

# DETERMINATION OF DAMAGE VARIABLE BY MEASUREMENT OF DIFFERENTIAL PERMEABILITY

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The aim of this paper is to introduce a modified damage variable for control and security purposes and to determine relation between the modified damage variable and a certain component of the plastic part of the linear strain tensor. Two relations for determination of the above-mentioned relation were derived. The first one is a relation between the effective differential permeability and the modified damage variable. The second one is the time evolution law for the modified damage variable.

**Key words:** modified damage variable, magnetic adaptive testing, nondestructive evaluation, differential permeability

## 1 INTRODUCTION

Many types of methods for determination of damage variables of engineering materials are described in the paper [1]. It is well-known that properties of the engineering materials that have been used for extended periods of time, are deteriorated as a result of cyclic stress, accidental impact damage, and/or unplanned overloading. Such service-induced damage processes lead to increasing structural alterations of the materials. The structural alterations of the materials are considered from the viewpoint of gradual creation and development of discontinuities in the solid media (micro-cracks and micro-cavities in metals). To detect the increasing accumulation of defects it is necessary to use a sensitive non-destructive method of measurement of the properties of the damaged material during its industrial employment. Following requirements need to be formulated:

- An evolution law for the damage variable. This evolution law is then used to predict failure of the material in the industrial employment.
- Determination “in situ” of the damaged variable for control and security purposes.
- Evaluation failure, at least qualitatively, within broken components, in order to find causes and mechanisms of the fracture.

The creation and development of defects correspond to various mechanisms and to various kinds of damage:

- Brittle damage, which corresponds to defects created without measurable macroscopic plastic strain.
- Ductile damage, which corresponds to defects induced by large plastic strains.
- Creep damage, which corresponds to intergranular decohesions at a high temperature.

- Fatigue damage, which mostly corresponds to a transgranular crashing phenomenon due to repetition of loadings.

In this paper we will be interested mainly in the ductile damage. For characterization of the stage of damage of the materials it is necessary to define a damage variable and to determine how the damage variable, in the case of the ductile damage, depends on the components of the plastic part of the linear strain tensor  $\epsilon$ . The linear strain tensor  $\epsilon$  can be divided into two parts,  $\epsilon = \epsilon_e + \epsilon_p$ , where  $\epsilon_e$  and  $\epsilon_p$  are the elastic and plastic parts of the strain tensor, respectively. At definition of a damage variable one requires to know mechanism of the damage and geometrical nature of the defects, which in the most cases is not very well-known. It can be shown, however, that a simple mean value of a defect characteristic in a certain domain is sufficient. There is a question: on which domain is a mean value? The domain is characterized by a representative volume element  $\Delta V$ . The size of  $\Delta V$  as the case may change. That depends on the mean value of quantity which characterizes the defect. In the case of micro-cracks and micro-cavities, the representative volume element is the smallest volume on which a density may represent a field of discontinuous properties. In this case the linear dimension for metal is equal to  $0.05 \rightarrow 0.5$  mm.

In paper [1], the damage variable  $\tilde{D}$  is defined as the effective surface fraction of micro-defects on the representative volume element  $\Delta V$

$$\tilde{D}_{(\mathbf{n})} \stackrel{\text{def}}{=} \frac{\Delta S}{\Delta S_0}, \quad (1)$$

$\mathbf{n}$  is the unit vector perpendicular to the surface  $\Delta S_0$  of intersection of a volume element  $\Delta V$  and  $\Delta S$  is the

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effective surface of intersection of micro-cracks and micro-cavities with the plane of  $\Delta S_0$ . The idea of this definition of the damage variable is due to Kachanov [2]. The advantage of this definition of the damage variable  $\tilde{D}_{(n)}$  is that it can be determined by direct measurement from observation of micro-graphic pictures but it is evident that the parameter  $\tilde{D}_{(n)}$  can be used only in the case of an advanced stage of damage. For this reason, we introduce a new definition of the damage variable  $D$  which can be used in the case of initial but also of advanced stage of damage. Our definition of the parameter  $D$  is based on the following idea:

On the microscopic level, matter consists of atoms which are linked by binding forces. The dominant role is played by the binding forces between the nearest neighbour atoms. The defects in solids will be modelled as the sequence of damaged bonds between the nearest neighbours. For that reason, we divide the sample into small representative volume elements,  $\Delta V$ , the linear dimension of which was mentioned before. The damage variable is defined as follows:

$$D \stackrel{\text{def}}{=} \frac{N_d}{N}, \quad (2)$$

where  $N_d$  is the number of *damaged bonds* and  $N$  is the total number of bonds in the representative volume element,  $\Delta V$ . If the material is homogeneous on the macroscopic level, then the parameter  $D$  is constant through the whole sample. By increasing the plastic deformation, mainly linear defects (dislocations) are generated at the beginning and they are followed by the gradual formation of micro-cracks and micro-cavities and, finally, by the fracture of the material. If  $\Delta V_b$  is the volume belonging to one bond, then relation (2) can be rearranged in the following way:

$$D = \frac{N_d \Delta V_b}{N \Delta V} = \frac{\Delta V_d}{\Delta V}, \quad (3)$$

where  $\Delta V_d$  is the volume of micro-cracks and micro-cavities in the representative volume element  $\Delta V$ . It is evident that relation (2) is more general than (1).

In reality it is not possible to measure the damage variable  $D$  directly but only indirectly, measuring some quantities which depend on the damage variable  $D$ . These quantities are not equally sensitive to the change of variable  $D$ . Therefore, they depend on the modified damage variable  $D_{mod}$ . The detailed discussion about this problem will be given in Section 4.

In the case of ductile damage of materials, when the sample is subjected to a uniform axial tension test, value of the damage variable  $D$  will be determined from the dependence between  $D$  and the component of the plastic part of the strain tensor  $\epsilon_{p11}$  ( $\epsilon_{11} = \epsilon_{e11} + \epsilon_{p11}$ , where  $\epsilon_{e11}$  and  $\epsilon_{p11}$  are the components of the elastic and plastic parts of the strain tensor, respectively). The concept for determining the damage variable  $D$  is as follows. First, we will use the relation between the effective relative differential permeability  $\mu_{eff}$  and the damage variable  $D$ . The

meaning “effective” will be given in Section 2. Secondly, we will derive the relation between the damage variable  $D$  and the component of the plastic strain tensor  $\epsilon_{p11}$  with the help of generalized non-equilibrium thermodynamics. The unknown material parameters, which appear in the last relation, will be determined in the following way: We introduce the parameter  $D$  (the second relation between  $D$  and  $\epsilon_{p11}$ ) into the first relation (relation between the  $\mu_{eff}$  and  $D$ ) and in this way we obtain a relation between the  $\mu_{eff}$  and  $\epsilon_{p11}$ . Now the last relation contains the unknown material parameters. If one has the experimental data from the measurement of  $\mu_{eff}$  in dependence on  $\epsilon_{p11}$ , then from the best fit of the experimental data to the theoretical relation between  $\mu_{eff}$  and  $\epsilon_{p11}$  the values of the unknown material parameters can be determined. This is the idea of the experimental determination of the  $D$  on the  $\epsilon_{p11}$  by the indirect method.

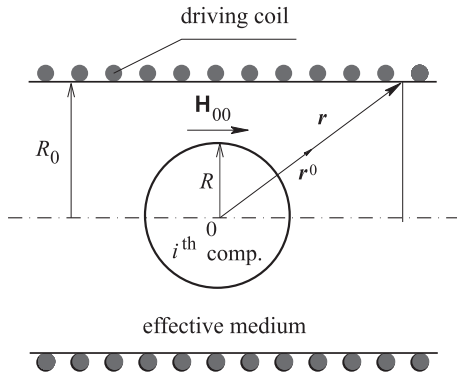
## 2 EFFECTIVE DIFFERENTIAL PERMEABILITY

For an easy introduction of an effective differential permeability  $\mu_{eff}$  of the damage ferromagnetic material, it will be suitable to model it as a binary system. Generally, the binary system is heterogeneous on the sub-macroscopic level (on the length scale of the linear dimension of clusters of the damaged bonds) and it is composed of clusters of individual components, which, on one hand, are spatially separated from each other and, on the other hand, are randomly distributed over the whole sample. Cluster consists of a set of damaged bonds which are connected with each other. The individual components are characterized by the quantities  $f_i$  where  $f_i$  characterizes the  $i^{\text{th}}$ -component. We will suppose that the quantities  $f_i$  rapidly but continuously change between the clusters belonging to the different components. Due to the distributional randomness, the physical quantities  $f(\mathbf{r})$  of the binary system on the sub-macroscopic level are not only continuous function of the space coordinates but they are also random quantities. For the averaging of the random quantities, one needs to know statistics of the structure of the binary system on the sub-macroscopic level. If the binary system on the macroscopic level is homogeneous, one can consider the following assumption:

**ASSUMPTION I.** *The probability to find a cluster of the  $i^{\text{th}}$ -component in the certain place is equal to the volume fraction  $c_i$  of the  $i^{\text{th}}$ -component in the binary system. The probability of finding a cluster in a certain place is independent of the probability of finding another cluster in another place.*

*If the binary system on the macroscopic level is homogeneous, then it can be characterized by effective parameters.*

The effective parameter can be obtained by a homogenization technique. In connection with the concept of the effective parameters the question arises how the effective



**Fig. 1.** Infinitely long solenoid with a large radius  $R_0$  filled with the binary system

parameters depend on the statistics of the structure of the binary system on the sub-macroscopic level and on the properties of individual components, as well. This is the reason why we will derive the relation for the effective parameters.

The derivation of the relation for effective parameter is a very difficult problem. When solving this problem, one has to deal with two difficulties. The first one is connected with the necessity to know the statistics of the damaged structure of the binary system on the sub-macroscopic level, which is usually unknown. The second one is connected with the mathematical difficulties of exact calculation of effective parameters, and, therefore, one is obliged to use an approximate method. One of these approximate methods is the mean field approximation method (MFAM), which is used very often. Now we are coming to the derivation of the relation for the effective differential permeability. As it was mentioned before the binary system is heterogeneous on the *sub-macroscopic* level, but it is very often homogeneous and isotropic on the *macroscopic* level. Therefore, in this case we can characterize the binary system by the effective parameters. Let us consider an infinitely long solenoid with a large radius  $R_0$  (Fig. 1). If we fill up the inside of the solenoid with the binary system, then its magnetic state is characterized by the effective magnetic quantities  $\mathbf{B}_e$  and  $\mathbf{H}_e$  which are parallel to the axis of the solenoid. The space which is occupied by the binary system will be called the effective medium. If the electric current in the solenoid is  $I_0$ , then the magnetic state of the effective medium is characterized by  $\mathbf{B}_{e0}$  and  $\mathbf{H}_{e0}$ . If we slightly enhance the electric current by  $\Delta I$ , then the magnetic state is characterized by  $\mathbf{B}_e$  and  $\mathbf{H}_e$ . Due to the axial symmetry of the solenoid the quantities  $\mathbf{B}_e$ ,  $\mathbf{B}_{e0}$ ,  $\mathbf{H}_e$  and  $\mathbf{H}_{e0}$  are parallel to each other and are also parallel to the axis of the solenoid. Let be the material relation

$$B_e = B_e(H_e) \quad (4)$$

We expand this material relation into Taylor's series. If the enhancement of the electric current is small ( $\Delta I \ll I_0$ ), then it is enough to restrict the series only to the

linear term. In the frame of this approximation we can write

$$\begin{aligned} \mathbf{B}_e(H_e) - \mathbf{B}_e(H_{e0}) &= \left( \frac{dB_e(H_e)}{dH_e} \right)_{H_e=H_{e0}} \mathbf{k}(H_e - H_{e0}) \\ &= \mu_0 \mu_{eff}(H_{e0}) (\mathbf{H}_e - \mathbf{H}_{e0}) \end{aligned} \quad (5)$$

where  $\mathbf{k}$  is the unit vector parallel to the axis of the solenoid,  $\mu_{eff}$  is the relative differential permeability and  $\mu_0$  is the permeability of vacuum. Now we put simultaneously both the binary system and a homogeneous and isotropic  $i^{\text{th}}$ -component of a globular form with radius  $R$  ( $R \ll R_0$ ) into the solenoid so that the center of the  $i^{\text{th}}$ -component is situated on the axis of the solenoid (see Fig. 1). If the electric current in the solenoid is  $I_0$ , then the quantities of the magnetic state inside the  $i^{\text{th}}$ -component are  $\mathbf{B}_{i0}$  and  $\mathbf{H}_{i0}$  and if the electric current in the solenoid is  $I_0 + \Delta I$ , then the quantities of the magnetic state inside the  $i^{\text{th}}$ -component are  $\mathbf{B}_i$  and  $\mathbf{H}_i$ , respectively. Again, due to the axial symmetry of the solenoid and to the spherical symmetry of the  $i^{\text{th}}$ -component the quantities  $\mathbf{B}_i$ ,  $\mathbf{H}_i$ ,  $\mathbf{B}_{i0}$  and  $\mathbf{H}_{i0}$  are parallel to the axis of the solenoid and they are constant through the whole  $i^{\text{th}}$ -component, but they are also random vectors due to the presence of  $i^{\text{th}}$ -component. Proceeding analogously as in the previous case we can write

$$\begin{aligned} \mathbf{B}_i(H_i) - \mathbf{B}_i(H_{i0}) &= \left( \frac{dB_i(H_i)}{dH_i} \right)_{H_i=H_{i0}} \mathbf{k}(H_i - H_{i0}) \\ &= \mu_0 \mu_i(H_{i0}) (\mathbf{H}_i - \mathbf{H}_{i0}) \end{aligned} \quad (6)$$

where  $\mu_i(H_{i0})$  is the relative differential permeability of the  $i^{\text{th}}$ -component. Using relation (76) (Appendix) we can write

$$\mathbf{B}(H_2) - \mathbf{B}(H_1) = \mu_0 \mu_{eff}(H_{e0}) (\mathbf{H}_2 - \mathbf{H}_1) \quad (7)$$

Due to the hysteresis the relative effective differential permeability  $\mu_{eff}$  is non-uniquely determined because it depends on the history of the magnetization process. But the obtained results will be applied on the sample of the ring shape and in this case the differential permeability will be defined uniquely on the hysteresis loop if the starting point of the magnetization process is well defined, eg by thorough demagnetization. It is obvious that  $\mu_{eff}$  is independent of  $|\mathbf{r}|$ . Due to the presence of  $i^{\text{th}}$ -component in the effective medium the quantities  $\mathbf{B}(H_2)$ ,  $\mathbf{H}_2$ ,  $\mathbf{B}(H_1)$ ,  $\mathbf{H}_1$ ,  $\mathbf{B}_{i0}$ ,  $\mathbf{H}_{i0}$ ,  $\mathbf{B}_i$  and  $\mathbf{H}_i$  are random vectors. If the rate of change of the electric current in the driving coil is sufficiently low, then we can neglect the eddy currents and in this case in both the binary system and the  $i^{\text{th}}$ -component there are no macroscopic currents, and, therefore, the magnetic field strength  $\mathbf{H}$  can be expressed as

$$\mathbf{H} = -\nabla \varphi_m, \quad (8)$$

where  $\varphi_m$  is the scalar magnetic potential. According to relation (8) one can write

$$\mathbf{H}_i - \mathbf{H}_{i0} = -\nabla \varphi_i \quad (9)$$

and

$$\mathbf{H}_2 - \mathbf{H}_1 = -\nabla\varphi, \quad (10)$$

where  $\varphi = \varphi_m - \varphi_{m0}$  and  $\varphi_i = \varphi_{mi} - \varphi_{mi0}$ . Using the Maxwell's equation  $\nabla \cdot \mathbf{B} = 0$ , then it follows immediately from relations (6) and (7)

$$\Delta\varphi_i = 0 \quad (11)$$

and

$$\Delta\varphi = 0, \quad (12)$$

where we consider  $\mu_{eff}(H_{e0})$  and  $\mu_i(H_{i0})$  as constants. The whole sample is divided into two regions. The first region is the  $i^{\text{th}}$ -component and the second one is the effective medium (see Fig. 1). The solution of equation (11) and (12) in the first and second region reads

$$\varphi_i = -\mathcal{B}\mathbf{H}_{00} \cdot \mathbf{r} \quad (13)$$

and in the effective medium

$$\varphi = -\mathbf{H}_{00} \cdot \mathbf{r} - \mathcal{G}\mathbf{H}_{00} \cdot \frac{\mathbf{r}}{r^3}. \quad (14)$$

The physical meaning of the vector  $\mathbf{H}_{00}$  will be given later on. The solutions (13) and (14) have the same mathematical structure as in the case when a dielectric component of the globular form is situated in the effective dielectric medium and both are in external homogeneous electrostatic field [3]. From relations (9), (10), (13) and (14) it follows:

$$\mathbf{H}_i - \mathbf{H}_{i0} = \mathcal{B}\mathbf{H}_{00} \quad (15)$$

and

$$\mathbf{H}_2 - \mathbf{H}_1 = \mathbf{H}_{00} + \frac{\mathcal{G}}{r^3}\mathbf{H}_{00} - \frac{\mathcal{G}}{r^5}\mathbf{H}_{00} \cdot \mathbf{r} \mathbf{r} \quad (16)$$

respectively. The constants  $\mathcal{G}$  and  $\mathcal{B}$  are determined from the boundary conditions at  $r = R$ :

$$\varphi_i(R) = \varphi(R) \quad (17)$$

and

$$-\mu_i \text{grad } \varphi_i(R)\mathbf{r}^0 = -\mu_{eff} \text{grad } \varphi(R)\mathbf{r}^0, \quad (18)$$

where  $\mathbf{r}^0$  is the unit vector perpendicular to the sphere of the radius  $R$ . Relation (18) expresses the fact that the normal components of the magnetic inductions ( $[\mathbf{B}(H_2) - \mathbf{B}(H_1)]\mathbf{r}^0 = [\mathbf{B}_i(\mathbf{H}_i) - \mathbf{B}_i(\mathbf{H}_{i0})]\mathbf{r}^0$ ) do not change at  $r = R$ . From (13)–(18) one obtains

$$\mathcal{B} = \frac{1}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} \quad (19)$$

and

$$\frac{\mathcal{G}}{R^3} = \frac{1}{3\mu_{eff}} \frac{\mu_{eff} - \mu_i}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} \quad (20)$$

where  $g = \frac{1}{3}$ . Using (6), (13), (15) and (19) one obtains

$$\mathbf{B}_i(H_i) - \mathbf{B}_i(H_{i0}) = \frac{\mu_0\mu_i}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}}\mathbf{H}_{00} \quad (21)$$

and

$$-\nabla\varphi_i = \mathbf{H}_i - \mathbf{H}_{i0} = \frac{1}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}}\mathbf{H}_{00}. \quad (22)$$

Considering Assumption I we can average (21) and (22) in the case of the  $n$ -component system. After averaging one can write

$$\begin{aligned} \langle \mathbf{B}_i(H_i) \rangle - \langle \mathbf{B}_i(H_{i0}) \rangle &= \left\langle \frac{\mu_0\mu_i}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} \right\rangle \mathbf{H}_{00} \\ &= \sum_{i=1}^n c_i \frac{\mu_0\mu_i}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} \mathbf{H}_{00} \end{aligned} \quad (23)$$

$$\begin{aligned} \text{and } -\nabla\langle\varphi_i\rangle = \langle \mathbf{H}_i \rangle - \langle \mathbf{H}_{i0} \rangle &= \left\langle \frac{1}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} \right\rangle \mathbf{H}_{00} \\ &= \sum_{i=1}^n c_i \frac{1}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} \mathbf{H}_{00} \end{aligned} \quad (24)$$

where  $c_i$  is the volume fraction of the  $i^{\text{th}}$ -component. From (24) we calculate  $\mathbf{H}_{00}$  and introduce it into (23). After introducing one obtains

$$\begin{aligned} \langle \mathbf{B}_i(H_i) \rangle - \langle \mathbf{B}_i(H_{i0}) \rangle &= \\ &= \frac{\left\langle \frac{\mu_i}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} \right\rangle}{\left\langle \frac{1}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} \right\rangle} (\langle \mathbf{H}_i \rangle - \langle \mathbf{H}_{i0} \rangle). \end{aligned} \quad (25)$$

The quantities  $\langle \mathbf{B}_i(H_i) \rangle$ ,  $\langle \mathbf{B}_i(H_{i0}) \rangle$ ,  $\langle \mathbf{H}_i \rangle$  and  $\langle \mathbf{H}_{i0} \rangle$  describe the magnetic field on the macroscopic level and, therefore, we can write that  $\langle \mathbf{B}_i(H_i) \rangle = \mathbf{B}_e$ ,  $\langle \mathbf{B}(H_{i0}) \rangle = \mathbf{B}_{e0}$ ,  $\langle \mathbf{H}_i \rangle = \mathbf{H}_e$ ,  $\langle \mathbf{H}_{i0} \rangle = \mathbf{H}_{e0}$ . In this case, relations (25) and (5) have to be the same. From this fact it follows that

$$\mu_{eff} = \frac{\sum_{i=1}^n c_i \frac{\mu_i}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}}}{\sum_{i=1}^n c_i \frac{1}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}}} \quad (26)$$

From (26) it immediately follows that

$$\sum_{i=1}^n c_i \frac{\mu_i - \mu_{eff}}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} = 0. \quad (27)$$

Equation (26) determines the  $\mu_{eff}$  if one knows  $\{c_i\}$  and  $\{\mu_i\}$  and it is similar to the relation for the effective thermal conductivity in [4], [5], and [6]. According to relation (27) the average value of fluctuating field  $\langle \mathbf{h}_2 - \mathbf{h}_1 \rangle$  is zero. From (27) it can be shown that

$$\sum_{i=1}^n c_i \frac{1}{1 + g \frac{\mu_i - \mu_{eff}}{\mu_{eff}}} = 1. \quad (28)$$

Introducing (28) into (22) one obtains

$$\langle \mathbf{H}_i \rangle - \langle \mathbf{H}_{i0} \rangle = \mathbf{H}_e - \mathbf{H}_{e0} = \mathbf{H}_{00}. \quad (29)$$

From (29) it immediately follows that  $\mathbf{H}_{00}$  is parallel with the axis of solenoid and this is the physical meaning of the  $\mathbf{H}_{00}$ . Further from (16), (20) and (27) it follows that

$$\langle \mathcal{G} \rangle = 0, \\ \langle \mathbf{H}_2 \rangle - \langle \mathbf{H}_1 \rangle = \mathbf{H}_{00} = \langle \mathbf{H}_i \rangle - \langle \mathbf{H}_{i0} \rangle = \mathbf{H}_e - \mathbf{H}_{e0}.$$

### 3 NON-EQUILIBRIUM THERMODYNAMICS OF DAMAGE

The aim of this part is to formulate an approach to plastic deformation and damage of materials based on non-equilibrium thermodynamics. With the help of non-equilibrium thermodynamics on the one hand, we derive a phenomenological equation of the time evolution of the damage variable  $D$  and on the other hand from its solution we obtain the dependence of damage variable  $D$  on the component of plastic part of the strain tensor  $\epsilon_{p11}$ . In the relation derived, there are unknown material parameters which in the frame phenomenological theory will be determined from the best fit of the theoretical relation to measured relative effective differential permeability and  $\epsilon_{p11}$  data. In order to determine material parameters, we will use the experimental results of the relative effective differential permeability and its dependence on the component of the plastic part of the strain tensor  $\epsilon_{p11}$ . The relations (30) and (48) will be derived by applying non-equilibrium thermodynamics which is nowadays commonly accepted in the continuum mechanics [7], [8] and [9].

We will start from the two general equations.

- The continuity equation of the specific internal energy:

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{J}_q + \boldsymbol{\sigma} : \frac{d\boldsymbol{\epsilon}}{dt} \quad (30)$$

- And the generalized Gibbs's relation [7] and [8]

$$\rho \frac{du}{dt} = \rho T \frac{ds}{dt} + \boldsymbol{\sigma} : \frac{d\boldsymbol{\epsilon}_e}{dt} + \boldsymbol{\beta} : \frac{d\boldsymbol{\alpha}}{dt} + Y \frac{dD}{dt}, \quad (31)$$

where  $u$  is the specific internal energy (internal energy per unit mass),  $s$  is the specific entropy,  $\boldsymbol{\sigma}$  is the symmetric Cauchy's stress tensor,  $\boldsymbol{\alpha}$  is the internal tensorial state parameter,  $\boldsymbol{\beta}$  is the tensorial affinity to  $\boldsymbol{\alpha}$ ,  $\mathbf{J}_q$  is the heat current density,  $\rho$  is the mass density,  $Y$  is the affinity to  $D$ ,  $T$  is the thermodynamic temperature.

Subtracting equation (30) from equation (31) one obtains the continuity equation for the specific entropy

$$\rho \frac{ds}{dt} = -\nabla \cdot \mathbf{J}_s + \sigma_s, \quad (32)$$

where  $\mathbf{J}_s = \frac{d\mathbf{J}_q}{dT}$  is the flux of entropy,

$$\sigma_s = -\frac{\mathbf{J}_q}{T^2} \cdot \text{grad } T + \frac{\boldsymbol{\sigma}}{T} : \frac{d\boldsymbol{\epsilon}_p}{dt} - \frac{\boldsymbol{\beta}}{T} : \frac{d\boldsymbol{\alpha}}{dt} - \frac{Y}{T} \frac{dD}{dt} \geq 0 \quad (33)$$

is the entropy production per unit volume and per unit time.

The entropy production plays an important role in the formulation of non-equilibrium thermodynamics. Relation (33) consists of a sum of products of two factors. One of these factors in each term is a flow quantity ( $\mathbf{J}_q$ ,  $\frac{d\boldsymbol{\epsilon}_p}{dt}$ ,  $\frac{d\boldsymbol{\alpha}}{dt}$  and  $\frac{dD}{dt}$ ). The other factors in each term are the thermodynamic forces ( $-\frac{\text{grad } T}{T^2}$ ,  $\frac{\boldsymbol{\sigma}}{T}$ ,  $-\frac{\boldsymbol{\beta}}{T}$  and  $-\frac{Y}{T}$ ). The phenomenological equations in the framework of linear non-equilibrium thermodynamics are based on the following assumption:

ASSUMPTION II. *The fluxes depend linearly on the thermodynamic forces. In principle any Cartesian component of a flux can be a linear function of the Cartesian components of all thermodynamic forces.*

For an isotropic system the fluxes and the thermodynamic forces of different tensorial character do not couple (Curie symmetry principle) [9]. Before we start the formulation of phenomenological equations we have to take the Curie symmetry principle into account. Therefore, we split up all tensors in two parts

$$\mathcal{A} = \mathcal{A}^d + \frac{1}{3} A \mathcal{I} \quad (34)$$

where  $\mathcal{A}^d$  is the deviator and  $A = \sum_{i=1}^3 A_{ii}$ ,  $\mathcal{I}$  is the unit tensor.

Using relation (34) for all tensors and introducing them into relation (33) we obtain:

$$\sigma_s = \sigma_{s1} + \sigma_{s2} + \sigma_{s3} \quad (35)$$

where

$$\sigma_{s1} = \frac{\boldsymbol{\sigma}^d}{T} : \frac{d\boldsymbol{\epsilon}_p^d}{dt} - \frac{\boldsymbol{\beta}^d}{T} : \frac{d\boldsymbol{\alpha}^d}{dt}, \quad (36)$$

$$\sigma_{s2} = -\frac{\mathbf{J}_q}{T^2} \cdot \text{grad } T, \quad (37)$$

and

$$\sigma_{s3} = -\left(\frac{1}{3}\right)^2 \beta \frac{d\alpha}{dt} - \frac{Y}{T} \frac{dD}{dt}. \quad (38)$$

In relation (38) we have put  $\epsilon_p = 0$ , due to the fact that the plastic deformation does not contribute to the change of the volume. According to the Curie symmetry principle from relation (38) in the framework of the linear non-equilibrium thermodynamics and from Assumption II it follows that:

$$\frac{d\alpha}{dt} = -L_{\beta\beta}\beta - L_{\beta Y}Y, \quad (39)$$

and

$$\frac{dD}{dt} = -L_{Y Y}Y - L_{Y\beta}\beta. \quad (40)$$

The terms  $L_{\beta Y}Y$  and  $L_{Y\beta}\beta$  describe the cross-effects between two irreversible phenomena. In hereon, we will not consider the cross-effect ( $L_{\beta Y} = 0$  and  $L_{Y\beta} = 0$ ). In this case relation (40) can be written as

$$\frac{dD}{dt} = -L_{Y Y}Y. \quad (41)$$

In addition, we need one more equation (*ie*, a state equation) apart from equation (41). We obtain this equation from Gibbs's relation (31). For this purpose, we introduce the specific free energy  $f = u - Ts$  into (31). After introducing it we obtain

$$\frac{df}{dt} = -s \frac{dT}{dt} + \frac{\sigma}{\varrho} : \frac{d\epsilon_e}{dt} + \frac{\beta}{\varrho} : \frac{d\alpha}{dt} + \frac{Y}{\varrho} \frac{dD}{dt}. \quad (42)$$

From (42), it immediately follows:

$$-s = \frac{\partial f}{\partial T}; \quad \frac{\sigma_{ij}}{\varrho} = \frac{\partial f}{\partial \epsilon_{eij}}; \quad \frac{\beta_{ij}}{\varrho} = \frac{\partial f}{\partial \alpha_{ij}} \quad \text{and} \quad \frac{Y}{\varrho} = \frac{\partial f}{\partial D}. \quad (43)$$

We assume that the specific free energy  $f(T, \epsilon_e, \alpha, D)$  can be divided into two parts

$$f(T, \epsilon_e, \alpha, D) = f_1(T, D) + f_2(T, \epsilon_e, \alpha). \quad (44)$$

The  $f_1(T, D)$  is expanded into a Taylor's series

$$f_1(T, D) = \tilde{C}_0(T) + \tilde{C}_1(T)D + \frac{\tilde{C}_2(T)}{2}D^2 + \frac{\tilde{C}_3(T)}{3}D^3 + \dots \quad (45)$$

Using relations (43) and (45), we can write

$$Y = \varrho[\tilde{C}_1(T) + \tilde{C}_2(T)D + \tilde{C}_3(T)D^2]. \quad (46)$$

From relations (41) and (46), we obtain

$$\frac{dD}{dt} = C_1(T) - C_2(T)D - C_3(T)D^2 \quad (47)$$

where  $C_i(T) = L_{Y Y} \varrho \tilde{C}_i(T)$ . The solution of equation (47) is the following:

$$D(t) = \frac{\alpha_1 - \alpha_2 \mathcal{H} e^{-C_3(\alpha_1 - \alpha_2)(t-t_0)}}{1 - \mathcal{H} e^{-C_3(\alpha_1 - \alpha_2)(t-t_0)}} \quad (48)$$

where  $\alpha_1$  and  $\alpha_2$  are the roots of the equation

$$\alpha^2 + \frac{C_2}{C_3}\alpha + \frac{C_1}{C_3} = 0,$$

$\mathcal{H} = \frac{D_0 - \alpha_1}{D_0 - \alpha_2}$  and  $D_0 = D(t_0)$ . where  $\mathcal{H} = \frac{D_0 - \alpha_1}{D_0 - \alpha_2}$  and  $D_0 = D(t_0)$ . Material parameters  $\alpha_1$ ,  $\alpha_2$ ,  $C_3$  and  $\mathcal{H}$  will be determined from a best fit of the theoretical relation to the measured relative effective differential permeability and  $\epsilon_{p11}$  data.

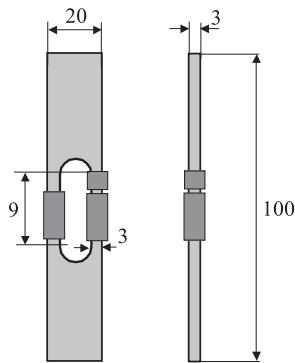
#### 4 EXPERIMENTALLY DETERMINED DAMAGE VARIABLE $D(H_a, H_b)$

The structural aspects of the samples cover material non-uniformities (structural defects), such as grain boundaries, material inclusions, micro-cracks, micro-cavities, clusters of dislocations and many others. Their presence, their distributions and their magnitudes control the details of magnetization processes, and there is no reason to expect that the influence of each structural defect affects all the regions of each magnetization mechanism with the same intensity. That means that the relative effective differential permeabilities  $\mu_{eff}(H_a, H_b)$  are not equally sensitive to the same damage state of the material. The  $H_a$  and  $H_b$  are the magnetic coordinates of the point on the minor hysteresis loop. Every minor hysteresis loop is uniquely determined by its amplitude,  $H_b$ . Measurement of the relative effective differential permeability confirms this fact and it can be interpreted in the following way: Let us consider only those defects which can be modelled by damaged bonds. We divide all damaged bonds into groups so that the damaged bonds belonging to a certain group affect the magnetization process with the same intensity. Influence of the damaged bonds of the  $i^{\text{th}}$ -group is characterized by a parameter  $\xi_i(H_a, H_b)$ , the value of which is situated in the interval  $(0, 1)$ . The value  $\xi_i = 1$  means that the damaged bonds influence the magnetization process to a full extent. For this reason the effective number of the damaged bonds of the  $i^{\text{th}}$ -group is expressed by the relation

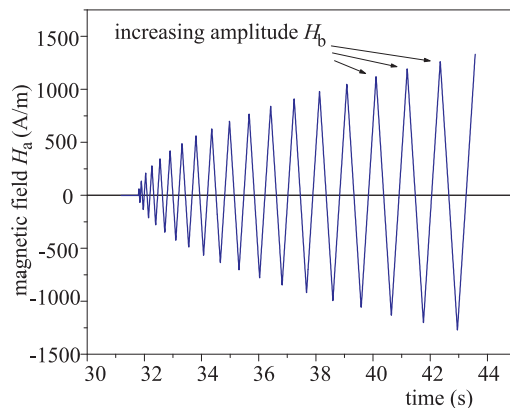
$$N_{di\text{eff}} = \xi_i(H_a, H_b)N_{di}, \quad (49)$$

where  $N_{di}$  is the real number of damaged bonds belonging to the  $i^{\text{th}}$ -group. The average value of the parameter  $\xi_i(H_a, H_b)$  is expressed by the relation

$$\langle \xi_i(H_a, H_b) \rangle = \frac{\sum_i \xi_i(H_a, H_b)N_{di}}{\sum_i N_{di}} = \frac{N_{d1}(H_a, H_b)}{N_d}, \quad (50)$$



**Fig. 2.** The window-shaped steel sample. The top and bottom pads of the sample were clamped into the Instron holders. Two driving coils are positioned on each prismatic legs, a pick-up coil sits on one.



**Fig. 3.** Shape of the typical magnetizing. Note the field varies linearly with time which implies the induce voltage is proportional to effective differential permeability of the material.

where  $N_{d1}(H_a, H_b)$  is the effective number of the damaged bonds which influence the magnetization process. The effective number of the damaged bonds which do not influence the magnetization process is expressed as

$$N_{d2}(H_a, H_b) = [1 - \langle \xi_i(H_a, H_b) \rangle] N_d. \quad (51)$$

Now all is prepared to define the modified damage variable

$$D_{\text{mod}}(H_a, H_b) \stackrel{\text{def}}{=} \frac{N_{d1}(H_a, H_b)}{N}. \quad (52)$$

As it was mentioned in Section 2 the damaged material is considered as a binary system. The first component consists of the non-damaged bonds and the effective number of damaged bonds  $N_{d2}$ . The second component consists of the effective number of the damaged bonds  $N_{d1}$ . The volume fraction of the first component is expressed by the relation

$$c_1 = \frac{[(N - N_d) + N_{d2}]}{N} = \frac{N - N_{d1}(H_a, H_b)}{N} = 1 - D_{\text{mod}}(H_a, H_b). \quad (53)$$

Because the damaged bonds  $N_{d2}$  do not influence the magnetization process, the differential permeability of the first component is equal probably to the differential permeability of the non-damaged material  $\mu_1$ .

The volume fraction of the second component is expressed by the relation

$$c_2 = \frac{N_{d1}(H_a, H_b) \Delta V_b}{N \Delta V_b} = D_{\text{mod}}(H_a, H_b). \quad (54)$$

It is obvious that the relation  $c_1 + c_2 = 1$  is valid. As it was mentioned before the damaged material is considered as a binary system. From (27) for binary system the quadratic equation follows

$$x^2 - \frac{1 - g - gr + D_{\text{mod}}(H_a, H_b)(r - 1)}{1 - g} x - \frac{g}{1 - g} r = 0 \quad (55)$$

where  $x = \frac{\mu_{\text{eff}}(h_a, H_b)}{\mu_1(H_a, H_b)}$  and  $r = \frac{\mu_2(H_a, H_b)}{\mu_1(H_a, H_b)}$ .

The second component (damaged bonds, which influence the magnetization process) may be characterized in the best case as a para-magnetics. In this case, the relative differential permeability  $\mu_2(H_{ai}, H_{bj})$  is in order of about one. The first component is characterized as a ferro-magnetics. The relative differential permeability  $\mu_1(H_{ai}, H_{bj})$  is in order of about  $10^3 \div 10^5$ . For this reason  $r$  in equation (55) can be neglected. For  $r = 0$  equation (55) simplifies in the following form

$$\frac{\mu_{\text{eff}}(H_{ai}, H_{bj})}{\mu_1(H_{ai}, H_{bj})} \left\{ \frac{\mu_{\text{eff}}(H_{ai}, H_{bj})}{\mu_1(H_{ai}, H_{bj})} - \frac{1}{1 - g} (1 - g - D_{\text{mod}}(H_{ai}, H_{bj})) \right\} = 0. \quad (56)$$

We consider that  $\frac{\mu_{\text{eff}}(H_{ai}, H_{bj})}{\mu_1(H_{ai}, H_{bj})} > 0$  and, therefore, the solution of equation (56) is the following:

- If  $1 - g - D_{\text{mod}}(H_{ai}, H_{bj}) \leq 0$  or  $c_1 \leq g$ , then

$$\mu_{\text{eff}}(H_{ai}, H_{bj}) = 0 \quad (57)$$

because the second term in (56) cannot be zero due to the fact that  $\frac{\mu_{\text{eff}}(H_{ai}, H_{bj})}{\mu_1(H_{ai}, H_{bj})} > 0$ . The clusters of the first component are separated by the clusters of the second component, and, therefore, the material is totally broken up ( $\mu_{\text{eff}}(H_{ai}, H_{bj}) = 0$ ).

- If  $1 - g - D_{\text{mod}}(H_{ai}, H_{bj}) > 0$  or  $c_1 > g$ , then

$$\frac{\mu_{\text{eff}}(H_{ai}, H_{bj})}{\mu_1(H_{ai}, H_{bj})} = \frac{1}{1 - g} \left\{ 1 - g - D_{\text{mod}}(H_{ai}, H_{bj}) \right\}. \quad (58)$$

If damage variable  $D_{\text{mod}}(H_{ai}, H_{bj})$  is equal to  $D_k$ , where

$$D_k = 1 - g \quad \text{or} \quad c_{1k} = g. \quad (59)$$

At  $c_1 = c_{1k}$  some clusters of the first component connect with each other and form a percolation cluster,

which is spread through the whole sample. From this moment  $\frac{\mu_{\text{eff}}(H_{ai}, H_{bj})}{\mu_1(H_{ai}, H_{bj})}$  increases linearly with  $c_1$ . This effect is called a percolation and at  $c_1 = c_{1k}$  (a percolation threshold) a percolation phase transition takes place. The detailed overview of percolation is given in [11]. The values of the parameter  $D_{\text{mod}}(H_{ai}, H_{bj})$  are situated in the interval  $(0, 1)$ , where  $D_{\text{mod}}(H_{ai}, H_{bj}) = 1$  evidently corresponds to a fully damaged material, in other words, to a fracture. In the reality the material will break up before the value of  $D_{\text{mod}}(H_{ai}, H_{bj})$  reaches the value 1 since atomic decohesion will occur. Probably the fracture initiation may happen near bellow  $D_k$ . The method MFAM, as it was shown, gives the value of the  $g$  equal to  $\frac{1}{3}$ . The value of  $g$  can be calculated numerically by the Monte Carlo method on the lattice model [12]

- For simple cubic lattice  $g = 0.254 \pm 0.013$ ,
- For body-centered cubic lattice  $g = 0.178 \pm 0.005$ ,
- For face-centered cubic lattice  $g = 0.119 \pm 0.002$ .

## 5 INTERPRETATION OF EXPERIMENTAL RESULTS

The tested material was ferrite-pearlite low-alloyed steel which after normalized annealing was plastically and elastically deformed in a loading machine Instron 1362. The sample was prepared in a shape, which allowed it to be deformed by the tensile stress and, at the same time, to be magnetized in a closed magnetic circuit. Shape of the sample is sketched in Fig. 2. All samples were subjected to a uniform axial tension test so that  $\sigma_{11} \neq 0$  and other  $\sigma_{ij} = 0$ . The axial tension was realized under the condition that  $\frac{d\epsilon_{11}}{dt} = \gamma$ , where  $\gamma$  is the constant. After unloading, the component  $\epsilon_{11}$  of the strain tensor is equal only to the plastic part  $\epsilon_{p11}$  ( $\epsilon_{11} = \epsilon_{e11} + \epsilon_{p11}$ ). The samples were prepared at various  $\epsilon_{p11}$ . After that the relative effective differential permeability was measured and in this way the dependence of the  $\mu_{\text{eff}}(H_{ai}, H_{bj})$  on the  $\epsilon_{p11}$  was obtained. Every magnetic measurement was started at a thoroughly pre-demagnetized sample, and then it proceeded by magnetizing the sample “around the window” by the driving coils, and picking the voltage signal  $u_{\text{ind}}$  induced in the pick-up coils. The shape of the magnetizing field is depicted in Fig. 3, *ie* triangular sweeps of the field with step-by-step increasing amplitude  $H_b$ , from 0 up to about 1500 A/m, which was close to the saturation field,  $H_{\text{sat}}$ , for the sample. The amplitudes begin with a finite value,  $H_{b0}$ , and are increased by a suitably chosen step  $\Delta H_b$ . In this way the family of minor hysteresis loops was realized. Every minor hysteresis loop is uniquely determined by its amplitude,  $H_b$ . The sweeping field  $H_a \epsilon \langle -H_{bj}, +H_{bj} \rangle$  is also varied by the steps  $\Delta H_a$  ( $= \Delta H_b$ , usually) and the steps are denoted as  $H_{ai}$ . According to relation (5) one can write

$$B_e(H) - B_e(H_{ai}) = \mu_{\text{eff}}(H_{ai}, H_{bj})(H - H_{ai}), \quad (60)$$

assuming  $|H - H_{ai}|$  to be small. The relative effective differential permeability  $\mu_{\text{eff}}(H_{ai}, H_{bj})$ , as it was

mentioned, is uniquely determined (however, for each  $(H_{ai}, H_{bj})$  it is different).

Similarly to [10], this paper proposes to adapt the measurements optimally to each task and always to collect the region in which the relative effective differential permeability reflects the structural changes in the most sensitive way. For this purpose a family of minor hysteresis loops was used.

The relative effective differential permeability  $\mu_{\text{eff}}(H_{ai}, H_{bj})$  was measured by the induction method, which is based on the Faraday’s law

$$u_{\text{ind}} = -\frac{d\phi}{dt} = -\mu_0 \mu_{\text{eff}}(H_{ai}, H_{bj}) \frac{Sn}{l_c} \frac{dI}{dt} \quad (61)$$

where  $S$  is the cross-section of the window,  $n$  is the number of turns,  $l_c$  is the mean length of the magnetic circuit,  $I$  is the electric current in the driving coil wound on the ring. The detailed information about the experimental method are given in [10]. The appraisal of the experimental results was done by the following procedure:

Integrating relation  $\frac{d\epsilon_{11}}{dt} = \gamma$  we obtain

$$\gamma(t - t_0) = \epsilon_{11} - \epsilon_{c11} \quad (62)$$

where  $\epsilon_{c11}$  is the plasticity threshold,  $t_0$  is the time of the beginning of the plastic deformation. If we use the relation  $\epsilon_{11} = \epsilon_{e11} + \epsilon_{p11}$ , we can write

$$\gamma(t - t_0) = \epsilon_{p11} + \epsilon_{e11} - \epsilon_{c11}. \quad (63)$$

From the measured uniaxial strain-stress curve it follows that for  $\epsilon_{p11} > 0.01$  one can neglect the term  $\epsilon_{e11} - \epsilon_{c11}$ . In this approximation relation (63) has the form

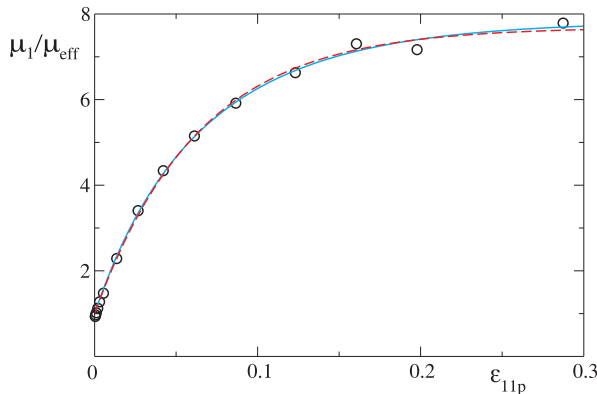
$$\gamma(t - t_0) \approx \epsilon_{p11}. \quad (64)$$

We will suppose that the time evolution of  $D(H_{ai}, H_{bj})$  is expressed by the relation which is similar to relation (48) with the difference that the parameters  $\alpha_1$ ,  $\alpha_2$ ,  $C_3$  and  $\mathcal{H}$  are now the functions of the two field-coordinates  $H_{ai}$  and  $H_{bj}$ . Considering this fact relation (18) can be written in this way

$$D(t, H_{ai}, H_{bj}) = \frac{\alpha_1(H_{ai}, H_{bj}) - \alpha_2(H_{ai}, H_{bj}) \mathcal{H}(H_{ai}, H_{bj}) e^{-A_1(H_{ai}, H_{bj}) \epsilon_{p11}}}{1 - \mathcal{H}(H_{ai}, H_{bj}) e^{-A_1(H_{ai}, H_{bj}) \epsilon_{p11}}} \quad (65)$$

where  $A_1(H_{ai}, H_{bj}) = C_3(H_{ai}, H_{bj}) \frac{\alpha_1(H_{ai}, H_{bj}) - \alpha_2(H_{ai}, H_{bj})}{\gamma}$  and where we used relation (64). It is evident that for  $\epsilon_{p11} = 0$ ,  $D(H_{ai}, H_{bj})$  must be equal to 0, too. Considering this fact we obtain





**Fig. 4.** The symbol  $\circ$  represents the experimental data of  $\mu_1/\mu_{eff}$ , a best fit of experimental data to theoretical formula (67) is depicted by the full line and the dashed line represents the best fit to simplified formula (68).

from (65) that  $\alpha_1(H_{ai}, H_{bj}) = \alpha_2(H_{ai}, H_{bj})\mathcal{H}(H_{ai}, H_{bj})$ . Introducing this result into (58) and (65) one obtains

$$D_{mod}(\epsilon_{p11}, H_{ai}, H_{bj}) = \frac{(1-g)[1 - A_2(H_{ai}, H_{bj})]\{1 - e^{-A_1(H_{ai}, H_{bj})\epsilon_{p11}}\}}{1 - [1 - A_2(H_{ai}, H_{bj} + A_0(H_{ai}, H_{bj}))e^{-A_1(H_{ai}, H_{bj})\epsilon_{p11}}]} \quad (66)$$

and

$$\frac{\mu_1(H_{ai}, H_{bj})}{\mu_{eff}(H_{ai}, H_{bj})} = \frac{1 - [A_0(H_{ai}, H_{bj}) - A_2(H_{ai}, H_{bj}) + 1]e^{-A_1(H_{ai}, H_{bj})\epsilon_{p11}}}{A_2(H_{ai}, H_{bj}) - A_0(H_{ai}, H_{bj})e^{-A_1(H_{ai}, H_{bj})\epsilon_{p11}}} \quad (67)$$

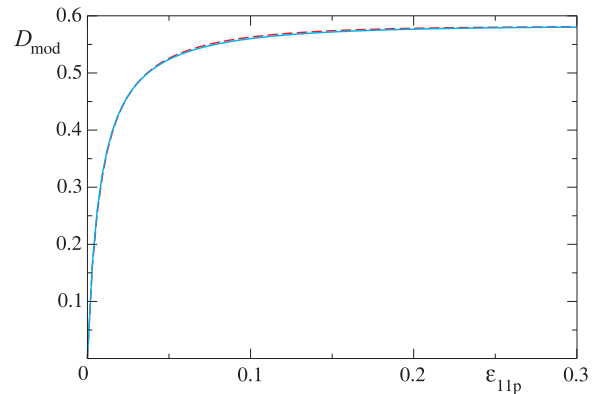
From the best fit of the experimental data to relation (67) we have obtained the values of the material parameters  $A_0 = 0.0290819$ ,  $A_1 = 12.7553$  and  $A_2 = 0.127697$ . The experimental data of  $\frac{\mu_1(H_{ai}, H_{bj})}{\mu_{eff}(H_{ai}, H_{bj})}$  were considered at  $H_{ai} = 250$  A/m and  $H_{bj} = 450$  A/m. For the calculation of  $\frac{\mu_1(H_{ai}, H_{bj})}{\mu_{eff}(H_{ai}, H_{bj})}$  we used  $g = \frac{1}{3}$ . The dependences of  $\frac{\mu_1}{\mu_{eff}(H_{ai}, H_{bj})}$  and  $D_{mod}(H_{ai}, H_{bj})$  on  $\epsilon_{p11}$  are depicted in Figs. 4 and 5. The parameter  $A_0$  is about one order lower than the parameter  $A_2$  and, therefore, we can neglect it. Neglecting  $A_0$  one can simplify relation (67) as

$$\frac{\mu_1(H_{ai}, H_{bj})}{\mu_{eff}(H_{ai}, H_{bj})} = \frac{1 - (1 - A_2)e^{A_1(H_{ai}, H_{bj})\epsilon_{p11}}}{A_2(H_{ai}, H_{bj})} \quad (68)$$

and  $D_{mod}(\epsilon_{p11}, H_{ai}, H_{bj}) =$

$$\frac{(1-g)\left(1 - A_2(H_{ai}, H_{bj})\right)\left\{1 - e^{-A_1(H_{ai}, H_{bj})\epsilon_{p11}}\right\}}{1 - [1 - A_2(H_{ai}, H_{bj})e^{-A_1(H_{ai}, H_{bj})\epsilon_{p11}}]} \quad (69)$$

The experimental data were best fitted also to relation (68). From the best fit we have obtain  $A_1 = 15.7816$  and  $A_2 = 0.127696$ . Both dependences (68) and (69) are depicted in Figs. 4 and 5. From Figs. 4 and 5 it is evident that the simplified relations (68) and (69) describe the dependences of  $\frac{\mu_1(H_{ai}, H_{bj})}{\mu_{eff}(H_{ai}, H_{bj})}$  and  $D_{mod}(H_{ai}, H_{bj})$  on  $\epsilon_{p11}$  quite well.



**Fig. 5.** The full line represents damage variable  $D_{mod}$  calculated according to relation (66) and the dashed line represents damage variable  $D_{mod}$  calculated according to relation (69).

## 6 DISCUSSION

The permeability measurement as described in Section 5, takes advantage of application of the method of Magnetic Adaptive Testing (MAT) [10], which collects a large representative data-pool of the differential permeability values,  $\mu_{eff}(H_a, H_b)$ , from a large range of the magnetizing field amplitudes,  $H_b$ , and the sweeping field values,  $-H_b \leq H_a \leq +H_b$ . Then, according to any intentions of the experimenter, MAT picks up the most convenient field-coordinates-point, where  $\mu_{eff}(H_a^{opt}, H_b^{opt})$  indicates degradation of the inspected material with an optimum sensitivity. The optimum field-coordinates for the experiment described in this paper were  $H_a^{opt} = 250$  A/m and  $H_b^{opt} = 450$  A/m. This means that the fitting of the curves in Figs. 4 and 5 was carried out for  $\mu_1$  (250 A/m, 450 A/m),  $\mu_{eff}$  (250 A/m, 450 A/m) and  $D_{mod}$  (250 A/m, 450 A/m). The curves could successfully fit also for any other field-coordinates-point, however, the sensitivity would be lower.

Measurement of the permeability was performed on magnetically closed thoroughly demagnetized samples, which made it possible to calculate the sample *internal* field from the electric current in the driving coil and made the permeability simply proportional to the voltage signal induced in the pick-up coil. Application of the described method to magnetically open samples makes the evaluation more complicated. Analysis of the open sample configurations is presently under work and will be published in a short time elsewhere.

The principal advantage of the presented approach is the description of the damaged material by means of an experimentally *accessible* differential permeability  $\mu_{eff}(H_a, H_b)$  instead of the experimentally *inaccessible* damage variable,  $D_{mod}$ . Finally,  $D_{mod}$  is calculated by means of formula (66), using the experimentally obtained data of  $\mu_{eff}(H_a, H_b)$ .

We investigated the sensitivity of certain quantities to the change of the component of the plastic part of the strain tensor  $\epsilon_{p11}$ . There were the mean lifetime of

positron, ultrasonic attenuation and velocity measurements [13], electrical resistivity [14] and relative differential permeability [10]. The measurements showed that the sensitivity of the relative effective differential permeability for field-coordinates, namely  $H_{ai} = 250$  A/m and  $H_{bj} = 450$  A/m was more than one order higher as in the other above-mentioned cases.

The time-evolution equation for  $D$  was derived in approximation of expansion of the free energy  $F_1(T, D)$  up to the third order according to  $D$ . It was shown that this approximation is sufficient in the used range of  $\epsilon_{p11}$ . Both dependences (66) and (67) demonstrate the saturated behavior. But in the advanced stage of the damage this approximation will probably not be sufficient.

One of the limitations of the described approach is that it can be applied to ferromagnetic substances only, where the magnetic permeability is significantly modified by damage of the material. However, as ferromagnetic steel and cast iron form a very substantial part of industrial construction materials, application of the presented ideas can be found helpful in many practical problems.

Now we present the comparison between our results and those ones already published. From our measurement it follows that both  $\frac{\mu_1}{\mu_{eff}}$  and  $D_{mod}$  saturate by increasing the value of the  $\epsilon_{p11}$ . But the dependence  $D_{mod}$  versus  $\epsilon_{p11}$  may also have a monotonously increasing behaviour. This fact depends on which region of plastic deformation the experiment was done. There are two regions: In the first region there are missing micro-cracks and micro-cavities. The second region is characterized by creation of micro-cracks and micro-cavities. If these types of defects are missing the dependences  $\frac{\mu_1}{\mu_{eff}}$  and  $D_{mod}$  versus  $\epsilon_{p11}$  exhibit the saturation behaviour as it is seen from Figs. 4 and 5 in this paper. Similar behaviour can be found also in other papers which will be quoted later on.

In the second region, namely in the case of an advanced stage of damage, the dependence of  $D$  versus  $\epsilon_{p11}$  may exhibit a monotonously increasing behaviour as it can be seen in [8] (Fig. 7.4(a) on page 354) and in [1] (Fig. 5 on page 651). There is presented an experiment of a ductile plastic damage: copper Cu/Al at the conditions  $T = 20^\circ\text{C}$  and  $\frac{d\epsilon_{11}}{dt} = 10^{-4}\text{s}^{-1}$ . The surfacic damage variable  $\tilde{D}$  was determined by the non-direct method. The dependence of the Young's modulus versus  $\epsilon_{p11}$  was measured from the stress-strain curve during unloadings. There is observed a threshold  $\epsilon_{11}^*$  at which the change of the Young's modulus versus  $\epsilon_{p11}$  becomes perceptible. From Fig. 7.4(a) it is seen that for  $\epsilon_{11} > \epsilon_{11}^*$  the  $\tilde{D}$  changes approximately linearly with  $\epsilon_{11}$ . From the condition that the deformation rate is constant it follows immediately that  $\epsilon_{11} \approx \epsilon_{p11} = \gamma(t - t_0)$ , relation (46), so one can state that the damage rate is an increasing function of time. But this situation differs from with that one in this paper. The measurements of the relative effective differential permeability  $\mu_{eff}$  versus  $\epsilon_{p11}$  was realized in the first region of the plastic deformation for  $\epsilon_{p11} < \epsilon_{p11}^*$ . This region is important from the point of

view of the safety of engineering materials and, therefore, it is necessary to monitor the damage variable  $D$  and its time-evolution. When micro-cracks and micro-cavities already exist the engineering materials become unreliable. In the first region, according to our knowledge, the other measured quantities are substantially less sensitive to the change of damage as the relative effective differential permeability  $\mu_{eff}$ .

It is interesting to note that the saturation behaviour is also observed in the second region. This fact can be found in [15] and [1] (Fig. 4 on page 650) where is described an experiment of pure ductile damage on steel. The decrease of the relative density mass according to the  $\epsilon_{p11}$  was measured with apparatus based on the Archimedean principle. The measurement of the relative density mass versus  $\epsilon_{p11}$  was done for different strain hardenings. The dependences of the relative density mass and the surfacic damage variable  $\tilde{D}$  versus  $\epsilon_{11}$  exhibit the saturation behaviour. The saturation behaviour was also observed in [14], [7] (Fig. 10.2 on page 210) and in [16]. In the two last papers there are presented the experimental results not from ductile damage but from the fatigue-creep test for material, stainless steel A316 at temperature  $T = 550^\circ\text{C}$ . If one considers that the number of loading cycles is proportional to the time then there exists a region of the time interval where damage rate is a decreasing function of time. But after formation of micro-cracks and micro-cavities due to an instability the damage rate is an increasing function of time. From this experimental results it follows that at the interpretation of experimental results one has to consider the region of the plastic deformation in which the experiment was done.

The saturation behaviour may be interpreted by the following way — the determination of the damage variable is based on the two facts: In section 2 there was derived the unambiguous relation between  $\mu_{eff}$  and  $D$ . Further, there was experimentally found a relation between  $\mu_{eff}$  and  $\epsilon_{p11}$ . From these two relation one obtains immediately the relation between  $D$  and  $\epsilon_{p11}$ . It follows logically that if the dependence  $\mu_{eff}$  versus  $\epsilon_{p11}$  manifests the saturation behaviour (experimental fact) then the dependence  $D$  versus  $\epsilon_{p11}$  has to manifest the saturation behaviour too. The above-mentioned reasonings are valid only in the first region of damage. In the second region the time-evolution relation of the damage variable  $D$  has to be derived with the help of non-linear non-equilibrium thermodynamics and there has to be used the higher order in Taylor's expansion of the free energy.

In this connection the following question can arise: How to interpret the saturation behaviour of the experimentally obtained relation between  $\mu_{eff}$  and  $\epsilon_{p11}$ ? It is evident that by increasing the  $\epsilon_{eff}$  the density of dislocations  $\psi(\epsilon_{p11})$  also increases. But change of the relative density of dislocations

$$\beta = \frac{\psi(\epsilon_{p11}^{(2)}) - \psi(\epsilon_{p11}^{(1)})}{\psi(\epsilon_{p11}^{(1)})}$$

decreases. The  $\epsilon_{p11}^{(1)}$  and  $\epsilon_{p11}^{(2)}$  are measured values of the successive measurements. One can assume that if the sensibility of the  $\mu_{eff}$  to damage is proportional to  $\beta$  then by decreasing  $\beta$  the sensibility also decreases. The decreasing of the sensibility outwardly manifests the saturation behaviour.

In conclusion one can suppose that in a certain sense it may be chosen a proper time-evolution formula for the damage variable  $D$  which can fit more or less suitably the experimental data. But time-evolution formula of the damage variable  $D$  derived in this paper was justified in the framework of the generalized non-equilibrium thermodynamics, so it is a physical relation.

### 7 CONCLUSION

- Non-equilibrium thermodynamics which takes simultaneously into account both damage and elastoplasticity was formulated.
- Considering the Curie principle, the phenomenological equations which describe time evaluation of damage variables and isotropic part of the internal tensorial state parameter  $\alpha$  were derived.
- The cross-effect between the damage variable  $D$  and  $\alpha$  was neglected in the first approximation.
- The free energy was expanded into Taylor's series up to the third order of  $D$  and the non-linear time evolution equation was obtained in this way.
- The theoretical relations were tested by experimental data of the measured  $\frac{\mu_1}{\mu_{eff}}$ .
- It was shown that by measuring  $\frac{\mu_1}{\mu_{eff}}$  it is possible to obtain the dependence of the damage variable  $D$  on  $\epsilon_{p11}$ .
- Both parameters  $\frac{\mu_1}{\mu_{eff}}$  and  $D$  saturate by increasing the value of  $\epsilon_{p11}$ .

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### APPENDIX

The presence of the  $i^{\text{th}}$ -component creates an additional magnetic field strength  $\mathbf{h}$  in the effective medium. If the electric current in solenoid is  $I_0$ , then the magnetic quantities of the magnetic state inside the effective medium are  $\mathbf{B}(H_1) = \mathbf{B}(H_{e0}) + \mathbf{b}_1$  and  $\mathbf{H}_1 = \mathbf{H}_{e0} + \mathbf{h}_1$ . If the electric current in solenoid is  $I + \Delta I$ , then the magnetic quantities of the magnetic state inside the effective medium are  $\mathbf{B}(H) = \mathbf{B}_e(H_e) + \mathbf{b}_2$  and  $\mathbf{H}_2 = \mathbf{H}_e + \mathbf{h}_2$ . We will consider the radius of the  $i^{\text{th}}$ -component so small

but still it can be described by a phenomenological electromagnetic theory.

ASSUMPTION II. In the above-mentioned case we can assume that the magnetic quantities  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ ,  $\mathbf{h}_1$  and  $\mathbf{h}_2$  due to the small radius  $R$  are substantially smaller than the magnetic quantities  $\mathbf{H}_{e0}$ ,  $\mathbf{H}_e$ ,  $\mathbf{B}_e(H_{e0})$  and  $\mathbf{B}_e(H_e)$ .

According to Assumption II we can write

$$\frac{\sqrt{(h_{1x} - h_{2x})^2 + (h_{1y} - h_{2y})^2}}{H_e - H_{e0} + h_{2z} - h_{1z}} \ll 1, \quad \frac{|\mathbf{b}_1|}{B_e(H_{e0})} \ll 1 \quad (70)$$

where  $z$  axis is parallel to the axis of the solenoid.

Using relation (70) one can write

$$\begin{aligned} \mathbf{H}_2 - \mathbf{H}_1 &= \mathbf{H}_e - \mathbf{H}_{e0} + \mathbf{h}_2 - \mathbf{h}_1 = H_e - H_{e0} + (h_{2z} - h_{1z})\mathbf{k} \\ &+ \sqrt{(h_{2x} - h_{1x})^2 + (h_{2y} - h_{1y})^2}\mathbf{e} \\ &\cong \mathbf{H}_e - \mathbf{H}_{e0} + (h_{2z} - h_{1z})\mathbf{k} \quad (71) \end{aligned}$$

and

$$\mathbf{B}(H_1) = \mathbf{B}(H_{e0}) + \mathbf{b}_1 \cong \mathbf{B}(H_{e0}), \quad (72)$$

where  $\mathbf{k}$  and  $\mathbf{e}$  are the unit vectors parallel and orthogonal to the axis of the solenoid, respectively.

Now we expand  $\mathbf{B}(H_2)$  into Taylor's series. If the enhancement of the electric current is small ( $\Delta I \ll I_0$ ), then it is enough to restrict the series only to the linear term. In the frame of this approximation we can write

$$\begin{aligned} \mathbf{B}(H_2) &= \mathbf{B}(H_1) + \left\{ (H_{2x} - H_{1x}) \frac{\partial \mathbf{B}}{\partial H_{2x}} + (H_{2y} - H_{1y}) \frac{\partial \mathbf{B}}{\partial H_{2y}} \right. \\ &\left. + (H_{2z} - H_{1z}) \frac{\partial \mathbf{B}}{\partial H_{2z}} \right\}_{H_2=H_1} + \dots \quad (73) \end{aligned}$$

In the similar way we can write terms for  $\mathbf{j}B_y(H_2)$  and  $\mathbf{k}B_z(H_2)$ . Introducing

$$\frac{\partial H_2}{\partial H_{2x}} = \frac{H_{2x}}{H_2}, \quad \frac{\partial H_2}{\partial H_{2y}} = \frac{H_{2y}}{H_2}, \quad \frac{\partial H_2}{\partial H_{2z}} = \frac{H_{2z}}{H_2},$$

into (73) we obtain

$$\mathbf{B}(H_2) - \mathbf{B}(H_1) = (\mathbf{H}_2 - \mathbf{H}_1) \cdot \left( \frac{\mathbf{H}_2}{H_2} \frac{d\mathbf{B}(H_2)}{dH_2} \right)_{H_2=H_1}. \quad (74)$$

Using relation (70) one can write

$$\frac{\mathbf{H}_1}{H_1} = \frac{(H_{e0} + h_{1z})(\mathbf{k} + \frac{\sqrt{h_{1x}^2 + h_{1y}^2}}{H_{e0} + h_{1z}}\mathbf{e})}{(H_{e0} + h_{1z})\sqrt{1 + \frac{h_{1x}^2 + h_{1y}^2}{(H_{e0} + h_{1z})^2}}} \cong \mathbf{k}. \quad (75)$$

Introducing relations (71), (72) and (75) into (74) one obtains

$$\mathbf{B}(H_2) - \mathbf{B}(H_1) = \mu_0 \mu_{eff}(H_{e0})(\mathbf{H}_2 - \mathbf{H}_1). \quad (76)$$

Relation (76) according to Assumption II is valid with the sufficient accuracy.

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