A NEW APPROACH TO THE ROBUST PID CONTROLLER SYNTHESIS FOR SYSTEMS WITH UNKNOWN MATHEMATICAL MODEL

Štefan Bucz * — Ladislav Harsányi** — Vojtech Veselý**

The presented paper deals with a new engineering method applicable to real plants with unknown mathematical model and varying parameters. Most of the existing 150 to 200 engineering PI(D) controller design methods are closely associated with the knowledge of the structure and parameters of the controlled plant [1], [2]. If neither the mathematical model of the controlled plant nor its parameters are available, only a negligible portion of the methods are applicable. The control law is PID based. The design procedure requires a minimum of a priori experiment-based information about the real uncertain plant. The proposed three-stage design technique can be integrated into industrial controllers as a method for tuning of robust PID controller parameters.

**Keywords:** frequency domain, plant perturbation, uncertainty, PID tuning, robustness, robust performance

1 INTRODUCTION

Parameters of every real plant are varying due to aging of material, changes of technological quantities and of the operation conditions as well. New cost-effective control algorithms use robust controllers designed by means of engineering methods; parameters of such controllers can be constant independently of varying technological quantities if we are able to describe or at least estimate the range of their changes. Individual stages of the proposed engineering method and their mutual relations are depicted in Fig. 1.

To start the synthesis procedure, only the number of uncertainties of the controlled plant and specification of robust control objectives in terms of stability margin are required. Solution of the three stages yields a robust PID controller guaranteeing robust stability and performance for the given uncertain plant.

Analysis of classical frequency domain design methods proved that for a successful PID controller design it is crucial to know the Bode plots of the controlled system over the frequency range around the crossover frequency (crossover region) [3] given by \( \Omega : (\omega_{\text{min}}, \omega_{\text{max}}) \), where \( \omega_{\text{min}} = 2\omega_a - \omega_f ; \omega_{\text{max}} = \omega_f \). The crossover region of the Bode plot of a general plant is depicted in Fig. 2.

It is represented by an infinite number of couples \([\omega_i, M_i]\) (Bode magnitude plot) and \([\omega_i, P_i]\) (Bode phase plot), where \( \omega_i \in \Omega \), \( M \in (M_{\text{min}}, M_{\text{max}}) \), \( P \in (P_{\text{min}}, P_{\text{max}}) \). The \( M \)-versus-frequency and \( P \)-versus-frequency plots can be obtained via interpolating \([\omega_i, M_i]\) and \([\omega_i, P_i]\) for \( \omega_i \), \( i = 1, 2, \ldots, n \) selected from the interval \( \Omega \). Let \( n = 3 \); let us denote the points \([\omega_i, M_i]\) and \([\omega_i, P_i]\), \( i = 1, 2, 3 \) as significant. Then the objective of identification is to obtain three significant points of the Bode magnitude and phase plots of the plant.

2 IDENTIFICATION OF THE UNCERTAIN PLANT: 1st STAGE

The main idea of identifying the three significant points consists in closing the loop comprising the controlled plant having transfer function \( \tilde{G}(j\omega) \) and an ideal relay having the equivalent transfer function \( G_R(j\omega) \) (Fig. 3), and exciting it up to the limit of instability. Parameters of the critical oscillations are further used to find the coordinates of the significant points of the Bode plots. A point of the Bode magnitude and phase plot plotted with a log-frequency axis is determined by the critical frequency of the closed loop in Fig. 3.

The filter \( G_F(j\omega) \) is used to affect parameters of critical oscillations so as to obtain coordinates of other two different points of the Bode magnitude and phase plots. The interpolation curve of the three identified significant points of the plant Bode plots will be considered the crossover region of the plant Bode plots. For the transfer function of the controlled plant

\[
\tilde{G}(j\omega) = \frac{G_0(j\omega)}{G_R(j\omega)G_F(j\omega)}
\]

the following relations for the magnitude and phase can be put down for three selected values of \( \omega_i \in \Omega \), \( i = 1, 2, 3 \)

\[
20 \log|\tilde{G}(j\omega_i)| = 20 \log|G_0(j\omega_i)| - 20 \log|G_R(j\omega_i)| - 20 \log|G_F(j\omega_i)|
\]

(2)

\[
\arg(\tilde{G}(\omega_i)) = \arg(G_0(\omega_i)) - \arg(G_R(\omega_i)) - \arg(G_F(\omega_i))
\]

(3)
The control loop in Fig. 3 satisfies the Nyquist criterion dies away. The amplitude of critical oscillations is at the limit of instability after the transient dies away. The amplitude of critical oscillations satisfies

\[ |G_i(j\omega)| = \omega \sin^{-1} \left( \frac{1}{\omega B_i} \right) \]

around the working point \( G_i \). The angular frequency of critical oscillations satisfies \( \omega_i = \frac{\pi}{T_i} \) where \( T_i \) is their period. The control loop in Fig. 3 satisfies the Nyquist condition

\[ 20 \log |G_0(j\omega_i)| = 0, \quad \arg \{ G_0(\omega_i) \} = -\pi. \]  

Relations (4), (5) and (6) represent contributions to the right-hand sides of (2) and (3) for calculating the coordinates of the significant points of plant Bode plots. Note that the parameters of critical oscillations were read out from the response of the output \( y(t) \) and the relay amplitude \( B_i \) was set prior to the identification process.

Substituting (4), (5) and (6) into (2) and (3) yields the following general relation that holds for individual significant points \( M_i: [\omega_i, A_i] \) of the Bode magnitude plot and \( P_i: [\omega_i, P_i] \) of the Bode phase plot of the plant

\[ M_i: \left[ \omega_i; -20 \log \left( \frac{4B_i}{\pi C_i} \right) \right], \quad P_i: \left[ \omega_i; -\pi - \arctg \{ -T_{F_i} \omega_i \} \right]. \]  

To identify the first significant point it is recommended to adjust \( T_{F1} = 0 \).

According to (7), its coordinates are

\[ M_1: \left[ \omega_1; -20 \log \left( \frac{4B_1}{\pi C_1} \right) \right], \quad P_1: \left[ \omega_1; -\pi \right]. \]  

Note that the coordinate \( A_1 \) specifies the plant gain margin \( D \) in dB and the point \( P_1 \) specifies the phase crossover frequency \( \omega_f \) of the plant (cross point of the Bode phase and the \(-180^\circ \) line). According to Fig. 2, the critical frequency \( \omega_1 \) is at the same time the upper limit \( \omega_{\text{max}} \) for the crossover region of the plant. The first significant point identified for \( T_{F1} = 0 \) provides information about the plant stability.

To identify the other two significant points it is recommended to select \( T_{F3} > T_{F2} > 0 \); this choice ensures that the critical frequency is shifted towards lower frequencies with respect to the critical frequency \( \omega_1 \) obtained for \( T_{F1} = 0 \). However in this case, the filter will contribute a nonzero magnitude and phase to \( M_i \) and \( P_i \) in (7), respectively. If the plant Bode plots is monotonous, the magnitude and phase of the second and third significant points will drop correspondingly with respect to the first point, and also the coordinates of the third significant point will decrease with respect to the second one. It is worth noting that the filter time constants \( T_{F2} \) and \( T_{F3} \) are always adjusted before identification of the particular significant point and the recommended values for the relay amplitudes are \( B_1 = B_2 = B_3 \). Mutual position of the resulting significant points is depicted in Fig. 4.

Interpolation curves of the significant points \( M_i \) and \( P_i \), \( i = 1, 2, 3 \) of the plant Bode plots given as follows

\[ M_s: \left[ \omega; -20 \log \left| a \right| -20 \log \left( \frac{1}{b^2 \omega^2 + 1} \right) \right], \quad P_s: \left[ \omega; -\pi - \arctg \{ -b \omega \} \right]. \]  

Fig. 1. Hierarchical stages of the engineering method

Fig. 2. Crossover region \( \Omega \) of the general plant

where \( G_0(j\omega) \) is the open-loop transfer function of the control loop in Fig. 3.

In addition, each industrial controller includes a 1st order low-pass filter with the time constant \( T_2 \), given by

\[ 20 \log |G_F(j\omega)| = 20 \log \sqrt{\frac{1}{T_2^2 \omega_i^2 + 1}}, \]

\[ \arg \{ G_F(\omega_i) \} = \arctg \{ -T_{F_i} \omega_i \}. \]  

The ideal relay satisfies

\[ 20 \log |G_R(j\omega)| = 20 \log \left( \frac{4B_i}{\pi C_i} \right), \]

\[ \arg \{ G_R(\omega_i) \} = 0. \]  

After adjusting the working point (by means of the set-point \( y_{sp} \)) and switching on the switch SB (Fig. 3), the closed loop is at the limit of instability after the transient dies away. The amplitude of critical oscillations \( C_i \) around the working point \( y_{sp} \) can be affected by changing the amplitude \( B_i \) of the relay. The angular frequency of critical oscillations satisfies \( \omega_i = 2\pi/T_i \) where \( T_i \) is their period. The control loop in Fig. 3 satisfies the Nyquist condition

\[ 20 \log |G_0(j\omega_i)| = 0, \quad \arg \{ G_0(\omega_i) \} = -\pi. \]
where \( a_N, b_N \) are nominal interpolation coefficients and \( \delta a, \delta b \) are interpolation coefficients uncertainties. Let \( p \) denote the number of plant uncertainties. If the lower and upper bounds of individual uncertainties are known then (11) can be interpreted as the uncertainty polytope in the \( p \)-dimensional space with the plant nominal model in its center and vertices being the plant models corresponding to the \( 2^p \) combinations of the upper and lower limit values of uncertainties. If the described identification procedure of the \( M_s \) and \( P_s \) plots is repeated \( 2^p \)-times by combining the uncertainty limits (eg by physically setting the minimum and maximum loads for the plant), then \( 2^p \) plots \( M_s^j, P_s^j \) for \( j = 1, \ldots, 2^p \) and the uncertain plant model are obtained. Moreover, according to the transformation (12) the following relations hold

\[
M_s^j(\omega, a^j, b^j) = M_{sN}(\omega, a_N, b_N) + \delta M_s(\omega, \delta a, \delta b),
\]

\[
P_s^j(\omega, b^j) = P_{sN}(\omega, b_N) + \delta P_s(\omega, \delta b)
\]  

where \( a^j, b^j \) are interpolation coefficients of the identified plots of the plant.

The Bode plots \( M_s^j, P_s^j \) for \( j = 1, \ldots, 2^p \), were obtained by applying the identification procedure for a real physical servo system with two slave control loops and two parametric uncertainties, ie \( p = 2 \). Four characteristics \( M_s^j, P_s^j \) for \( j = 1, \ldots, 4 \) have been identified for all limit value combinations of the both uncertainties; the fifth plot "Nom" is the nominal plant Bode plot \( M_{sN}(\omega, a_N, b_N), P_{sN}(\omega, b_N) \), calculated analytically according to the following relation

\[
M_{sN} : \left[ \omega; -20 \log |a_N| - 20 \log \sqrt{\frac{1}{b_N^2 \omega^2 + 1}} \right],
\]

\[
P_{sN} : [\omega; -\pi - \arctg(-b_N \omega)]
\]

where the nominal interpolation coefficients and the maximum uncertainties of the interpolation coefficients are given as

\[
a_N = \sum_{i=1}^{2^p} a^i, \quad b_N = \sum_{i=1}^{2^p} b^i,
\]

\[
\delta a_{\text{max}} = \max_{i=1}^{2^p} \{ a^i - a_N \}, \quad \delta b_{\text{max}} = \max_{i=1}^{2^p} \{ b^i - b_N \}.
\]  

Similar relations are valid also for the \( \delta M_{\text{max}} \) and \( \delta P_{\text{max}} \) characteristics. They are obtained by substituting \( \delta a_{\text{max}}, \delta b_{\text{max}} \) for \( a, b \) in (9). The equivalence \( PB_j \equiv M_s^j \) holds for \( j = 1, \ldots, 2^p \) in Fig. 5.

In the presented case study the stable nominal plant is represented by the plot "Nom" in Fig. 5 with the nominal phase and gain margins \( \varphi_{\text{a,N}} = 38.4^\circ \) and \( D_{\text{N}} = 90.9 \) d, respectively, and the nominal gain and phase crossovers \( \omega_{\text{a,N}} = 1.1 \) rads\(^{-1} \) and \( \omega_{\text{f,N}} = 421.1 \) rads\(^{-1} \), respectively.

For the robust PID controller synthesis it is necessary to introduce the concept of the maximum phase margin uncertainty of the plant which is defined as follows

\[
\delta \varphi_{\text{a,max}} = \max_{i=1}^{2^p} \{ \varphi^i_{\text{a}} - \varphi_{\text{a,N}} \}.
\]

In particular, for the given uncertain plant its value is \( 17.8^\circ \).
The controller frequency domain design method that follows the implemented in the parallel (non-interacting) form is the standard task of simultaneous design of the three parallel correcting terms of the PID controller will be converted to rectifying terms of the 1D controller given by:

$$G_{R}^{(PID)}(s) = r_0^{(PID)} + \frac{r_{-1}^{(PID)}}{s} + r_1^{(PID)} s$$

and it is equal to the series (interacting) PI-ID controller with the transfer function

$$G_{R}^{(PL)}^{(1D)}(s) = \left[ r_0^{(PI)} + \frac{r_{-1}^{(PI)}}{s} r_1^{(1D)} \right] + \frac{r_{-1}^{(PI)}}{s} + \left[ r_0^{(PI)} r_1^{(1D)} \right] s.$$  

Robust PI controller design

The objective of the robust PI controller design is to guarantee the required open-loop phase margin $\varphi^*$. By increasing the $P$-gain of the PI controller, the magnitude crossover $\omega_a$ of the plant is shifted to the frequency point $\omega_0^*$ in which $\varphi(\omega_0^*)$ achieves the required open loop phase margin $\omega_a^*$ augmented by the maximum uncertainty of the phase margin $\delta \varphi_{max}$. Then $\omega_a^*$ becomes the new gain crossover and the corresponding magnitude $D(\omega_a^*)$, indicating how much the magnitude plot with $P$-action was shifted with respect to the magnitude plot of the controlled system in $\omega_a^*$, will be equal to the PI controller gain $20 \log r_0$. The relation for computing the proportional gain of the PI controller results from this equality

$$20 \log \frac{r_0^{(PI)}}{r_0} = D(\omega_a^*) \implies r_0^{(PI)} = 10^{\frac{D(\omega_a^*)}{20}}.$$  

To achieve the necessary closed-loop bandwidth it is beneficial to place the cutoff frequency $r_{-1}/r_0$ of the PI controller magnitude plot to the frequency one decade lower than the gain crossover frequency $\omega_0^*$ of the open loop. Then, from the relation for the cutoff frequency of the PI controller magnitude plot results

$$\frac{r_0^{(PI)}}{r_0} = 0.1 \omega_a^* \implies r_{-1}^{(PI)} = 0.1 r_0^{(PI)} \omega_a^*.$$  

Note that the described design procedure will be applied to the plots $M_{sN}$ and $F_{sN}$ identified in the 1st stage of the proposed engineering method.

1D Controller Design

The main idea of the 1D controller design is to integrate several design objectives into the selection of a single parameter. To satisfy the noise tolerance of the robust control loop it is recommended to apply the rule of thumb limit value for the D-term of the PID controller, i.e. $r_1 \leq 5$. Improvement of the closed-loop dynamics and simultaneous preservation of its robustness is achieved by placing the cutoff frequency $1/r_1$ of the Bode magnitude plot of the 1D controller one half decade lower than the gain crossover of the open loop with the PI controller. This yields its lower limit. The resulting relations obtained using (17)–(20)

$$r_1^{(1D)} = \frac{5}{r_0^{(PI)}} \quad r_1^{(1D)} = \frac{1}{5} r_0^{(PI)}$$  

define the interval of the so-called feasible derivative terms of the 1D controller given by $r_1 \in (r_{1min}, r_{1max})$. Unlike the classical approach where only one particular D-term is designed it is advantageous to design three derivative terms instead

$$r_{k}^{(1D)} = r_{1min} + \frac{k}{4}(r_{1max} - r_{1min}), \quad k = 1, 2, 3$$

which are mutual limit values of individual segments of a linearly segmented interval of feasible derivative terms depicted in Fig. 6. During the design procedure such a D-term is selected that provides the most appropriate robustness and dynamics of the closed loop. Table 1 shows the coefficients of the robust PID controller (18) for the considered servo system and three different values of the required phase margin $\varphi_a^*$.  

![Fig. 5. Identified Bode plots of the servo drive](image-url)  

![Fig. 6. Interval of feasible derivative terms of the 1D controller](image-url)
4 ROBUST CLOSED-LOOP STABILITY AND ITS VERIFICATION: 3rd STAGE

The controller designed for the nominal plant using the proposed approach also guarantees the nominal performance [4]. As the controller design was performed to provide a required phase margin $\phi^*_a$ augmented by the maximum phase margin uncertainty $\delta \phi_{a_{max}}$, which represents the “worst case” controller design, the closed loop robust stability is ensured [5]. The derivative terms $r^{(PID)}_1$ were calculated using the coefficient selected from the three terms $r^{(PID)}_1$ of the 1D controller that provided the largest gain margin of the closed-loop. This, in addition, secures also the robust closed-loop performance under all three controllers with coefficients given in Table 1.

Table 1. PID controller coefficient for various values of $\phi^*_a$.

<table>
<thead>
<tr>
<th>$\phi^*_a$ [°]</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^{(PID)}_0$</td>
<td>0.1370</td>
<td>0.0420</td>
<td>0.0270</td>
</tr>
<tr>
<td>$r^{(PID)}_1$</td>
<td>0.0087</td>
<td>0.0034</td>
<td>0.0017</td>
</tr>
<tr>
<td>$r^{(PID)}_2$</td>
<td>1.0960</td>
<td>1.0430</td>
<td>1.0110</td>
</tr>
</tbody>
</table>

5 CONCLUSION

The procedure of plant identification, robust PID controller design and verification of its robustness has been computer-supported using Matlab-Simulink and Real-Time Workshop. When implementing the digital form of the control law, it is recommended to set the sampling period $T_s$ according to Wittermark and Middleton (23), where $\omega_c$ is the closed loop critical frequency.

$$\omega_c T_s \in (0.2 ; 0.6) \quad \text{or} \quad \omega_c T_s < \frac{2\pi}{10}.$$ (23)

Time responses of the controlled variable (angular position of the motor) as well as of the auxiliary variables of the servo system have proved the achieved robust stability and robust performance of the designed control loop as well, for both small and large angular positions. Each controller with coefficients from Table 1 was able to cope with the additive perturbation of the servosystem while maintaining a 100% steady-state accuracy of the robust control. Implementation of individual controllers in the control loop is given in the legends, in Fig. 7.
Acknowledgement

The work has been supported by Grants N 1/3841/06 of the Slovak Scientific Grant Agency and APVV- 20-023505 of the Slovak Research and Development Agency.

REFERENCES


Received 27 September 2007

Štefan Bucz (Ing) was born in Komárno, Slovakia, in 1978. He graduated from the Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Bratislava, in 2005. At present he is a PhD student in the Department of Automatic Control Systems at the Faculty of Electrical Engineering and Information Technology. The main field of his study is robust control, control of servo systems and engineering methods for real industrial processes with unknown mathematical model and varying parameters.

Ladislav Harsányi (Doc, Ing, CSc) was born in Ohrady, Slovakia, in 1936. He graduated from the Faculty of Electrical Engineering, Slovak University of Technology, Bratislava, in 1960. He gained the PhD degree in Technical Cybernetics from the Faculty of Electrical Engineering of Slovak Technical University, in 1974. At present he is Associate Professor at the Institute of Control and Industrial Informatics of Faculty of Electrical Engineering and Information Technology. The main field of his research and teaching activities are the theory of control, robust control and control of power systems and the use computers in education.

Vojtech Veselý (Prof, Ing, DrSc) was born in 1940. Since 1964 he has worked at the Department of Automatic Control Systems at the Faculty of Electrical Engineering and Information Technology, Slovak University of Technology in Bratislava, where he has supervised up today 18 PhD students. Since 1986 he has been Full Professor. His research interests include the areas of power system control, decentralized control of large-scale systems, process control and optimization. He is author and coauthor of more than 250 scientific papers.