbf{K} & \mathbf{G}\\ \mathbf{G}^T & \mathbf{0}\end{bmatrix}\begin{bmatrix}A\\ \lambda\end{bmatrix}=\begin{bmatrix}F\\ q\end{bmatrix}, \tag{30}$$

where

$$k_{ij}=\iint_\Omega\Big[(\phi_i)_r\frac{1}{r\mu_r}(\phi_j)_r+(\phi_i)_z\frac{1}{r\mu_r}(\phi_j)_z +\frac{j\omega\sigma}{r}\phi_i\Big]\mathrm{d}r\mathrm{d}z\,, \tag{31}$$

$$g_{i,j}=\int_{\Gamma_u}\phi_iN_j\mathrm{d}s\,, \tag{32}$$

$$f_i=\iint_\Omega\mu_0J\phi_i\mathrm{d}r\mathrm{d}z+\int_{\Gamma_q}\overline{q}\phi_i\mathrm{d}s\,, \tag{33}$$

$$q_i=\int_{\Gamma_u}N_i\overline{A}\mathrm{d}s\,. \tag{34}$$

ϕ_r and ϕ_z are the derivatives of ϕ in terms of r and z, respectively. In order to obtain the integrals above (equations 31–34) by Gauss-quadrature formula, a cell structure which is independent of the nodes is used (see Fig. 1).

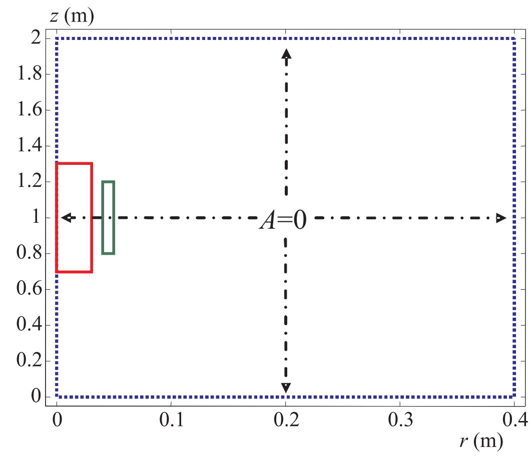

Fig. 3. The model of the studied domain ($A = 0$ on boundaries).

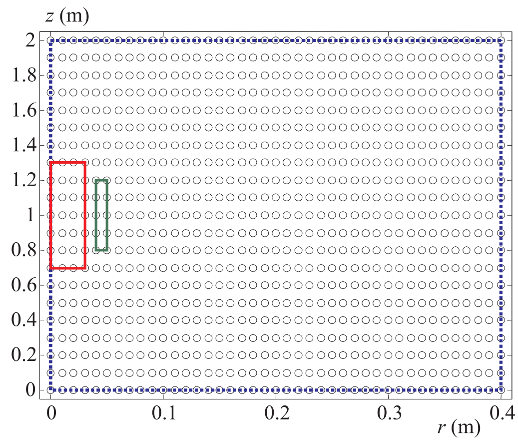

Fig. 4a. Uniform nodes spacing $n_r = [3\ 1\ 1\ 35]$; $n_z = [7\ 1\ 4\ 1\ 7]$; $41 \times 21 = 861$ nodes.

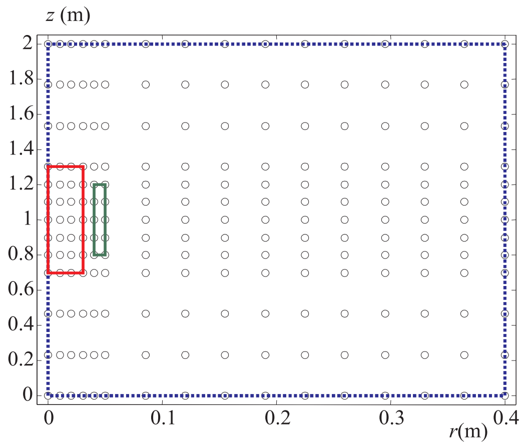

Fig. 4b. Uniform per interval nodes spacing $n_r = [3\ 1\ 1\ 10]$; $n_z = [3\ 1\ 4\ 1\ 3]$; $16 \times 13 = 208$ nodes.

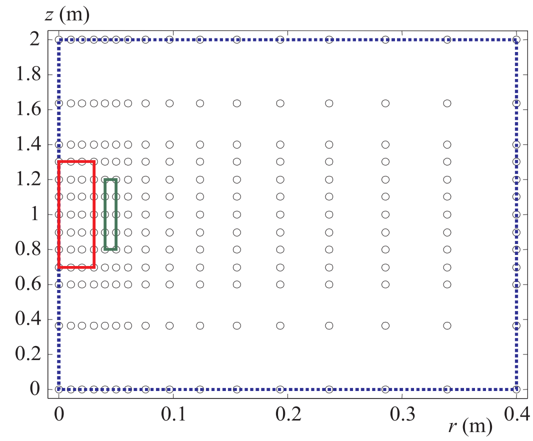

Fig. 4c. Non-uniform nodes spacing $n_r = [3\ 1\ 1\ 10]$; $n_z = [3\ 1\ 4\ 1\ 3]$; $16 \times 13 = 208$ nodes.

4 NUMERICAL RESULTS

The test problem is an axisymmetric heating device (Fig. 2) which presents three main regions:

1. The inductor for which the excitation current density has 50 Hz frequency and 5×10^6 A/m^2 magnitude.
2. The non-magnetic load in which the eddy current is developed has an electric conductivity of 5×10^6 S/m.
3. The free space surrounds the inductor and the load.

The Dirichlet condition $\overline{A} = 0$ is applied on all boundaries shown in Fig. 3. In this paper we study the effect of the nodes spacing, the EFG results are compared with those obtained by FEM. Since this problem has no analytical solution, a large number of nodes (34669 nodes) are used in FEM to approximate almost exactly the exact solution.

In this case uniform, uniform per interval (region) and non-uniform node spacings are used, the last one is obtained by using arithmetic suite in the regions where the phenomenon is less important (free space). These three types of nodal distributions are shown in Figures 4a, 4b and 4c, respectively. The distributions are done using vectors n_r and n_z whose components define the nodes spacing in each interval. The number of nodes in the entire domain is given by

$$M = \Big(1 + \sum_{i=1}^{4} n_r(i)\Big)\Big(1 + \sum_{j=1}^{5} n_z(j)\Big) . \tag{35}$$

The sensitivity of EFG solution to the nodes spacing type studied by comparison with FE method's results is obtained by using the error norm defined as

$$\|e_M\| = \frac{1}{M}\sqrt{\sum_{j=1}^{M}(u - u_{FEM})^2} . \tag{36}$$

where M is the total number of nodes used by the EFGM in the studied domain and u_{FEM} is the solution interpolated from the FEM solution on the nodes used in EFGM, since the number of nodes is very different in the two methods.

To obtain good accuracy it is necessary to increase the number of nodes, in particularly in the regions, where the phenomenon is greatly important (inductor and load).

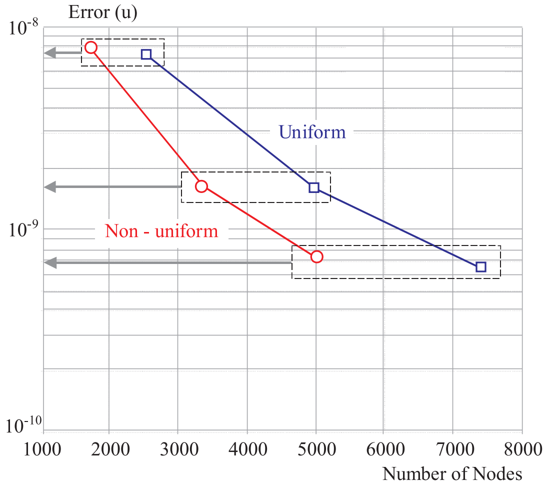

Fig. 5a. Estimation error in uniform and non-uniform nodes spacing.

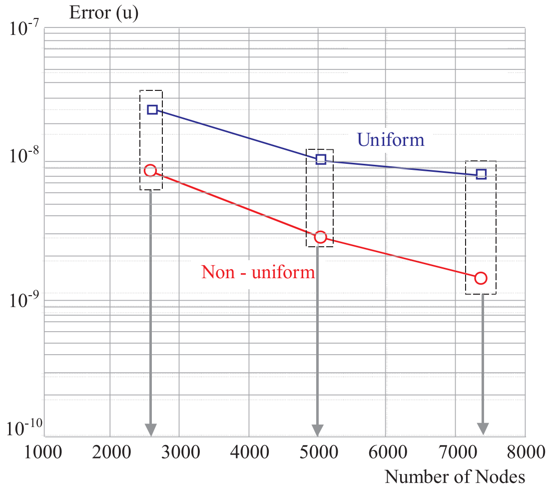

Fig. 5b. Estimation error in non-uniform and uniform per interval nodes spacing.

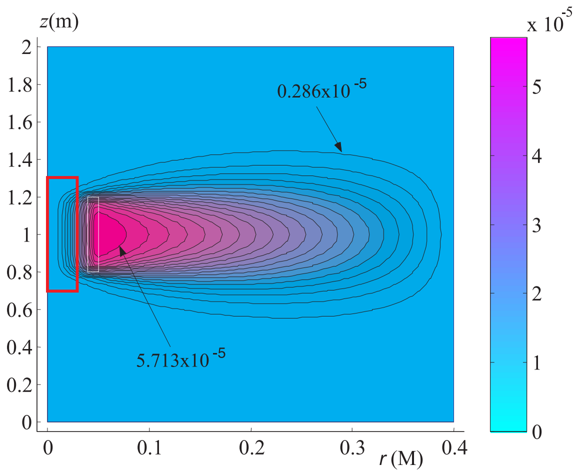

Fig. 6a. EFG solution in term of potential $n_r = 3\,[3\;1\;1\;25]$; $n_z = 3\,[6\;1\;4\;1\;6]$.

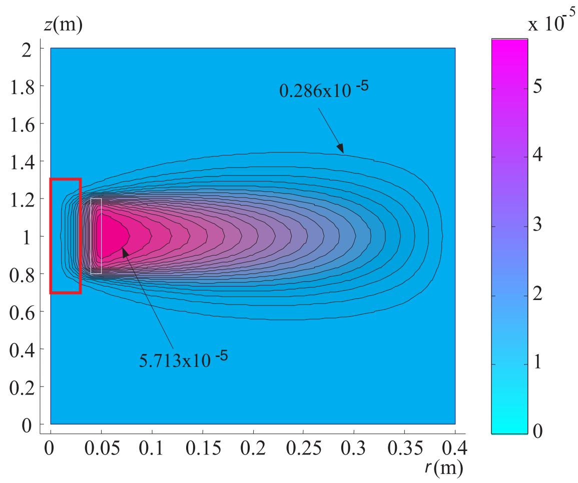

Fig. 6b. MEF solution in term of potential.

For this, the uniform distribution leads to a large number of nodes also in the free space region, while the non-uniform technique proposed allows using few nodes in this region utilizing the arithmetic suite. This property involves a less computation time but with more accuracy as shown in Fig. 5a. On the other hand, we propose the use of a uniform nodal distribution per region, in this case the same number of nodes per region as in the non-uniform distribution case is used. The results given in Fig. 5b show that for the same total number of nodes, the accuracy is better in the second case.

Figures 6a and 6b illustrate the comparison of the equipotential contours obtained both by the EFGM with non-uniform nodes spacing (a) and by the FEM (b). This result proves that EFGM with the nodal distribution proposed can provide similar computation results to those of FEM.

5 CONCLUSION

An axisymmetric 2D electromagnetic problem is treated by means of the EFG method. This study which concerns with a heating induction device is based on three types of nodal distributions: uniform, uniform per region and non-uniform using the arithmetic suite. It is proved that this latter gives more accurate results with a reduced number of nodes and consequently with a lower computation time since it permits optimization of the number of nodes in the regions, where the phenomenon is not greatly important. Indeed, using arithmetic suite allows the concentration of the nodes in the parts, where the phenomena vary more rapidly, *ie* , near the load and the inductor.

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