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A note on "A note on the magnetic vector potential"

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In the recently published short paper author deals with the derivation of the scalar potential pertaining to the point charge as well as of the vector potential pertaining to the point current. He shows his alternative approach and compares it to the "traditional" methods commonly used in textbooks. Here we want to show that use of the generalised functions (symbolic functions, distributions) in the domain of electromagnetic field theory provides more straightforward and more rigorous approach to the problem.

K e y w o r d s: magnetic vector potential, Helmholtz equation

1 Introduction

The problem stated in [1], with a slight generalisation, is to solve:

- The Poisson equation for scalar and vector potential $V(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ of time-constant field

$$\nabla^2 V(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon, \quad \nabla^2 \mathbf{A}(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r}), \tag{1}$$

where $\rho(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$ are the charge and current densities and ∇^2 is the Laplace operator.

- The Helmholtz equation for the complex representation of the time-harmonic scalar and vector potential

$$\begin{aligned} \nabla^2 \tilde{V}(\mathbf{r}) + k_0^2 \tilde{V}(\mathbf{r}) &= \tilde{\rho}(\mathbf{r})/\varepsilon, \\ \nabla^2 \tilde{\mathbf{A}}(\mathbf{r}) + k_0^2 \tilde{\mathbf{A}}(\mathbf{r}) &= -\mu \tilde{\mathbf{J}}(\mathbf{r}), \end{aligned} \tag{2}$$

where for all complex representattions $\tilde{V}$, $\tilde{\mathbf{A}}$, $\tilde{\rho}$, and $\tilde{\mathbf{J}}$ generally holds $V(\mathbf{r},t) = \mathrm{Re}\left\{\tilde{V}(\mathbf{r})\exp(j\omega_0 t)\right\}$, and where $k_0 = \omega_0/c,\ c = 1/\sqrt{\mu\varepsilon}$,

- The wave equation for arbitrary time-dependance of potentials $V(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$

$$\begin{aligned} \nabla^2 V(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 V(\mathbf{r},t)}{\partial t^2} &= -\rho(\mathbf{r},t)/\varepsilon, \\ \nabla^2 \mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} &= -\mu \mathbf{J}(\mathbf{r},t). \end{aligned} \tag{3}$$

2 Convolutional formulation

Solution of (1) and (2) can be formulated as the three-dimensional convolution integrals over the infinite space,

$$\begin{aligned} V(\mathbf{R}) &= \frac{1}{\varepsilon}\iiint_\infty \rho(\mathbf{r}) g(\mathbf{R}-\mathbf{r})\mathrm{d}v, \\ \mathbf{A}(\mathbf{R}) &= \mu \iiint_\infty \mathbf{J}(\mathbf{r}) g(\mathbf{R}-\mathbf{r})\mathrm{d}v. \end{aligned} \tag{4}$$

where $g(\mathbf{r})$, is called the Green function for the Poisson equation in the unbounded infinite space. The Green function is given as the solution of

$$\nabla^2 g(\mathbf{r}) = -\delta(\mathbf{r}), \tag{5}$$

where $\delta(\mathbf{r})$ is the three-dimensional Dirac delta-function belonging to the class of so called generalised functions (symbolic functions, distributions). In Cartesian coordinates it equals $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$. The main property of the δ -function is its sifting property

$$f(\mathbf{R}) = \iiint_\infty f(\mathbf{r})\delta(\mathbf{R}-\mathbf{r})\mathrm{d}v,$$

ie the convolution of the function with the delta-function leads to the same function. One can easily see that applying the Laplace operator ∇_R^2 in variable $\mathbf{R}$ to (4)

$$\begin{aligned} \nabla_R^2 V(\mathbf{R}) &= \frac{1}{\varepsilon}\iiint_\infty \rho(\mathbf{r})\nabla_R^2 g(\mathbf{R}-\mathbf{r})\mathrm{d}v = \\ &= -\frac{1}{\varepsilon}\iiint_\infty \rho(\mathbf{r})\delta(\mathbf{R}-\mathbf{r})\mathrm{d}v, \end{aligned}$$

indeed yields the result $\nabla_R^2 V(\mathbf{R}) = -\rho(\mathbf{R})/\varepsilon$ *ie* (1) in variable $\mathbf{R}$.

Analogously holds for (2)

$$\begin{aligned} \tilde{V}(\mathbf{R}) &= \frac{1}{\varepsilon}\iiint_\infty \tilde{\rho}(\mathbf{r})\tilde{g}(\mathbf{R}-\mathbf{r})\mathrm{d}v, \\ \tilde{\mathbf{A}}(\mathbf{R}) &= \mu \iiint_\infty \tilde{\mathbf{J}}(\mathbf{r})\tilde{g}(\mathbf{R}-\mathbf{r})\mathrm{d}v, \end{aligned} \tag{6}$$

where $\tilde{g}(\mathbf{r})$ is given as the solution of

$$\nabla^2 \tilde{g}(\mathbf{r}) + k_0^2 \tilde{g}(\mathbf{r}) = -\delta(\mathbf{r}). \tag{7}$$

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The solution of (3) can be expressed as a four-dimensional space-time convolution

$$\tilde{V}(\mathbf{R},t)=\frac{1}{\varepsilon}\iiint\limits_{\infty}\left\{\int\limits_{-\infty}^{\infty}\rho(\mathbf{r},\tau)G(\mathbf{R}-\mathbf{r},t-\tau)\mathrm{d}\tau\right\}\mathrm{d}v,$$
$$\mathbf{A}(\mathbf{R},t)=\mu\iiint\limits_{\infty}\left\{\int\limits_{-\infty}^{\infty}\mathbf{J}(\mathbf{r},\tau)G(\mathbf{R}-\mathbf{r},t-\tau)\mathrm{d}\tau\right\}\mathrm{d}v, \tag{8}$$

where the Green function $G(\mathbf{r},t)$ is given as the solution of

$$\nabla^2 G(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2 G(\mathbf{r},t)}{\partial t^2}=-\delta(\mathbf{r})\delta(t). \tag{9}$$

3 Solution of equations for Green functions

As shown in [2] and [3] the three-dimensional spatial direct and inverse Fourier transform is introduced by

$$g(\varkappa_x,\varkappa_y,\varkappa_z)=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}g\left(x,y,z\right)\times,$$
$$\times\exp\left\{j\left(\varkappa_x x+\varkappa_y y+\varkappa_z z\right)\right\}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$g\left(x,y,z\right)=\frac{1}{8\pi^3}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}g\left(\varkappa_x,\varkappa_y,\varkappa_z\right)\times$$
$$\times\exp\left\{-j\left(\varkappa_x x+\varkappa_y y+\varkappa_z z\right)\right\}\mathrm{d}\varkappa_x\mathrm{d}\varkappa_y\mathrm{d}\varkappa_z$$

The solution of (5), (7) and (9) in Fourier domain reads

$$g(\varkappa_x,\varkappa_y,\varkappa_z)=1/\varkappa^2,\quad \tilde{g}(\varkappa_x,\varkappa_y,\varkappa_z)=1/(\varkappa^2-k_0^2),$$
$$G(\varkappa_x,\varkappa_y,\varkappa_z,\omega)=1/(\varkappa^2-\omega^2/c^2),$$
$$\varkappa=\sqrt{\varkappa_x^2+\varkappa_y^2+\varkappa_z^2}.$$

As shown in [2] and [3] the inverse Fourier transforms of the last formulae yield the Green functions

$$g(\mathbf{r})=\frac{1}{4\pi r},\quad \tilde{g}(\mathbf{r})=\frac{\exp(-jk_0 r)}{4\pi r},$$
$$G(\mathbf{r},t)=\frac{\delta(t-r/c)}{4\pi r},\ r=\sqrt{x^2+y^2+z^2}.$$

The convolution integrals (4), (6) and (8) for *eg*; vector potential then read

$$\mathbf{A}(\mathbf{R})=\frac{\mu}{4\pi}\iiint\limits_{\infty}\frac{\mathbf{J}(\mathbf{r})}{|\mathbf{R}-\mathbf{r}|}\mathrm{d}v, \tag{10}$$

$$\tilde{\mathbf{A}}(\mathbf{R})=\frac{\mu}{4\pi}\iiint\limits_{\infty}\frac{\tilde{\mathbf{J}}(\mathbf{r})}{|\mathbf{R}-\mathbf{r}|}\exp\left(-jk_0|\mathbf{R}-\mathbf{r}|\right)\mathrm{d}v, \tag{11}$$

$$\mathbf{A}(\mathbf{R},t)=\frac{\mu}{4\pi}\iiint\limits_{\infty}\frac{\mathbf{J}(\mathbf{r},t-|\mathbf{R}-\mathbf{r}|/c)}{|\mathbf{R}-\mathbf{r}|}\mathrm{d}v, \tag{12}$$

the last integral being well known as the retarded potential.

The point quantities as the point charge Q and the infinitesimal point current element $I\mathrm{d}\mathbf{r}$, placed in the origin are represented by the charge and current densities $\rho(\mathbf{r})=Q\delta(\mathbf{r})$, $\mathrm{d}\mathbf{J}(\mathbf{r})=I\mathrm{d}\mathbf{r}\delta(\mathbf{r})$. Then one simply obtains well known formulas

$$V(r)=\frac{1}{4\pi\varepsilon}\frac{Q}{r},\quad \mathrm{d}\mathbf{A}(r)=\frac{\mu}{4\pi}\frac{I}{r}\mathrm{d}\mathbf{r},$$
$$\tilde{V}(r)=\frac{1}{4\pi\varepsilon}\frac{Q}{r}\exp(-jk_0 r),$$
$$\mathrm{d}\tilde{\mathbf{A}}(r)=\frac{\mu}{4\pi}\frac{\tilde{I}}{r}\exp(-jk_0 r)\mathrm{d}\mathbf{r},$$
$$V(r,t)=\frac{1}{4\pi\varepsilon}\frac{Q(t-r/c)}{r},$$
$$\mathrm{d}\mathbf{A}(r,t)=\frac{\mu}{4\pi}\frac{I(t-r/c)}{r}\mathrm{d}\mathbf{r}.$$

4 Conclusion

The author in [1] in fact first solves the homogeneous equation (2) $\nabla^2\tilde{V}(\mathbf{r})+k_0^2\tilde{V}(\mathbf{r})=0$ in spherical coordinates for centro-symmetrical function, *ie* the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\right\}\tilde{V}(r)+k_0\tilde{V}(r)=0,$$

with the result

$$\tilde{V}(\mathbf{r})=C\frac{\exp(-jk_0 r)}{r}.$$

Then he looks for the value of the constant C by volume integrating (2) within the sphere with the center in origin, subsequently limiting the sphere radius to zero. We believe that the method presented here is more rigorous, straightforward and up-to-date.

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