MODELLING OF UNCERTAINTY AND BI–VARIABLE MAPS

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The paper gives an overview and compares various bi-varilable maps from orthomodular lattices into unit interval. It focuses mainly on such bi-variable maps that may be used for constructing joint probability distributions for random variables which are not defined on the same Boolean algebra.

K e y w o r d s: orthomodular lattice, probability measure, copula, triangular norm, s-map

1 INTRODUCTION

Studies of alternative models for uncertainty (random events) are usefull. We can find many papers on these topics (*eg* [15, 24, 31]). This paper is devoted to modeling of bi-variate functions for non-compatible or for unsharp events.

In classical (Kolmogorovian) probability theory one works with a single probability space $(\Omega, \mathcal{A}, P)$, where Ω is a set of elementary events, $\mathcal{A}$ is a Boolean σ-algebra of subsets of Ω (random events), and P is a probability measure defined on $\mathcal{A}$. If X and Y are two random variables defined on $(\Omega, \mathcal{A}, P)$ then it is tacitly assumed that X and Y can be measured simultaneously, so their joint distribution function

$$F_{X,Y}(t,s) = P(X^{-1}(-\infty, t) \cap Y^{-1}(-\infty, s))$$

is well-defined both from theoretical and 'experimental' point of view. Moreover, the famous Sklar's theorem (see, *eg*, [23]) says that even if one knows only marginal distribution functions $F_X(t) = P(X^{-1}(-\infty, t))$ and $F_Y(s) = Y^{-1}(-\infty, s)$ then the joint distribution function $F_{X,Y}$ can be reconstructed.

However, there are situations in which one deals with two or more random variables defined on different probability spaces and, nevertheless, wants to construct a kind of a "joint probability distribution" of them. In such a situation one can build a horizontal sum of Boolean algebras that represent families of random events in these different probability spaces and treat this horizontal sum of Boolean algebras as a single structure that represent random events. However, a horizontal sum of Boolean algebras is not a Boolean algebra but an orthomodular lattice. This fact is a motivation for studying probability measures and random variables on orthomodular lattices. The other motivation comes from quantum physics where orthomodular lattices of closed subspaces of Hilbert spaces represent properties of quantum objects

and the existence of random variables (here usually called "observables") that cannot be measured simultaneously is a well-established fact (see, *eg*, [4, 27] and references listed therein).

Investigations presented in this paper have their origin in studies of Nánásiová and her collaborators (see [18, 19, 20] and, first of all, [1]). We address here the problem of the existence of some useful bi-variable maps introduced in these papers like s-maps and QL-copulas. We also define in this paper bi-variable maps that are generated by classical triangular norms, and study relations between these various maps.

For relevant facts concerning copulas we refer to [23]. Triangular norms were studied in numerous books and papers, *eg*, in [7, 14, 30].

2 BASIC NOTIONS

We begin with the definition and basic properties of orthomodular lattices [4, 13, 27].

DEFINITION 1. Let L be a lattice with I (the greatest element), O (the smallest element) and partial ordering $\leq$, endowed with a unary operation $\perp\colon a \mapsto a^{\perp}$ such that the following conditions are satisfied:

(i) $a^{\perp\perp} := (a^{\perp})^{\perp} = a$;

(ii) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$;

(iii) $a \vee a^{\perp} = 1$;

(iv) $a \leq b$ implies $b = a \vee (a^{\perp} \wedge b)$.

The system $\mathcal{L} = (L, O, I, \leq, \vee, \wedge, \perp)$ is said to be an *orthomodular lattice* (briefly OML).

Condition (iv) is called the *orthomodular law*. If it is replaced by the stronger distributivity law: $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$, then an OML becomes a Boolean algebra. If an OML $\mathcal{L}$ is closed under countable lattice operations, then $\mathcal{L}$ is called a σ-*orthomodular lattice* (σ-OML).

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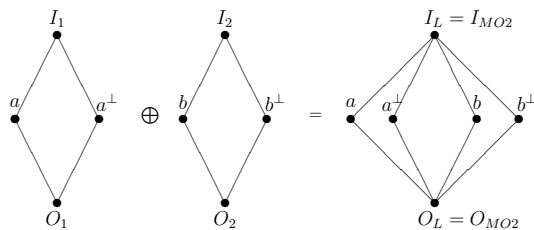
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Let $a, b \in L$. We say that a, b are *orthogonal* $(a \perp b)$, if $a \leq b^{\perp}$ and we say that a, b are *compatible* $(a \leftrightarrow b)$ if $a = (a \wedge b) \vee (a \wedge b^{\perp})$, $b = (a \wedge b) \vee (a^{\perp} \wedge b)$. Let L' be a subset of L containing O and I. If for each $a, b \in L'$ $a \leftrightarrow b$, then $\mathcal{L}'$ is a Boolean subalgebra of an OML $\mathcal{L}$, and conversely: every two elements of a Boolean subalgebra of $\mathcal{L}$ are compatible.

DEFINITION 2. Let $\mathcal{L}_i = (L_i, O_i, I_i, \leq_i, \vee_i, \wedge_i, \perp_i)$ be a family of orthomodular lattices. The set-theoretic union of all L_i in which all smallest elements O_i and all greatest elements I_i are, respectively, identified, and partial order, lattice operations, and orthocomplementation are inherited from original OMLs $\mathcal{L}_i$, is called a *horizontal sum* of the family of OMLs $\{\mathcal{L}_i\}_i$.
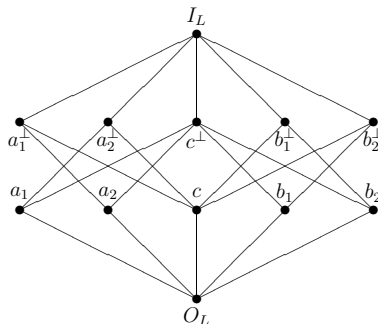
It is easy to check that a horizontal sum of OMLs is an OML as well. Therefore, a horizontal sum of Boolean algebras is an OML, although, except of the trivial cases, it is not a Boolean algebra. We show it on the following example.

EXAMPLE 1. Let $\{O_1, a, a^{\perp}, I_1\}$ and $\{O_2, b, b^{\perp}, I_2\}$ be two different 4-element Boolean algebras 2^2. Their horizontal sum is a 6-element orthomodular lattice usually denoted MO2:

One easily notes that $(a \vee a^{\perp}) \wedge b = I_{MO2} \wedge b$ while $(a \wedge b) \vee (a^{\perp} \wedge b) = O_{MO2} \vee O_{MO2} = O_{MO2}$ so MO2 is not a Boolean algebra.

EXAMPLE 2. Let $\mathcal{B}_a = \{I_L, O_L, a_1, a_2, c, a_1^{\perp}, a_2^{\perp}, c^{\perp}\}$ and $\mathcal{B}_b = \{I_L, O_L, b_1, b_2, c, b_1^{\perp}, b_2^{\perp}, c^{\perp}\}$. It is easy to see that they are Boolean algebras. However, $\mathcal{B}_a \cup \mathcal{B}_b$ is not a Boolean algebra, but it is an orthomodular lattice.

It is possible to see the following facts: $a_1 \perp a_2$ because $a_1 \leq a_2^{\perp}$, $a_1 \leftrightarrow a_2$ because $(a_1 \wedge a_2) \vee (a_1 \wedge a_2^{\perp}) = O_L \vee a_1 = a_1$ and since

$$(a_1 \wedge b_1) \vee (a_1 \wedge b_1^{\perp}) = O_L < a_1$$

we see a_1 and b_1 are not compatible.

DEFINITION 3. Let $\mathcal{L}$ be an σ-orthomodular lattice. A map $m: L \to [0, 1]$ such that
  (i) $m(I) = 1$,
  (ii) $m(\vee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$, if $a_i \perp a_j$ for all $i \neq j$, $a_i \in L$

is called a *probability measure* on $\mathcal{L}$.

Let us note that so defined map on an OML $\mathcal{L}$ is also often called a *state* on $\mathcal{L}$. If for every $a \in L$, $m(a) \in \{0, 1\}$, then m is called a *two-valued* probability measure.

DEFINITION 4. A set of probability measures M on an OML $\mathcal{L}$ is called *unital* if for every $a \in L$, $a \neq O$ there exists $m \in M$ such that $m(a) = 1$, and it is called *full* or *ordering* if the following implication holds:

$$m(a) \leq m(b) \quad \forall m \in M \quad \Rightarrow \quad a \leq b.$$

It is known, that there exist orthomodular lattices without any probability measure [12, 21] and also such that have only one probability measure [21]. If there are at least two different probability measures defined on an OML $\mathcal{L}$, then the set of probability measures defined on $\mathcal{L}$ is obviously uncountable since any convex combination of probability measures is a probability measure as well.

The reader is warned that in [1] an OML with at least one probability measure is called a *quantum logic*. However, various authors use this name to denote different objects, such as just orthomodular lattices (without mentioning probability measures at all) [32], or orthomodular posets with ordering sets of probability measures [25]. Therefore, we decided not to use this name in the present paper.

3 S-MAPS

Functions for simultaneous measurements (abbr. *s*-maps) were introduced in [18] in order to construct joint distribution also for random variables that cannot be measured simultaneously. The definition of an s-map is as follows:

DEFINITION 5. Let $\mathcal{L}$ be an OML. The map $p: L \times L \to [0, 1]$ is called an *s-map* if the following conditions hold:
  (S1) $p(I, I) = 1$,
  (S2) if $a \perp b$, then $p(a, b) = 0$,
  (S3) if $a \perp b$, then for any $c \in L$

$$p(a \vee b, c) = p(a, c) + p(b, c),$$
$$p(c, a \vee b) = p(c, a) + p(c, b).$$

An *s*-map is called *commutative* iff for any $a, b \in \mathcal{L}$ $p(a, b) = p(b, a)$.

Numerous properties of *s*-maps were studied (see, *eg*, [1, 18–20, 28]). Here we list two of these properties that will be used in the sequel.

PROPOSITION 1. *Let $\mathcal{L}$ be an OML and let p be an s-map on $\mathcal{L}$. Then the following statements are true:*

(1) *A map $m\colon L \to [0,1]$, such that $m(a) = p(a,a)$ for any $a \in L$, is a probability measure on $\mathcal{L}$.*

(2) *If $a \leftrightarrow b$, then $p(a,b) = p(a \wedge b, a \wedge b) = m(a \wedge b) = p(b,a)$.*

Application of s-maps, in particular their utilization for constructing joint probability distributions of random variables that are not defined on the same probability space, were studied in numerous papers mentioned above. In the present paper we study the problem of the very existence of s-maps, since this problem was a little bit neglected in the previous papers.

In the papers mentioned above numerous examples of s-maps defined on various OMLs were given. On the other hand the fact that any s-map p generates a probability measure by the formula $m(a) = p(a,a)$ implies that orthomodular lattices with no probability measures also do not admit any s-map to be defined on them. The impossibility of defining an s-map on some OMLs that have only one probability measure follows from the following lemma.

LEMMA 1. *Let $\mathcal{L}$ be an OML and let p be an s-map defined on $\mathcal{L}$. If $p(b,b) \neq 0$ then $m_b(\cdot)\colon L \to [0,1]$ defined by $m_b(a) = \frac{p(a,b)}{p(b,b)}$ is a probability measure on $\mathcal{L}$ and if $p(b,b) \neq 1$, then the probability measure m_b is different from the probability measure $m(\cdot) = p(\cdot,\cdot)$.*

P r o o f . Since $I \leftrightarrow b$, we have by properties of s-maps

$$m_b(I) = \frac{p(I \wedge b, I \wedge b)}{p(b,b)} = \frac{p(b,b)}{p(b,b)} = 1 \,.$$

The fact that $a \perp c \Rightarrow m_b(a \vee c) = m_b(a) + m_b(c)$ follows immediately from the definition of an s-map. Finally, we have $m_b(b) = \frac{p(b,b)}{p(b,b)} = 1$, so it differs from $m(b) = p(b,b)$ if $p(b,b) \neq 1$.

COROLLARY 1. *If $\mathcal{L}$ is an OML with only one probability measure that is not a two-valued measure, then no s-map can be defined on $\mathcal{L}$.*

On the other hand every two-valued probability measure defined on an OML $\mathcal{L}$ generates a two-valued and commutative s-map on $\mathcal{L}$. Before proving the existence of such an s-map we have to remind the notion of the support of a probability measure on an OML (see, *eg*, [4]).

DEFINITION 6. *Let m be a probability measure on an OML $\mathcal{L}$. We say that $s(m) \in L$ is the support of m if $m(a) = 0 \Leftrightarrow a \perp s(m)$, ie,*

$$\{a \in L : m(a) = 0\} = \{a \in L : a \perp s(m)\} \,.$$

It is easy to check that

$$\{a \in L\colon m(a) = 1\} = \{a \in L\colon s(m) \leq a\} \,,$$

so the support of m, if exists, is unique.

R e m a r k 1 . Since in Boolean algebras of sets orthogonality of elements is equivalent to their disjointness, it easy to see that the notion of the support of a probability measure defined on an OML is a straightforward generalization of the notion of the support of a probability measure in a classical probability space $(\Omega, \mathcal{A}, P)$, where $s(P) = \cap\{A \in \mathcal{A} : P(A) = 1\}$. However, the support of a classical probability measure always exists, while, for example, a probability measure defined on an atomic OML that takes value 1 on two distinct atoms, has no support.

PROPOSITION 2. *Let $\mathcal{L}$ be an OML and let M be a set of probability measures on $\mathcal{L}$ such that each $a \in L, a \neq O$ is the support of exactly one $m_a \in M$. Moreover, let M be such that $\forall a,b,c \in L$ such that $a,b \neq O$ the following implication holds*

$$a \perp b \ \Rightarrow m_{a \vee b}(c) = m_a(c)m_{a \vee b}(a) + m_b(c)m_{a \vee b}(b) \,. \ (1)$$

Then the function $p\colon L^2 \to [0,1]$ defined as

$$p(a,b) = \begin{cases} m_I(a)m_a(b) & \text{if } a \neq O \,, \\ 0 & \text{if } a = O \end{cases}$$

is an s-map on $\mathcal{L}$.

P r o o f . The condition (S1) of the definition of an s-map is obvious:

$$p(I,I) = m_I(I)m_I(I) = 1 \,.$$

The condition (S2) follows from the definition of the support: If $a \perp b$ and $a \neq O$, then

$$p(a,b) = m_I(a)m_a(b) = m_I(a) \cdot 0 = 0 \,.$$

To prove the first equality of (S3) we utilize the assumption (1). Since $(a \vee b) \vee (a \vee b)^\perp = I$, we get from (1)

$$m_I(a) = m_{(a \vee b) \vee (a \vee b)^\perp}(a) = m_{a \vee b}(a)m_I(a \vee b)$$
$$+ m_{(a \vee b)^\perp}(a)m_I((a \vee b)^\perp) \,.$$

But $a \leq a \vee b = ((a \vee b)^\perp)^\perp$, so $a \perp (a \vee b)^\perp$ and from the definition of the support we get $m_{(a \vee b)^\perp}(a) = 0$, which means that

$$m_I(a) = m_{a \vee b}(a)m_I(a \vee b) \,.$$

Analogously we get

$$m_I(b) = m_{a \vee b}(b)m_I(a \vee b) \,,$$

and finally

$$p(a \vee b, c) = m_I(a \vee b)m_{a \vee b}(c)$$
$$= m_I(a \vee b)[m_a(c)m_{a \vee b}(a) + m_b(c)m_{a \vee b}(b)]$$
$$= m_I(a \vee b)m_{a \vee b}(a)m_a(c) + m_I(a \vee b)m_{a \vee b}(b)m_b(c)$$
$$= m_I(a)m_a(c) + m_I(b)m_b(c)$$
$$= p(a, c) + p(b, c).$$

It is a little bit surprising that the second equality of the condition (S3) is valid regardless of the assumption (1):

If $a \perp b$, then for any $c \neq O$

$$p(c, a \vee b) = m_I(c)m_c(a \vee b) = m_I(c)[m_c(a) + m_c(b)]$$
$$= m_I(c)m_c(a) + m_I(c)m_c(b) = p(c, a) + p(c, b).$$

R e m a r k 2. Since $m(s(m)) = 1$, the assumption that each $a \in L$, $a \neq O$ is the support of a measure $m_a \in M$ obviously means that the set of probability measures M is unital.

R e m a r k 3. Let $(\Omega, \mathcal{A}, P)$ be a classical probability space and let a probability measure P be such that $\forall A \in \mathcal{A}$, $P(A) \neq 0$ iff $A \neq \emptyset$. Then the set M of conditional probabilities

$$M = \left\{ P_A \colon P_A(B) = P(B \mid A) = \frac{P(A \cap B)}{P(A)} \right\}$$

fulfills assumptions of the previous proposition: Every $A \neq \emptyset$ is the support of exactly one probability measure $P_A \in M$, and if $A \cap B = \emptyset$, then for any $C \in \mathcal{A}$

$$P_{A \cup B}(C) = P(C \mid A \cup B) = \frac{P(C \cap (A \cup B))}{P(A \cup B)}$$
$$= \frac{P((C \cap A) \cup (C \cap B))}{P(A \cup B)} = \frac{P(C \cap A)}{P(A \cup B)} + \frac{P(C \cap B)}{P(A \cup B)}$$
$$= P(C \mid A)P(A \mid A \cup B) + P(C \mid B)P(B \mid A \cup B)$$
$$= P_A(C)P_{A \cup B}(A) + P_B(C)P_{A \cup B}(B),$$

so the condition (1) is fulfilled and the bi-variable map defined on $\mathcal{A}$ by

$$p(A, B) = \begin{cases} P(A)P(B \mid A) & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset \end{cases}$$

is an s-map defined on $\mathcal{A}$.

We finish this section with a much expected fact, the proof of which is a straightforward exercise.

PROPOSITION 3. *Any convex combination of s-maps defined on an OML $\mathcal{L}$ is an s-map defined on an OML $\mathcal{L}$.*

4 QL–COPULAS

In classical probability a copula function (see [23]) allows to simulate unknown joint distribution function when only marginal distribution functions are known.

DEFINITION 7. A two-dimensional copula is a function $C \colon [0,1]^2 \to [0,1]$ with the following properties

(C1) $C(0, u) = C(u, 0) = 0$ for all $u \in [0,1]$,

(C2) $C(u, 1) = C(1, u) = u$ for all $u \in [0,1]$,

(C3) if $u_1 \leq u_2$ and $v_1 \leq v_2$, then
$C(u_1, v_1) + C(u_2, v_2) \geq C(u_1, v_2) + C(u_2, v_1)$.

The importance of copulas in classical probability follows from Sklar's theorem [23]:

THEOREM 1. *Let F be a two-dimensional distribution function with marginal distributions F_1 and F_2. Then there exists a two-dimensional copula C such that*

$$F(t, s) = C(F_1(t), F_2(s)) \text{ for every } t, s \in R. \quad (2)$$

Conversely, for any distribution functions F_1 and F_2 and for every copula C, the function F given by (2) is a two-dimensional distribution function with marginals F_1 and F_2. Furthermore, if F_1 and F_2 are continuous, then C is unique.

The notion of a QL-copula was defined in [1] as follows.

DEFINITION. Let $\mathcal{L}$ be an OML. A function $C \colon L^2 \to [0,1]$ that satisfies the conditions

(C1) $C(O, a) = C(a, O) = 0$ for each $a \in L$,

(C2) $C(I, \cdot), C(\cdot, I)$ are probability measures on $\mathcal{L}$,

(C3) if $a \leq b$ and $c \leq d$, then $C(b, d) + C(a, c) \geq C(a, d) + C(b, c)$,

is called a *QL-copula.*

If p is an s-map defined on an OML $\mathcal{L}$, then the existence of a QL-copula is guaranteed by the following proposition proved in [1].

PROPOSITION 4. *Let $\mathcal{L}$ be an OML and let p be an s-map on it. Then p is a QL-copula.*

This proposition shows that the notion of a QL-copula is more general than the notion of an s-map. We shall show in the sequel in a specific example that there exist QL-copulas that are not not s-maps, *ie*, the set of all s-maps defined on an OML $\mathcal{L}$ is usually strictly included in the set of all QL-copulas defined on $\mathcal{L}$. The problem of finding general conditions under which a QL-copula would be an s-map is, for the time being, not solved.

Although Proposition 4 assures the existence of a QL-copula on an OML $\mathcal{L}$ when there exists an s-map on $\mathcal{L}$, one can construct a lot of QL-copulas on $\mathcal{L}$ in a straightforward way, without the neccesity of using s-maps.

PROPOSITION 5. *Let m_1, m_2 be two, not necessarily different, probability measures on an OML $\mathcal{L}$ and let $C\colon [0,1]^2 \to [0,1]$ be a two-dimensional copula. Then a function $C_{m_1,m_2}\colon L^2 \to [0,1]$ defined for any $a,b \in L$ by*

$$C_{m_1,m_2}(a,b) = C[m_1(a), m_2(b)]$$

is a QL-copula defined on $\mathcal{L}$.

The proof follows immediately from definitions of two-dimensional copulas and QL-copulas, and from properties of probability measures defined on orthomodular lattices.

R e m a r k 3. It was noticed in [1] that if m_1, m_2 are two probability measures on an OML, then $C(a,b) = m_1(a)m_2(b)$ is a QL-copula. Actually, this follows from the previous proposition since $C(x,y) = xy$ is a two-dimensional copula (see, *eg*, [23]).

EXAMPLE 3. Let $\mathcal{L}$ be an orthomodular lattice MO2 described in Example 1 and let m_1 and m_2 be two probability measures defined by the following table.

$x =$	a	$a^\perp$	b	$b^\perp$
$m_1(x)$	0.2	0.8	0.3	0.7
$m_2(x)$	0.4	0.6	0.5	0.5

Since bi-variable maps $T_0(x,y) = \min(x,y)$, $T_1(x,y) = xy$, and $T_\infty(x,y) = \max(x+y-1,0)$ are two-dimensional copulas (see, *eg*, [23]), they generate QL-copulas on MO2 by

$$C_{m_1,m_2}(a,b) = T_i[m_1(a), m_2(b)] \quad i = 0,1,\infty,$$

neither of these maps is an s-map on MO2 since it is easy to check that for the QL-copula generated by T_0 and $T_{/infty}$ the conditions (S2) and (S3) of Definition 5 are not fulfilled, for the QL-copula generated by T_1 the condition (S2) is not fulfilled.

Note that it is easy to check that the conditions (S2) and (S3) are not fulfilled both for OL-copulas generated by T_0 and by $T_{/infty}$ t-norms.

We finish this section with a fact, the proof of which is a simple exercise:

PROPOSITION 6. *Any convex combination of QL-copulas is a QL-copula.*

5 BI–VARIABLE MAPS ON OMLS GENERATED BY TRIANGULAR NORMS

Triangular norms, defined by K. Menger already in 1942 [16], are another maps from $[0,1]^2$ into $[0,1]$ that have been, and still are, intensively studied and utilized in various branches of mathematics, like probabilistic metric spaces [33,30], fuzzy sets [7,17], multivalued logic [29], mathematical statistics [9], capacity theory [10], or game theory [2,3,5,6], to mention a few.

DEFINITION 9. A *triangular norm* (abbr. *t-norm*) is a function $T\colon [0,1]^2 \to [0,1]$ which is commutative, associative, nondecreasing in each argument, and satisfies the boundary condition: $T(x,1) = x$.

The most important family of t-norms is the family of *fundamental t-norms* $\{T_s\}_{s\in[0,\infty]}$ obtained by M. D. Frank [10] as continuous solutions of the functional equation $F(x,y) + 1 - F(1-x, 1-y) = x + y$. This family is

$$T_s(x,y) = \begin{cases} \min(x,y) & \text{if } s = 0, \\ xy & \text{if } s = 1, \\ \max(x+y-1,0) & \text{if } s = \infty \\ \log_s\big[1 + \frac{(s^x-1)(s^y-1)}{s-1}\big] & \text{otherwise.} \end{cases}$$

This family is "continuous" in the sense that $\lim_{s\to t} T_s = T_t$, and t-norms T_0, T_1, and T_∞ encountered already in the previous section are distingushed also because they generate on fuzzy sets operations known as Zadeh, probabilistic, and Łukasiewicz intersection, respectively (see, *eg*, [7,14,17]).

Triangular norms and two-dimensional copulas are both functions from $[0,1]^2$ into $[0,1]$, but neither of these families of functions contains, nor is contained in the other. However, commutative and associative two-dimensional copulas are t-norms, and t-norms that satisfy the 1-Lipschitz condition, *ie*, $|T(w,x) - T(y,z)| \le |w - y| + |x - z|$ for all $w,x,y,z \in [0,1]$ are two-dimensional copulas [22]. In particular t-norms T_0, T_1, and T_∞ are two-dimensional copulas.

It occures [23] that t-norms T_0 and T_∞ are, respectively, the greatest and the least of all two-dimensional copulas since for all $x,y \in [0,1]$

$$T_\infty(x,y) = \max(x+y-1,0) \le C(x,y) \le$$
$$\min(x,y) = T_0(x,y).$$

Let us note that for a bi-variable function that maps an OML into $[0,1]$ the notion of associativity makes no sense. Therefore, in order to define bi-variable maps on OMLs that would be "functional counterparts" of classic triangular norms, we have to omit associativity. We adopt the following definition.

DEFINITION 10. Let $\mathcal{L}$ be an OML. A function $T\colon L^2 \to [0,1]$ that is commutative, nondecreasing in each argument, and satisfies the boundary condition

$$T(\cdot, I) \quad \text{is a probability measure on } \mathcal{L},$$

is called a *T-map*.

In the previous section we showed that if we have two (not necessarily different) probability measures on an OML, then any two-dimensional copula generates a QL-copula. The analogous proposition concerning t-norms and T-maps is as follows.

PROPOSITION 7. *Let m be a probability measure on an OML $\mathcal{L}$ and let $T : [0,1]^2 \to [0,1]$ be a triangular norm. Then a function $T_m : L^2 \to [0,1]$ defined for any $a, b \in L$ by*

$$T_m(a,b) = T[m(a), m(b)]$$

is a T-map defined on $\mathcal{L}$.

The proof, as in the case of QL-copulas, follows immmediately from definitions of a t-norm, T-map, and properties of probability measures defined on OMLs.

R e m a r k 4 . Since there is no counterpart of the notion of associativity in the definition of a T-map, it is obvious that any map $T : [0,1]^2 \to [0,1]$ that is commutative but not necessarily associative, nondecreasing in both arguments, and is such that $T(x,1) = x$, generates a T-map on an OML in a way described in the previous proposition.

R e m a r k 5 . A T-map cannot in general be constructed like a QL-copula was constructed in Proposition 5, *ie*, with the use of two different probability measures, because of the required commutativity. Although by commutativity of any t-norm we have $T[m_1(a), m_2(b)] = T[m_2(b), m_1(a)]$, this is in general different from $T[m_1(b), m_2(a)]$. In a full analogy to the relation between classical two-dimensional copulas and t-norms we have the following lemma.

LEMMA 2. *Any commutative QL-copula is a T-map.*

P r o o f . Commutativity and the boundary condition are obvious. In order to prove non-decreasingness it is enough to put $c = O$ in the condition (C3) of the definition of a QL-copula, which then takes the form

if $a \le b$ and $O \le d$, then

$$C(a,d) + C(b,O) \le C(b,d) + C(a,O),$$

and note that by (C1), $C(b,O) = C(a,O) = 0$.

Since by Proposition 4 any commutative s-map is a commutative QL-copula, we obtain the following corollary

COROLLARY 2. *Any commutative s-map on an OML is a T-map.*

The proof of the following much expected fact that is a T-map counterpart of Propositions 3 and 6, is a straightforward exercise.

PROPOSITION 8. *Any convex combination of T-maps is a T-map.*

Before closing this section let us note that there are well-known generalizations of classical triangular norms and conorms to bounded posets (*ie*, partially ordered sets with the least and the greatest elements) in the sense of *two-argument operations* on posets, not in the sense of *bi-variable functions* from a poset into the unit interval (*eg*, [8,34]). According to this approach a triangular norm T on a bounded poset $\mathcal{L} = \{L, \le, O, I\}$ is a two-argument operation on $\mathcal{L}$, *ie*, a map $T : \mathcal{L}^2 \to \mathcal{L}$ that is commutative, associative, nondecreasing, and satisfies the boundary condition $T(a,I) = a$, *ie*, it satisfies all properties of a t-norm defined on $[0,1]$. Since any OML is a bounded poset, this approach is applicable to OMLs as well, but in this paper we are interested in bi-variable maps from OMLs into $[0,1]$, not in operations on OMLs. Nevertheless, the following proposition can be easily proved.

PROPOSITION 9. *Let $\mathcal{L}$ be an OML. Then every probability measure m on $\mathcal{L}$ and every triangular norm $T : L^2 \to L$ defined on $\mathcal{L}$ generate a T-map $T_m : L^2 \to [0,1]$ by*

$$T_m(a,b) = m(T(a,b)).$$

6 TRANGULAR NORMS AND COPULAS IN FUZZY SET REPRESENTATIONS OF ORTHOMODULAR LATTICES

In order to simplify the notation we adopt in the present section the following conventions

- The t-norm T_∞ will be denoted $\sqcap$, *ie*, $\forall x, y \in [0,1]$, $x \sqcap y = \max(x + y - 1, 0) = T_\infty(x,y)$, and its dual t-conorm $S_\infty(x,y) = 1 - T_\infty(1-x, 1-y)$ will be denoted $\sqcup$, *ie*, $x \sqcup y = \min(x + y, 1) = S_\infty(x,y)$.

- The same symbols will be used to denote operations on fuzzy sets pointwisely defined by this t-norm and t-conorm, and usually called Łukasiewicz intersection and union:

$$(A \sqcap B)(x) = A(x) \sqcap B(x) = \max[A(x) + B(x) - 1, 0],$$

$$(A \sqcup B)(x) = A(x) \sqcup B(x) = \min[A(x) + B(x), 1].$$

- The standard fuzzy complement of a set A will be denoted A', *ie*, $A'(x) = 1 - A(x)$.

In [25] the following representation theorem was proved:

THEOREM 2. *Every orthomodular poset $\mathcal{L}$ with an ordering set of probability measures M can be isomorphically represented by a family Λ of fuzzy subsets of M endowed with the inclusion of fuzzy sets as partial order, the standard fuzzy set complementation as orthocomplementation, and such that*

(i) *$\emptyset \in \Lambda$,*

(ii) *$A \in \Lambda \Rightarrow A' \in \Lambda$,*

(iii) *if $\{A_i\} \subset \Lambda$ is at most countable sequence such that $A_i \sqcap A_j = \emptyset$ for $i \ne j$ (such fuzzy sets were called weakly disjoint by R. Giles in [11]), then $\sqcup_i A_i \in \Lambda$,*

(iv) *if $A \in \Lambda$ satisfies $A \sqcap A = \emptyset$, then $A = \emptyset$.*

The isomorphism is defined by: $a \mapsto A \Leftrightarrow A(m) = m(a)$.

Since any orthomodular lattice is an orthomodular poset in which operations $\wedge$ and $\vee$ are defined for all pairs of elements, Theorem 2 is obviously valid for any orthomodular lattice $\mathcal{L}$ with an ordering set of probability measures M.

R e m a r k . Orthogonality of two elements $a, b \in \mathcal{L}$ defined by $a \perp b \Leftrightarrow a \le b^{\perp}$ means that for any probability measure m one has $m(a) \le m(b^{\perp}) = 1 - m(b) \Leftrightarrow m(a) + m(b) \le 1 \Leftrightarrow m(a) \sqcap m(b) = 0$. Therefore, orthogonality of two elements of $\mathcal{L}$ is equivalent to weak disjointness of their fuzzy set representations.

Conditions (i) – (iv) of Theorem 2 can be linguistically expressed as follows.

(i) The empty set belongs to Λ,

(ii) Λ is closed with respect to the standard fuzzy set complementastion,

(iii) Λ is closed with respect to Łukasiewicz unions of sequences of pairwisely weakly disjoint sets,

(iv) the empty set is the only set in Λ that is weakly disjoint with itself.

Theorem 2 distinguishes Łukasiewicz operations from all other t-norm generated operations on fuzzy sets. Indeed, it can be easily checked that for any other t-norm T_s that belongs to the family of Frank's fundamental t-norms, and for any $x \in (0, 1)$ one has $T_s(x, 1 - x) \neq 0$ which means that a T_s-generated intersection of any genuine fuzzy (*ie*, non-crisp) set with its complement is nonempty, and which makes T_s useless for building fuzzy set representations of orthomodular posets or lattices.

Although Łukasiewicz intersection and union are in fuzzy set representation of an OML only partially defined (it was proved in [26] that if they are globally defined, then Λ in which all conditions (i) – (iv) are satisfied is necessarily a Boolean algebra), the fact that in the case of such representation one practically deals with numbers from the interval $[0, 1]$ allows now to apply to Λ pointwisely any "classical" two-dimensional copula or a triangular norm. It is an interesting question what structures can Λ generate in this way, of course if one deletes the very restrictive requirement that the empty set is the only set in the structure that is weakly disjoint with itself.

In the case of t-norms the answer is known: the obtained structure is a *t-tribe*, *ie*, a family of fuzzy sets that contains the empty set and is closed with respect to the standard fuzzy set complementation and countable intersections generated by a t-norm. Such families of fuzzy sets are basic structures in fuzzy probability theory and they were studied in numerous papers (see, *eg*, [7, 17]). The analogous structures based on the notion of a two-dimensional copula instead of a t-norm were not studied yet and the question remains open.

7 SUMMARY

In the paper we studied various bi-variable functions that map an OML into the unit interval $[0, 1]$ and are generalizations of bi-variable maps defined on $[0, 1]$ that are extensively used in classical probability. We showed the hierarchy of notions: each s-map is a QL-copula and each commutative s-map is a commutative QL-copula and a T-map. The ways in which these functions can be constructed from given probability measures on an

OML were shown. These constructions can be useful for constructing joint distributions of random variables that are not simultaneously measurable.

References

[1] AL-ADILEE, A. M.—NÁNÁSIOVÁ, O. : Copula and s-Map on a Quantum Logic, Information Sciences **179** (2009), 4199–4207.

[2] AUBIN, J. P. : Mathematical Methods of Games and Economic Theory, North-Holland, Amsterdam, 1979.

[3] AUMAN, R. J.—SHAPLEY, L. S. : Values of Non-Atomic Games, Princeton University Press, Princeton, 1974.

[4] BELTRAMETTI, E. G.—CASSINELLI, G. : The Logic of Quantum Mechanics, Addison-Wesley, Reading MA, 1981.

[5] BUTNARIU, D. : Non-Atomic Fuzzy Measures and Games, Fuzzy Sets and Systems **17** No. 1985, 39–52.

[6] BUTNARIU, D. : Values and Cores for Fuzzy Games with Infinitely Many Players, International Journal of Game Theory **16** (1987), 43–68.

[7] BUTNARIU, D.—KLEMENT, E. P. : Triangular Norm-Based Measures and their Markov Kernel Representation, Journal of Mathematical Analysis and Applications **162** (1991), 111–143.

[8] De BAETS, B.—MESIAR, M. : Triangular Norms on Product Lattices, Fuzzy Sets and Systems **104** (1999), 61–75.

[9] DVORETZKY, A.—WALD, A.—WOLFOWITZ, J. : Relations Among Certain Ranges of Vector Measures, Pacific Journal of Mathematics **1** (1951), 59–74.

[10] FRANK, M. D. : On the Simultaneous Associativity of $F(x, y)$ and $x + y - F(x, y)$, Aequationes Mathematicae **19** (1979), 194–226.

[11] GILES, R. : Łukasiewicz Logic and Fuzzy Set Theory, International Journal of Man-Machine Studies **67** (1976), 313–327.

[12] GREECHIE, R. J. : Orthomodular Lattices Admitting no States, Journal of Combinatorial Theory, Ser. A **10** (1971), 119–132.

[13] KALMBACH, G. : Orthomodular Lattices, Academic Press, London, 1983.

[14] KLEMENT, E. P.—MESIAR, R.—PAP, E. : Triangular Norms, Kluwer, Dordrecht, 2000.

[15] MARTINEK, R.—KELNAR, M.—VANUS, J. : A Robust Approach for Acoustic Noise Suppression in Speech using ANFIS, Journal of Electrical Engineering **66** (2015), 301–310.

[16] MENGER, K. : Statistical Metrics, Proceedings of the National Academy of Sciences of the USA **28** (1942), 353–537.

[17] MESIAR, R. : Fundamental Triangular Norm Based Tribes and Measures, Journal of Mathematical Analysis and Applications **177** (1993), 633–640.

[18] NÁNÁSIOVÁ, O. : Map for Simultaneous Measurements for a Quantum Logic, International Journal of Theoretical Physics **42** (2003), 1889–1903.

[19] NÁNÁSIOVÁ, O.—KHRENNIKOV, A. : Representation Theorem for Observables on a Quantum System, International Journal of Theoretical Physics **45** (2006), 469–482.

[20] NÁNÁSIOVÁ, O.—VALÁŠKOVÁ, Ľ.: Maps on a Quantum Logic, Soft Computing **14** (2010), 1047–1052.

[21] NAVARA, M.: Small Quantum Structures with Small State Spaces, International Journal of Theoretical Physics **47** (2008), 36–43.

[22] NAVARA, M.: Triangular Norms and Conorms, www.scholarpedia.org/article/Triangular_norms_and_conorms .

[23] NELSEN, R. B.: An Introduction to Copulas, Springer, New York, 1999.

[24] PETROVIĆ, I.—JOZSA, L.—BAUS, Z.: Use of Fuzzy Logic System for Assessment of Primary Faults, Journal of Electrical Engineering **66** (2015), 257–263.

[25] PYKACZ, J.: Fuzzy Quantum Logics and Infinite-Valued Łukasiewicz Logic, International Journal of Theoretical Physics **33** (1994), 1403–1416.

[26] PYKACZ,.: Łukasiewicz Operations in Fuzzy Set and Many-Valued Representations of Quantum Logics, Foundations of Physics **30** (2000), 1503–1524.

[27] PTÁK, P.—PULMANNOVÁ, S.: Orthomodular Structures as Quantum Logics, Kluwer, Dordrecht, 1991.

[28] PYKACZ, J.—VALÁŠKOVÁ, Ľ.—NÁNÁSIOVÁ, O.: Bell-Type Inequalities for Bivariate Maps on Orthomodular Lattices, Foundations of Physics **45** (2014), 900–913.

[29] ROSE, A.—ROSSER, J. B.: Fragments of Many Valued Statement Calculus, Transactions of the American Mathematical Society **87** (1958), 1–53.

[30] SCHWEIZER, B.—SKLAR, A.: Probabilistic Metric Spaces, North-Holland, New York, 1983.

[31] SOCHA, V.—KUTÍLEK, P.—VITEČKOVÁ, S.: The Evaluation of the Practical Adhesion Strenght of Biocompatible Thin Films by Fuzzy Logic Expert Sysstem and International Standards, Journal of Electrical Engineering **64** (2013), 354–360.

[32] TKADLEC, J.: Subadditivity of States on Quantum Logics, International Journal of Theoretical Physics **34** (1995), 1767–1774.

[33] WALD, A.: On Statistical Generalizations of Metric Spaces, Proceedings of the National Academy of Sciences of the USA **29** (1943), 196–197.

[34] ZHANG, D.: Triangular Norms on Partially Ordered Sets, Fuzzy Sets and Systems **153** (2005), 195–209.

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