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A note on magnetic vector potential

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This paper presents a non-traditional way of determining an unknown constant encountered in the expression for magnetic vector potential due to an elemental dipole antenna.

K e y w o r d s: infinitesimal current element, Hertzian dipole antenna, magnetic vector potential, Lorentz gauge condition, Helmholtz wave equation, divergence theorem, volume integral, surface integral

1 Introduction

We briefly review the derivation of the expression for magnetic vector potential due to a Hertzian dipole antenna, with an emphasis on the evaluation of a constant encountered in the expression. Then we describe a new method of evaluation of this constant, which is normally not dealt in the textbooks.

The concepts of time varying magnetic vector potential, $\boldsymbol{A}$, and the electric scalar potential, V, are invariably introduced in the very first few lectures while teaching a basic course on antennas [1, 2, 3, 4, 5, 6]. The time varying magnetic flux density, $\boldsymbol{B}$, is determined from the curl operation on magnetic vector potential, *ie*, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. The time varying electric field intensity, $\boldsymbol{E}$, is obtained from the relation, $\boldsymbol{E} = -\nabla V - \dot{\boldsymbol{A}}$. Elimination of electric and magnetic field terms from the Maxwell equations using the above mentioned relations between the potentials and the fields results in two coupled partial differential equations in $\boldsymbol{A}$ and V, which are difficult to solve. The introduction of Lorentz gauge condition in the coupled equations leads to the two uncoupled in-homogeneous Helmholtz wave equations in $\boldsymbol{A}$ and V. For sinusoidal time variations, these Helmholtz equations become,

$$(\nabla^2 + \omega^2\mu\varepsilon)\boldsymbol{A} = -\mu\boldsymbol{J}, \quad (\nabla^2 + \omega^2\mu\varepsilon)V = -\rho/\varepsilon, \tag{1}$$

where $\boldsymbol{J}$ is the current density; ω, μ, ε and ρ have the usual meaning as described in [1]. Many textbooks introduce z-directed infinitesimal current element $(I\mathrm{d}l)$ placed at origin as an elemental antenna, or, Hertzian dipole antenna, and proceed to determine the potential, $\boldsymbol{A}$, due to this source. As a result, the in-homogeneous Helmholtz wave equation (in potential $\boldsymbol{A}$) reduces to a scalar equation as shown below,

$$(\nabla^2 + k^2)A_z = -\mu J_z \tag{2}$$

where A_z is the scalar part of z-directed magnetic vector potential, $k^2 = \omega^2\mu\varepsilon$ and J_z is the scalar part of z-directed current density. At any point P in the source-free field region, see Fig. 1, equation (2) reduces to,

$$(\nabla^2 + k^2)A_z = 0 \tag{3}$$

which is homogeneous Helmholtz equation. Expressing $\nabla^2 A_z$ in spherical coordinate system, and noting that there is no variation of A_z in θ and ϕ directions, the solution of (3) is obtained as,

$$A_z = \frac{C}{r}e^{-jkr} \tag{4}$$

where r is the radial distance of P from the origin; and C is an unknown constant. The textbooks deal with either of the following two methods for evaluation of C.

The first method makes use of the similarity that exists between the inhomogeneous Helmholtz wave equation (2) when $k \to 0$ and the Poisson's equation,

$$\nabla^2 V = -\rho/\varepsilon$$

for the electrostatic potential, V. Notice that $k \to 0$ corresponds to $\omega \to 0$, which is electrostatic case [3, 4, 5, 6]. A comparison with the solution of the Poisson's equation, for the case of a point charge placed at origin, leads to evaluation of C as

$$C = \frac{\mu I\mathrm{d}l}{4\pi} \tag{5}$$

The second method employs divergence theorem for the evaluation of C as described below [1, 2].

Consider an infinitesimal spherical volume centered at the origin having radius r_0, enclosing the current element $I\mathrm{d}l$. Now, consider (2), which includes the source term. Taking the volume integral of (2) within the in

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nitesimal sphere results into,

$$\int_v \nabla^2 A_z \mathrm{d}v + \int_v k^2 A_z \mathrm{d}v = -\int_v \mu J_z \mathrm{d}v \qquad (6)$$

As the infinitesimal spherical volume starts shrinking (and in the limit becomes zero), the radius r_0; also tends to zero. So, applying this limit, in (6), the right-hand-side is determined as $-\mu I\mathrm{d}l$; and since $dv = r^2 sin\theta \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r$ the second term vanishes. Thus we have,

$$\lim_{r_0\to 0}\int_v \nabla^2 A_z \mathrm{d}v = -\mu I\mathrm{d}l$$

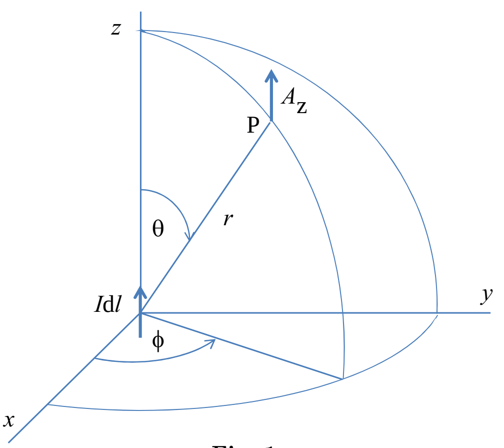

Fig. 1.

Rewriting $\nabla^2 A_z$ as $\nabla\cdot\nabla A_z$, and applying divergence theorem to convert the volume integral the above expression to a closed surface integral leads to,

$$\int_v \nabla\cdot\nabla A_z \mathrm{d}v = \oint_s \nabla A_z \cdot \mathrm{d}\boldsymbol{s}$$

and further it becomes,

$$\lim_{r_0\to 0}\oint_s \nabla A_z \cdot \mathrm{d}\boldsymbol{s} = -\mu I\mathrm{d}l \qquad (7)$$

where $\mathrm{d}\boldsymbol{s} = \boldsymbol{a}_r r_0^2 \sin\theta \mathrm{d}\theta\mathrm{d}\phi$; and because of spherical symmetry ∇A_z reduces to: $\nabla A_z = \boldsymbol{a}_r \partial A_z/\partial r$: Therefore

$$\nabla A_z \cdot \mathrm{d}\boldsymbol{s} = r_0^2 \sin\theta\,(\partial A_z/\partial r)\,\mathrm{d}\theta\mathrm{d}\phi$$

and since the integration is over the closed spherical surface of radius $r = r_0$,

$$\frac{\partial A_z}{\partial r}|_{r=r_0} = -\frac{C}{r_0^2}(1 + jkr_0)e^{-jkr_0}$$

and so (7) can be written as,

$$\lim_{r_0\to 0}\oint_s C\sin\theta(1 + jkr_0)e^{-jkr_0}\mathrm{d}\theta\mathrm{d}\phi = \mu I\mathrm{d}l \qquad (8)$$

After the integration and the application of the limit, (8) becomes, $4\pi C = \mu I\ dl$; yielding again (5).

A third method to evaluate C is presented in the next section, which is normally not dealt in the textbooks.

2 New method

A new method to determine C, which does not make use of the divergence theorem for the volume integral in (6) is described below.

Since the potential, A_z, has no variation in θ and ϕ directions, $\nabla^2 A_z$ is expressed as [1]:

$$\nabla^2 A_z = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_z}{\partial r}\right).$$

Use of the above expression in (6) and applying the limit leads to

$$\lim_{r_0\to 0}\int_v \frac{\partial}{\partial r}\left(r^2\frac{\partial A_z}{\partial r}\right)\sin\theta \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r = -\mu I\mathrm{d}l \qquad (9)$$

Since $(\partial A_z/\partial r) = -C(1+jkr)\exp(-jkr)/r^2$; using it in (9), and after the double integration with $\theta \in (0,\pi)$ and $\phi \in (0, 2\pi)$, one obtains,

$$lim_{r_0\to 0} 4\pi C\int \frac{\partial}{\partial r}(1 + jkr)\,e^{-jkr_0}\mathrm{d}r = \mu I\mathrm{d}l \qquad (10)$$

Note that integration of a differentiation of a function is the function itself, and since $r \le r_0$, as $r_0 \to 0$ we have $r \to 0$, so

$$\lim_{r\to 0} 4\pi C\,(1 + jkr)\,e^{-jkr} = \mu I\mathrm{d}l \qquad (11)$$

and after applying the limit, one gets the same result (5) as before.

3 Conclusion

A brief review of derivation of expression for magnetic vector potential, $\boldsymbol{A}$, is given; the two traditional methods to evaluate a constant (C) encountered in the expression for $\boldsymbol{A}$ are discussed along with a new method presented in this paper. The new method will provide the readers a broader insight into the subject beyond what they learn from the textbooks.

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