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COMMUNICATIONS

CONTROL OF UNCERTAIN SYSTEMS

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The paper deals with the frequency domain robust PID controller design for linear continuous-time systems. The uncertain system is given by a finite number of transfer functions. The design method evolves from the frequency domain robust stability criteria for unstructured uncertainty, and from the Small Gain Theorem based stability condition. The proposed design algorithm is illustrated on an example.

K e y w o r d s: structured and unstructured uncertainty, additive uncertainty, robust stability

1 INTRODUCTION

The dynamic system model uncertainty is usually considered as the difference between real performance and nominal performance. The deviation (uncertainty) can arise both in static and dynamic system qualities. We assume that the uncertainty bounds can be determined through estimate, measurement or computation. Mathematically, the uncertainties can be described by variations of system coefficients within certain intervals around the coefficients of the nominal model (structured uncertainty). Another mathematical representation of uncertainty is by a block of dynamic system set with bounded amplitude frequency response connected in series with a weighting transfer function. This weighting function represents the maximal dispersion of uncertain system frequency responses around the nominal system frequency response (unstructured uncertainty). Additive or multiplicative uncertainties can be considered, where the uncertainties are in parallel or in series with the nominal model, respectively. We assume that in all cases the uncertainties are bounded. For example, the lower and upper limits of uncertain coefficient intervals can be known, or the norms of the considered uncertain system variables are bounded.

Our aim is to design such a PID controller for an uncertain linear SISO system which would guarantee the closed loop system stability for the whole uncertainty region. In this case we refer to the robust control loop, robust stability and respective robust PID controller.

There are many approaches to the robust stability analysis, *eg*, the Kharitonov and the Bialas Theorems for structured uncertainties, or the frequency domain robust stability criteria based on weighting matrices, the Cypkin-Polyak Theorem, *etc*.

The robust controller synthesis approaches include, *eg*, the Small Gain Theorem, robust pole placement, F_2 and H_∞ norms based approaches, structured singular value μ with M-Δ structure *etc*.

This paper deals with a frequency domain robust PID controller design method for uncertain systems described by a family of transfer functions or Nyquist plots; such a family can describe, *eg*, a nonlinear system in its different operating points. An analytical and easy-to-use approach to the robust PI/PID controller design is proposed.

2 ROBUST STABILITY CONDITIONS

Consider an uncertain continuous-time SISO system $G(s)$ given by a family of transfer functions, and a controller $R(s)$, hence the loop transfer function is $L(s) = R(s)G(s)$. For uncertain nonlinear SISO the *small gain theorem* applies; according to it, the closed loop is robustly stable if the open-loop magnitude satisfies

$$|L(j\omega)| < 1\,, \quad \forall \omega \in [0, \infty]\,. \tag{1}$$

If the above inequality is satisfied, then the deviations of all signals in the closed-loop vanish in time and the system is closed-loop stable.

Suppose that the uncertainty is specified by a set of stable transfer functions

$$G_k(s) = \frac{B_k(s)}{A_k(s)}\,, \quad k = 1, 2, \dots, n \tag{2}$$

where a_{ki}, $i = 0, 1, \dots, n$ and b_{kj}, $j = 0, 1, \dots, m$ are coefficients of polynomials A_k and B_k, respectively. The nominal model is specified using the mean values of the respective coefficients

$$G_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{\sum_{j=0}^{m} \frac{1}{N} \sum_{k=1}^{N} [b_{kj}] s^j}{\sum_{i=0}^{n} \frac{1}{N} \sum_{k=1}^{N} [a_{ki}] s^i}\,. \tag{3}$$

The transfer function of the nominal model is also supposed to be stable but with constant coefficients. The

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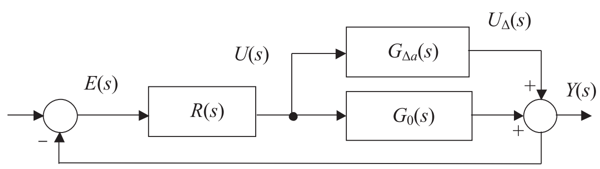

Fig. 1. Closed loop with additive uncertainty

variance of Nyquist plots of the family of transfer functions around the nominal model $G_0(s)$ is given in the form of an additive unstructured uncertainty

$$G_{\Delta a}(s) = w_a(s)\Delta_a(s)\,, \tag{4}$$

where $\Delta a(s)$ represents a class of an infinite number of transfer functions with the maximum magnitude satisfying $|\Delta a(j\omega)| \le 1\ \forall\omega$; $w_a(s)$ is the weighting transfer function. Hence the uncertain system is given using additive uncertainty

$$G(s) = G_0(s) + G_{\Delta a}(s) = G_0(s) + w_a(s)\Delta_a(s)\,. \tag{5}$$

The closed loop comprising the uncertain system (5) and controller $R(s)$ is depicted in Fig. 1. Let Γ denote the family of transfer functions $G(s)$. The weighting function is specified by the maximum magnitudes found for different ω as follows

$$\ell_a(\omega_k) = \max_{G(j\omega)\in\Gamma} |G(j\omega_k) - G_0(j\omega_k)|\,. \tag{6}$$

The weighting function $w_a(s)$ is chosen so as to satisfy the following condition

$$|w_a(j\omega)| \ge \ell_a(\omega) \quad \forall\omega\in[0,\infty]\,. \tag{7}$$

If $|\Delta_a(j\omega)| \le 1\ \forall\omega$, then also

$$|w_a(j\omega)|\,|\Delta_a(j\omega)| \le |w_a(j\omega)| \quad \forall\omega\in[0,\infty]\,. \tag{8}$$

By choosing the weight $w_a(j\omega)$ according to (7) we have specified the transfer function of the additive dynamics $G_{\Delta a}(s)$ as well.

Suppose that the nominal closed loop is stable. The closed-loop characteristic equation with the uncertain system is

$$1 + R(s)G(s) = 0\,. \tag{9}$$

Substituting (5) into (9) we obtain

$$1 + R(s)[G_0(s) + G_{\Delta a}(s)] = \\ [1 + R(s)G_0(s)]\Big[1 + V_0(s)\frac{G_{\delta a}(s)}{G_0(s)}\Big] = 0\,, \tag{10}$$

where

$$V_0(s) = \frac{R(s)G_0(s)}{1 + R(s)G_0(s)}\,. \tag{11}$$

Actually, (10) is a product of two characteristic equations. The necessary and sufficient robust stability condition for the closed-loop in Fig. 1 is that the nominal characteristic equation (11)

$$1 + R(s)G_0(s) = 0 \tag{12}$$

has to be stable and simultaneously the other characteristic equation in (10)

$$1 + V_0(s)\frac{G_{\Delta a}(s)}{G_0(s)} = 0 \tag{13}$$

has to satisfy the small gain theorem magnitude condition

$$\Big|V_0(j\omega)\frac{G_{\Delta a}(j\omega)}{G_0(j\omega)}\Big| < 1 \quad \forall\omega\in[0,\infty]\,. \tag{14}$$

Consider that worst case in (14), *ie*, $|\Delta_a(j\omega)| = 1\ \forall\omega$. By manipulating (14) we obtain

$$\frac{1}{|w_a(j\omega)|} > \Big|\frac{V_0(j\omega)}{G_0(j\omega)}\Big| \tag{15}$$

or

$$|w_a(j\omega)| < \Big|\frac{G_0(j\omega)}{V_0(j\omega)}\Big|\,. \tag{16}$$

The closed-loop robust stability conditions (15), (16) are to be satisfied by suitably choosing the structure and parameters of the transfer function $V_0(s)$.

3 ROBUST PID CONTROLLER DESIGN

The controller transfer function is obtained from (11)

$$R(s) = \frac{V_0(s)}{G_0(s)[1 - V_0(s)]}\,. \tag{17}$$

Individual transfer functions are expressed as ratios of pertinent polynomials

$$\frac{R_c(s)}{R_m(s)} = \frac{\frac{V_{0c}(s)}{V_{0m}(s)}}{\frac{G_{0c}(s)}{G_{0m}(s)}\big(1 - \frac{V_{0c}(s)}{V_{0m}(s)}\big)} \\ = \frac{G_{0m}(s)V_{0c}(s)}{G_{0c}(s)[V_{0m}(s) - V_{0c}(s)]}\,. \tag{18}$$

By manipulating (11) we obtain

$$V_0(s) = \frac{V_{0c}(s)}{V_{0m}(s)} = \frac{R_c(s)G_{0c}(s)}{R_m(s)G_{0m}(s) + R_c(s)G_{0c}(s)} \\ = \frac{G_{0c}(s)}{R_m(s)D_{0m}(s) + G_{0c}(s)}\,, \tag{19}$$

where

$$V_{0c}(s) = G_{0c}(s)\,, \tag{20a}$$

$$V_{0m(s)} = R_m(s)D_{0m}(s) + G_{0c}(s)\,, \tag{20b}$$

$$D_{0m}(s) = \frac{G_{0m}(s)}{R_c(s)}\,, \tag{20c}$$

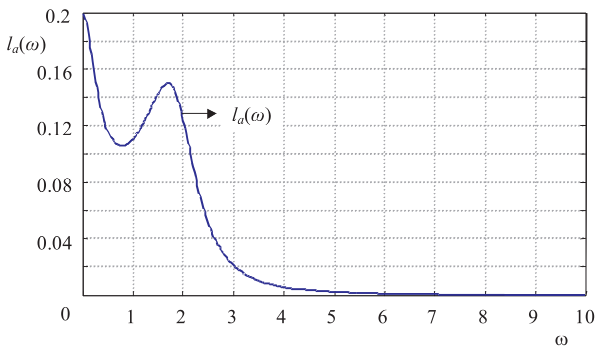

Fig. 2. Magnitude-versus-ω plots of $l_a(\omega)$.

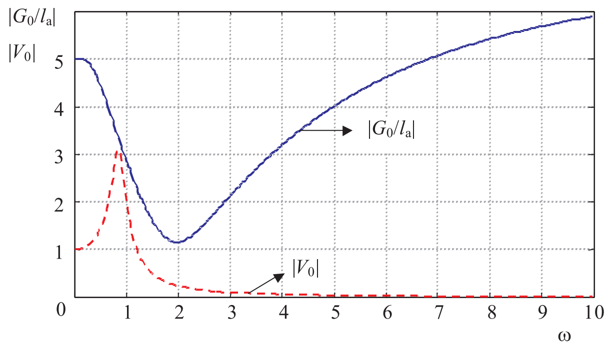

Fig. 3. Magnitude-versus-ω plots of $G_0(j\omega)/l_a(\omega)$ and $V_0(j\omega)$

Substituting (20a) into (18) yields

$$\frac{R_c(s)}{R_m(s)} = \frac{G_{0m}(s)}{V_{0m}(s) - V_{0c}(s)}\,. \tag{21}$$

Comparing (21) with the PID controller transfer function gives

$$\frac{R_c(s)}{R_m(s)} = \frac{1}{K}\frac{C(s)}{s} = \frac{1}{K}\frac{c_1 s^2 + c_0 s + c_{-1}}{s}\,. \tag{22}$$

It is necessary to choose its numerator and denominator in the following way

$$G_{0m}(s) = D_{0m}(s)R_c(s) = D_{0m}(s)C(s)$$
$$= D_{0m}(s)(c_1 s^2 + c_0 s + c_{-1})\,, \tag{23}$$

$$V_{0m}(s) - V_{0c}(s) = V_{0m}(s) - G_{0c}(s) \stackrel{!}{=} D_{0m}(s)Ks \tag{24}$$

where polynomial $D_{0m}(s)$ is to be chosen as well. The sum of its degree and of the degree of $C(s)$ gives the degree of $G_{0m}(s)$. Hence, both $D_{0m}(s)$ and $C(s)$ can be determined from (23). K is to be chosen; the method of its choice will be specified in the sequel.

$V_{0m}(s)$ as a function of K is calculated from (24)

$$V_{0m}(s) = G_{0c}(s) + D_{0m}(s)Ks\,. \tag{25}$$

Polynomial $V_{0m}(s)$ has to be stable, as it figures in the nominal closed-loop characteristic equation (cf. (11)). The transfer function $V_0(s)$ is fully specified by polynomials (20a) and (25).

Suppose that there is equality in (7) and consider the robust stability condition (16). Then the following inequality holds

$$|w_a(j\omega)| = \ell(\omega) < \left|\frac{G_0(j\omega)}{V_0(j\omega)}\right| \tag{26}$$

or after some manipulation

$$|V_0(j\omega)| < \frac{|G_0(j\omega)|}{\ell_a(\omega)}\,, \tag{27}$$

where there is a coefficient K in $V_0(j\omega)$. The inequality is satisfied due to its suitable choice.

The following relations hold for the robust PID controller coefficients

$$r_0 = \frac{c_0}{K}\,, \quad r_1 = \frac{c_1}{K}\,, \quad r_{-1} = \frac{c_{-1}}{K}\,. \tag{28}$$

In the case of PI controller just r0 and r-1 are considered. The PI/PID controller designed for the uncertain system G(s) guarantees robust closed-loop stability.

4 ILLUSTRATIVE EXAMPLE

Particular transfer functions (2) are as follows

$$G_1(s) = \frac{B_1(s)}{A_1(s)} = \frac{0.8}{0.7s^3 + s^2 + 3s + 1}\,,$$
$$G_2(s) = \frac{B_2(s)}{A_2(s)} = \frac{1}{s^3 + 2s^2 + 4s + 1}\,,$$
$$G_3(s) = \frac{B_3(s)}{A_3(s)} = \frac{1.2}{1.3s^3 + 3s^2 + 5s + 1}\,,$$

yielding the nominal model (3)

$$G_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{1}{s^3 + 2s^2 + 4s + 1}\,.$$

Let the uncertain system be controlled by a PID controller

$$R(s) = \frac{R_c(s)}{R_m(s)} = \frac{r_1 s^2 + r_0 s + r_{-1}}{s}\,.$$

PID controller parameters will be determined using (23)

$$s^3 + 2s^2 + 4s + 1 = D_{0m}(s)(c_1 s^2 + c_0 s + c_{-1})\,.$$

Polynomial $D_{0m}(s)$ is to be chosen

$$D_{0m}(s) = d_1 s + 1\,.$$

Equating coefficients of corresponding powers of s on both sides we are able to determine coefficients c_i, $i = 0, 1, -1$ and d_1, *ie*, both $C(s)$ and $D(s)$.

$$D_{0m}(s) = d_1 s + 1 = 3.5115a + 1\,,$$
$$C(s) = c_1 s^2 + c_0 s + c_{-1} = 0.2846s^2 + 0.4885s + 1\,.$$

$V_0(s)$ as a function of K is as follows

$$V_0(s) = \frac{G_{0c}(s)}{G_{0c}(s) + D_{0m}(s)Ks} = \frac{1}{3.5115s^2 + Ks + 1}\,.$$

Fulfillment of the robustness condition (27) has to be guaranteed by appropriately choosing K in this function.

The magnitude plot $|G_0(j\omega)/\ell_a(\omega)|$ generated according to the robust stability condition (27) is in Fig. 4. Coefficient K in $V_0(j\omega)$ is then changed until the magnitude

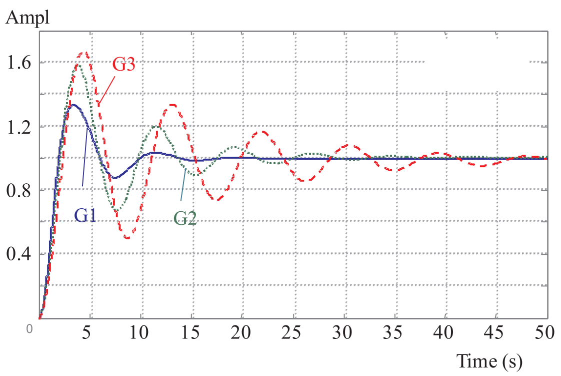

Fig. 4. Closed-loop step responses under the robust controller

$|V_0(j\omega)|$ lies below $|G_0(j\omega)/\ell_a(\omega)|$; in such a case the robust stability condition is satisfied. The ultimate value of K that satisfies (27) is $K = 0.4168$. Evaluation of $V_0(j\omega)$ gives

$$V_0(s) = \frac{1}{1.4635s^2 + 0.4168s + 1}\,.$$

Its magnitude is depicted in Fig. 3. The corresponding PID controller transfer function is

$$R(s) = \frac{r_1 s^2 + r_0 s + r_{-1}}{s} = \frac{\frac{c_1}{K}s^2 + \frac{c_0}{K}s + \frac{c_{-1}}{K}}{s} = \frac{0.6830s^2 + 1.715s + 2.3984}{s}\,.$$

Closed-loop step responses in three different working points in Fig. 4 verify the robust operation of the designed closed-loop.

5 CONCLUSION

The PI/PID controller design methodology presented in the paper is simple and easy to use in practice, hence it can be considered as an engineering method suitable especially for practitioners. It can be applied to design PID controllers for uncertain linear systems given as a set of stable transfer functions. To design the controller it is necessary to find $\ell_a(j\omega)$ according to (6) and solve simultaneous equations implied by (31). Afterwards, the magnitudes $|G_0(j\omega)/\ell_a(\omega)|$ and $V_0(j\omega)$ are shaped by means of the factor K so as to find their mutual position that satisfies the robustness condition (27). The closed loop performance can be improved by choosing such a K that extends the minimum distance between the magnitude plots in Fig. 3.

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