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ON SOME ASPECTS OF THE COMPLEX–ENVELOPE FINITE–DIFFERENCES SIMULATION OF WAVE PROPAGATION IN ONE–DIMENSIONAL CASE

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Some aspects of the numerical modeling of the electromagnetic waves propagation using the "complex-envelope" finite-differences formulation in the one-dimensional case are here reviewed and discussed in comparison with the standard finite-differences in time-domain (FDTD) approach. The main focus is put on the stability and the numerical dispersion issues of the "complex envelope" explicit and implicit methods.

K e y w o r d s: wave phenomena, computer simulation, finite-differences in time-domain, complex-envelope method

1 INTRODUCTION

Numerical simulations are routinely used for modelling electromagnetic wave propagation problems in the area of high frequency electromagnetics and photonics. The finite-difference time-domain (FDTD) method of explicit type designed by Yee [1] was for long years the preferred numerical technique for such simulations due to its flexibility, since it allows the inclusion of arbitrarily heterogeneous objects in the region to be simulated.

However, artificial artefacts are introduced in course of the numerical computations, such as numerical attenuation or amplification of the wave power-flow-density as well as the numerical dispersion. These are two main factors influencing electromagnetic-wave-propagation simulations. Especially for explicit-type methods special precautions concerning the stability of the calculations must be taken, *ie* the well-known CFL (Courant-Friedrichs-Lewy) condition must be met, limiting thus the step-length of the in-time-forward-marching algorithm. The extent of literature in this area is huge including several book publications *eg* [2, 3].

The implicit methods make possible to avoid the severe CFL condition, on the other hand they require inversion of tridiagonal matrices. The concise review of these methods has been given in [4] and [5].

The "complex envelope" (CE) formulation can be used to strip-off fast oscillations from the time dependence of high frequency waves. As shown further this formulation improves the stability properties of the explicit methods [6], as well as the signal deterioration due to numerical dispersion.

2 EXPLICIT AND IMPLICIT FORMULATION OF THE STANDARD FDTD METHOD IN ONE–DIMENSIONAL CASE

Let us consider the functions $f(z,t)$ and $g(z,t)$, representing similarly as in [4] the normalised wave amplitudes of the voltage $u(z,t)$ and current $i(z,t)$ on the transmission line, *ie* $f(z,t) = u(z,t)$, $g(z,t) = Z_0 i(z,t)$, $Z_0 = \sqrt{L_0/C_0}$, or the normalised transversal components of the electric and magnetic field vectors $E_x(z,t)$, $H_y(z,t)$ pertaining to the homogeneous plane wave, *ie* $f(z,t) = E_x(z,t)$, $g(z,t) = Z_0 H_y(z,t)$, $Z_0 = \sqrt{\mu/\varepsilon}$.

The wave propagation is either governed by a pair of coupled equations (formula (14) in [4]), *ie* for $f(z,t)$ and $g(z,t)$ holds

$$\frac{1}{c}\frac{\partial f(z,t)}{\partial t} = -\frac{\partial g(z,t)}{\partial z}\,, \tag{1}$$

$$\frac{1}{c}\frac{\partial g(z,t)}{\partial t} = -\frac{\partial f(z,t)}{\partial z}\,, \tag{2}$$

or by the separated second order wave equations of type (formula (15) in [4])

$$\frac{1}{c^2}\frac{\partial^2 f(z,t)}{\partial t^2} = \frac{\partial^2 f(z,t)}{\partial z^2}\,, \tag{3}$$

where $c = (L_0 C_0)^{-1/2}$, or $c = (\mu\varepsilon)^{-1/2}$.

The equidistant discrete representation of $f(z,t)$ and $g(z,t)$ [4] is formulated for the explicit discretisation method in points of a double staggered grid in time and space with the discrete values

$$f(z_m, t_n) = f(m\Delta_z, n\Delta_t) = f\big|_m^n, \tag{4}$$

$$g\big(z_{m+\frac{1}{2}}, t_{n+\frac{1}{2}}\big) = g\big([m+\tfrac{1}{2}]\Delta_z, [n+\tfrac{1}{2}]\Delta_t\big) = g\big|_{m+\frac{1}{2}}^{n+\frac{1}{2}}\,, \tag{5}$$

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or for the implicit discretisation method in the points of a single staggered grid along the space axis only with discrete values

$$f\big(z_m,t_n\big)=f\big(m\Delta_z,n\Delta_t\big)=f\big|_m^n\,, \tag{6}$$

$$g\big(z_{m+1/2},t_n\big)=g\big([m+1/2]\Delta_z,n\Delta_t\big)=g\big|_{m+1/2}^n, \tag{7}$$

where Δ_z and Δ_t, are discretisation intervals along the spatial and time axis.

Standard explicit discretisation of (1) through (3) leads to discretised equations on staggered grid (formulae (37), (38) and (39) in [4])

$$\frac{f|_m^{n+1}-f|_m^n}{c\Delta_t}=-\frac{g|_{m+1/2}^{n+1/2}-g|_{m-1/2}^{n+1/2}}{\Delta_z}\,, \tag{8}$$

$$\frac{g|_{m+1/2}^{n+1/2}-g|_{m+1/2}^{n-1/2}}{c\Delta_t}=-\frac{f_{m+1}^n-f|_m^n}{\Delta_z}\,, \tag{9}$$

$$\frac{f|_m^{n+1}-2f|_m^n+f|_m^{n-1}}{c^2\Delta_t^2}=\frac{f|_{m+1}^n-2f|_m^n+f|_{m-1}^n}{\Delta_z^2}\,. \tag{10}$$

In the operator form (8) through (10) can be written as

$$\frac{D^+-1}{c\Delta_t}f\big|_m^n=-\frac{1-D_-}{\Delta_z}g\big|_{m+1/2}^{n+1/2}\,, \tag{11}$$

$$\frac{1-D^-}{c\Delta_t}g\big|_{m+1/2}^{n+1/2}=-\frac{D_+-1}{\Delta_z}f_m^n\,, \tag{12}$$

$$\frac{D^+-2+D^-}{c^2\Delta_t^2}f\big|_m^n=\frac{D_+-2+D_-}{\Delta_z^2}f\big|_m^n\,, \tag{13}$$

where D^+ denotes the shift-operator adding 1 to the upper index and D^- subtracting 1 from the upper index. Similarly, the operators D_+ and D_- do the same with the lower index.

The standard FDTD using implicit Crank-Nicolson formulation leads either to two coupled equation of the first order as shown in [4] formulae (44), (45), that in the operator notation read

$$\frac{D^+-1}{c\Delta_t}f\big|_m^n=-\frac{1-D_-}{2\Delta_z}(1+D^-)g\big|_{m+1/2}^{n+1}\,, \tag{14}$$

$$\frac{1-D^-}{c\Delta_t}g\big|_{m+1/2}^{n+1}=-\frac{D_+-1}{2\Delta_z}(1+D^+)f\big|_m^n\,, \tag{15}$$

or to the equations of second order of type

$$\frac{D^+-2+D^-}{c^2\Delta_t^2}f\big|_m^n=\frac{D_+-2+D_-}{4\Delta_z^2}(D^++2+D^-)f\big|_m^n\,. \tag{16}$$

3 "COMPLEX ENVELOPE IN TIME" (CET) FDTD — EXPLICIT AND IMPLICIT FORMULATION

In order to strip-off the fast oscillations in time dependence of the wave amplitude let us further express $f(z,t)$ and $g(z,t)$ in the complex representation as

$$f(z,t)=\varphi(z,t)\exp(j\omega_0 t)\,, \tag{17}$$

$$g(z,t)=\psi(z,t)\exp(j\omega_0 t) \tag{18}$$

ie as the amplitude- and phase-modulated harmonic quantities, where $\varphi(z,t)$ and $\psi(z,t)$ is the slowly varying complex amplitude of the full wave amplitude $f(z,t)$ and $g(z,t)$ and ω_0 is the carrier frequency.

For the "complex envelopes" $\varphi(z,t)$, $\psi(z,t)$ one again obtains either two coupled equations of the first order

$$\frac{1}{c}\frac{\partial\varphi(z,t)}{\partial t}+j\frac{\omega_0}{c}\varphi(z,t)=-\frac{\partial\psi(z,t)}{\partial z}\,, \tag{19}$$

$$\frac{1}{c}\frac{\partial\psi(z,t)}{\partial t}+j\frac{\omega_0}{c}\psi(z,t)=-\frac{\partial\varphi(z,t)}{\partial z}\,, \tag{20}$$

or separated second order equations of the type

$$\frac{1}{c^2}\frac{\partial^2\varphi(z,t)}{\partial t^2}+\frac{2j\omega_0}{c^2}\frac{\partial\varphi(z,t)}{\partial t}-\frac{\omega_0^2}{c^2}\varphi(z,t)=\frac{\partial^2\varphi(z,t)}{\partial z^2}\,. \tag{21}$$

Explicit discretisation of the CET FDTD equations (19) through (21) leads to the discretised equations

$$\frac{\eta D^+-\eta^*}{c\Delta_t}\varphi\big|_m^n=-\frac{1-D_-}{\Delta_z}\psi\big|_{m+1/2}^{n+1/2}\,, \tag{22}$$

$$\frac{\eta-\eta^*D^-}{c\Delta_t}\psi\big|_{m+1/2}^{n+1/2}=-\frac{D_+-1}{\Delta_z}\varphi\big|_m^n\,, \tag{23}$$

$$\frac{\eta^2D^+-2|\eta|^2+\eta^*D^-}{c^2\Delta_t^2}\varphi\big|_m^n=\frac{D_+-2+D_-}{\Delta_z^2}\varphi\big|_m^n\,, \tag{24}$$

analogous to (11)–(13), where

$$\eta=1+j\gamma\,,\ \gamma=\omega_0\Delta_t/2 \tag{25}$$

and the asterisk denotes the complex conjugate.

Similarly for the implicit formulation of CET FDTD one obtains from (19), (20) and (21) the equations

$$\frac{\eta D^+-\eta^*}{c\Delta_t}\varphi\big|_m^n=-\frac{1-D_-}{2\Delta_z}(1+D^-)\psi\big|_{m+1/2}^{n+1}\,, \tag{26}$$

$$\frac{\eta-\eta^*D^-}{c\Delta_t}\psi\big|_{m+1/2}^{n+1}=-\frac{D_+-1}{2\Delta_z}(1+D^+)\varphi\big|_m^n\,, \tag{27}$$

$$\frac{\eta^2D^+-2|\eta|^2+\eta^{*2}D^-}{c^2\Delta_t^2}\varphi\big|_m^n=\frac{D_+-2+D_-}{4\Delta_z^2}(D^++2+D^-)\varphi\big|_m^n\,. \tag{28}$$

It is easily recognised that the form of equations (22) through (24) and (26) through (28) for the "complex envelopes in time" differs only in factors η, η^*, or $|\eta|^2$ on the left-hand sides from the equations (11) through (13), and (14) through (16) for the standard FDTD.

4 "COMPLEX ENVELOPE IN TIME AND SPACE" (CETS) FDTD — EXPLICIT AND IMPLICIT FORMULATION

Similarly as in previous paragraph, let us strip-off from the complex amplitude $\varphi(z,t)$ and $\psi(z,t)$ also the fast oscillations in space dependence due to propagating wave, by defining

$$\varphi(z,t) = \phi(z,t)\exp(-jk_0 z)\,, \tag{29}$$

$$\psi(z,t) = \chi(z,t)\exp(-jk_0 z)\,. \tag{30}$$

The equations (1) through (3) now take the form

$$\frac{1}{c}\frac{\partial\phi(z,t)}{\partial t} + j\frac{\omega_0}{c}\phi(z,t) = -\frac{\partial\chi(z,t)}{\partial z} + jk_0\chi(z,t)\,, \tag{31}$$

$$\frac{1}{c}\frac{\partial\chi(z,t)}{\partial t} + j\frac{\omega_0}{c}\chi(z,t) = -\frac{\partial\phi(z,t)}{\partial z} + jk_0\phi(z,t)\,, \tag{32}$$

$$\frac{1}{c^2}\frac{\partial^2\phi(z,t)}{\partial t^2} + \frac{2j\omega_0}{c^2}\frac{\partial\phi(z,t)}{\partial t} - \frac{\omega_0^2}{c^2}\phi(z,t) = \frac{\partial^2\phi(z,t)}{\partial z^2} - 2jk_0\frac{\partial\phi(z,t)}{\partial z} - k_0^2\phi(z,t)\,. \tag{33}$$

For the explicit discretisation one obtains the following equations in operator for m

$$\frac{\eta D^+ - \eta^*}{c\Delta_t}\phi\big|_m^n = \frac{\alpha^* - \alpha D_-}{\Delta_z}\chi\big|_{m+1/2}^{n+1/2}\,, \tag{34}$$

$$\frac{\eta - \eta^* D^-}{c\Delta_t}\chi\big|_{m+1/2}^{n+1/2} = -\frac{\alpha^* D_+ - \alpha}{\Delta_z}\phi\big|_m^n\,, \tag{35}$$

$$\frac{\eta^2 D^+ - 2|\eta|^2 + \eta^{*2} D^-}{c^2\Delta_t^2}\phi\big|_m^n = \frac{\alpha^{*2} D_+ - 2|\alpha|^2 + \alpha^2 D_-}{\Delta_z^2}\phi\big|_m^n, \tag{36}$$

where

$$\alpha = 1 + jk_0\Delta_z/2 = |\alpha|e^{j\delta}. \tag{37}$$

For the implicit discretisation one obtains

$$\frac{\eta D^+ - \eta^*}{c\Delta_t}\phi\big|_m^n = -\frac{\alpha^* - \alpha D_-}{2\Delta_z}(1 + D_-)\chi\big|_{m+1/2}^{n+1}\,, \tag{38}$$

$$\frac{\eta - \eta^* D^-}{c\Delta_t}\chi\big|_{m+1/2}^{n+1} = -\frac{\alpha^* D_+ - \alpha}{2\Delta_z}(1 + D^+)\phi\big|_m^n\,, \tag{39}$$

$$\frac{\eta^2 D^+ - 2|\eta|^2 + \eta^{*2} D^-}{c^2\Delta_t^2}\phi\big|_m^n = \frac{\alpha^{*2} D_+ - 2|\alpha|^2 + \alpha^2 D_-}{4\Delta_z^2}(D^+ + 2 + D^-)\phi\big|_m^n\,, \tag{40}$$

Observe that for $\alpha = 1$, *ie* $k_0 = 0$, (34)–(36) and (38)–(40) are identical with (22)–(24) and (26)–(28). Moreover for $\eta = 1$, *ie* $\omega_0 = 0$, (22)–(24) and (26)–(28) are identical with (11)–(13) and (14)–(16).

5 POWER CONSERVATION AND NUMERICAL DISPERSION FOR THE EXPLICIT METHODS

The von Neumann stability analysis of standard explicit one-dimensional FDTD method (10), as performed also in [4] formula (54) through (57), is done by substituting $f|_m^n$ in the form

$$f|_m^n = \exp(j\omega\Delta_t)\exp(-jkm\Delta_z) = \xi^n\exp(-jkm\Delta_z) \tag{41}$$

into (10) to obtain the equation

$$\xi^2 - 2[1 - 2A]\xi + 1 = 0\,, \tag{42}$$

with the solution

$$\xi = 1 - 2A^2 + j2A\sqrt{1 - A^2}\,, \tag{43}$$

Where

$$A = b\sin(k\Delta_z/2)\,, \quad b = c\Delta_t/\Delta_z\,. \tag{44}$$

Performing the von Neumann stability analysis of the explicit CET FDTD formulation, *ie* substituting $\phi\big|_m^n$ in the form

$$\varphi_m^n = \exp(j\Omega n\Delta_t)\exp(-jkm\Delta_z) = \zeta^n\exp(-jkm\Delta_z)\,, \tag{45}$$

where $\Omega = \omega - \omega_0$, into (24) yields the equation

$$\eta^2\zeta - 2\left[|\eta|^2 - 2A\right]\zeta + \eta^{*2} = 0\,, \tag{46}$$

with the solution

$$\zeta = \frac{1}{\eta^2}\{|\eta|^2 - 2A^2 + j2A\sqrt{|\eta|^2 - A^2}\}. \tag{47}$$

Observe that (46) and (47) yield for $\eta = 1$, *ie* $\omega_0 = 0$, the same result as (42) and (43).

For the CETS FDTD formulation one takes $\phi|_m^n$ in the following form

$$\phi\big|_m^n = \exp(j\Omega n\Delta_t)\exp(-j\kappa m\Delta_z) = \theta^n\exp(-j\kappa m\Delta_z)\,, \tag{48}$$

where $\kappa = k - k_0$, and after the substitution into (36) one arrives at the equation

$$\eta^2\theta^2 - 2\left[|\eta|^2 - 2|\alpha|^2 B^2\right]\theta + \eta^{*2} = 0\,, \tag{49}$$

where

$$B = b\sin\{(\kappa\Delta_z/2) + \delta\}\,, \tag{50}$$

and $\delta = \arctan(k_0\Delta_z/2)$ accordingly (37), with the solution

$$\theta = \frac{1}{\eta^2}\{|\eta|^2 - 2|\alpha|^2 B^2 + j2|\alpha|B\sqrt{|\eta|^2 - |\alpha|^2 B^2}\}. \tag{51}$$

Observe that (49) and (51) yield for $\alpha = 1$, *ie* $k_0 = 0$, the same result as (46) and (47).

For all three explicit FDTD methods the power conservation conditions $|\xi| = 1$, $|\zeta| = 1$, and $\eta| = 1$ are

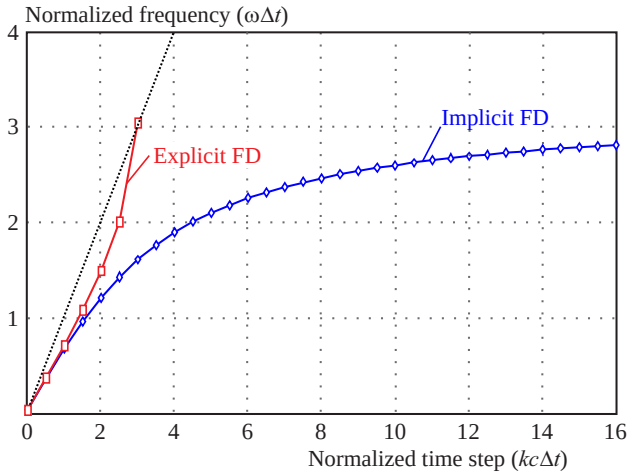

Fig. 1. The normalised numerical dispersion relation $\omega\Delta_t = \text{phase}(\xi)$ versus $kc\Delta_t$ for the explicit method, $k\Delta_z = 2.51$. The dashed line represents the true dispersion dependence $\omega = kc$

fulfilled only if the square roots in (43), (47) and (51) are real numbers. This leads to conditions

$$A \le 1\,, \quad A \le |\eta|\,, \quad B \le |\eta|/|\alpha| \tag{52}$$

for the respective cases.

Since the maximum value of sinus function in (44) and (50) is equal to one, (52) leads to the ultimate Courant-Friedrichs-Lewy (CFL) power conservation condition

$$c\Delta_t \le \Delta_z \tag{53}$$

for the standard FDTD, and further to the ultimate condition

$$c\Delta_t \big/ \sqrt{1 + \omega_0^2\Delta_t^2/4} \le \Delta_z \tag{54}$$

for the CET FDTD, and

$$c\Delta_t \big/ \sqrt{1 + \omega_0^2\Delta_t^2/4} \le \Delta_z \big/ \sqrt{1 + k_0^2\Delta_z^2/4} \tag{55}$$

for the CETS FDTD.

The ultimate condition (54) can also be written as

$$c\Delta_t/\Delta_z \le 1\big/\sqrt{1 - \omega_0^2\Delta_z^2/4c^2}\,, \tag{56}$$

and the ultimate condition (55) as

$$c\Delta_t/\Delta_z \le 1\big/\sqrt{1 + k_0^2\Delta_z^2/4 - \omega_0^2\Delta_z^2/4c^2}\,. \tag{57}$$

For the CET FDTD method (56) leads to the less stringent condition for the choice of time-step Δ_t in comparison with the CFL condition (53). It means that, if the appropriate choice of ω_0 fulfils the condition

$$\omega_0 \ge 2c\sqrt{1/\Delta_z^2 - 1/c^2\Delta_t^2}\,, \tag{58}$$

the CET FDTD method becomes power conserving for the case $c\Delta_t/\Delta_z > 1$ in contrary to the classical explicit FDTD method as it was shown in [6].

For the sufficiently high frequency ω_0

$$\omega_0 \ge 2c/\Delta_z\,, \quad \gamma \ge b\,, \tag{59}$$

the CET FDTD method becomes absolutely power-conserving for any arbitrary magnitude of the ratio $c\Delta_t/\Delta_z$ and all representable spatial harmonics.

For the CETS FDTD method (57) indicates that for the choice $\omega_0 = ck_0$ the condition of power conservation is equal to the CFL condition (53). For the choice $\omega_0 < ck_0$ the condition of power conservation leads to even more severe limitation than the CFL condition. For $\omega_0 > ck_0$ the condition of power conservation is less stringent than CFL condition but more stringent than (56) for the CET FDTD method

For the CETS FDTD method in the case $c\Delta_t/\Delta_z > 1$, the power conservation can be reached if the appropriate choice of ω_0 fulfils for given k_0 condition analogous to (58)

$$\omega_0 \ge c\sqrt{k_0^2 + 4[1/\Delta_z^2 - 1/c^2\Delta_t^2]}\,, \tag{60}$$

and for the sufficiently high frequency ω_0 the method becomes absolutely power conserving if

$$\omega_0 \ge c\sqrt{k_0^2 + 4/\Delta_z^2}\,. \tag{61}$$

The phase of ξ in (43) for the standard FDTD, provided the CFL condition (53) is met, equals

$$\text{phase}(\xi) = \omega\Delta_t = \arctan\big\{2A\sqrt{1 - A^2}/(1{-}2A^2)\big\}, \tag{62}$$

leading, instead of to $\omega = ck$, to the dispersion relation

$$\omega(k) = \frac{2}{\Delta_t}\arcsin\{b\sin(k\Delta_z/2)\}\,, \tag{63}$$

or, written in more familiar form [2, 3], to

$$\frac{\sin(\omega\Delta_t/2)}{c\Delta_t} = \frac{\sin(k\Delta_z/2)}{\Delta_z}\,. \tag{64}$$

Both (63) and (64) describe the numerical dispersion of the explicit numerical method of wave propagation simulation, with the phase velocity $v_p(k) = \omega(k)/k$ and the group velocity $v_g(k) = \mathrm{d}\omega(k)/\mathrm{d}k$ different from the correct physical values $v_f = v_g = c$.

The phase of ζ in (47) for CET FDTD provided the condition (56) is met, is given by

$$\text{phase}(\zeta) = \Omega\Delta_t = \arctan\big\{2A\sqrt{|\eta|^2 - A^2}/(|\eta|^2 - 2A^2)\big\} \tag{65}$$

with the pertaining numerical phase velocity $v_p(k) = \omega/k = (\Omega + \omega_0)/k$ and the group velocity $v_g(k) = \mathrm{d}\omega/\mathrm{d}k = \mathrm{d}\Omega/\mathrm{d}k$, for any value of the ratio $b = c\Delta_t/\Delta_z$, provided, in case $b = c\Delta_t/\Delta_z > 1$, the condition (58)

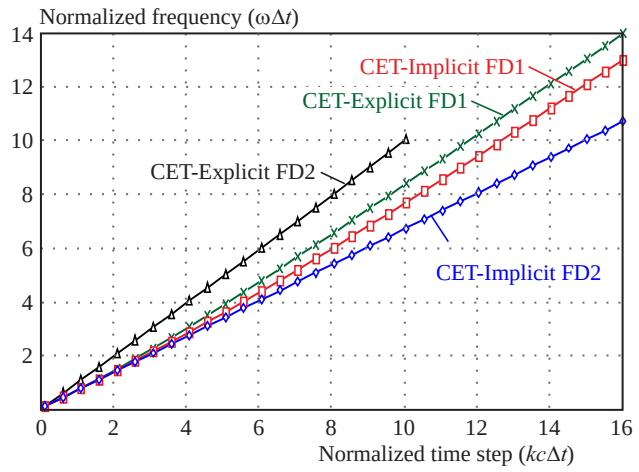

Fig. 2. The normalised numerical dispersion relation $\omega\Delta_t = \text{phase}(\zeta) + \omega_0\Delta_t = (\Omega + \omega_0)\Delta_t$ versus $kc\Delta_t$ for the explicit and implicit method, $k\Delta_z = 2.51$. The values of ω_0/c are 1) $\omega_0/c = 6$ and 2) $\omega_0/c = 4$

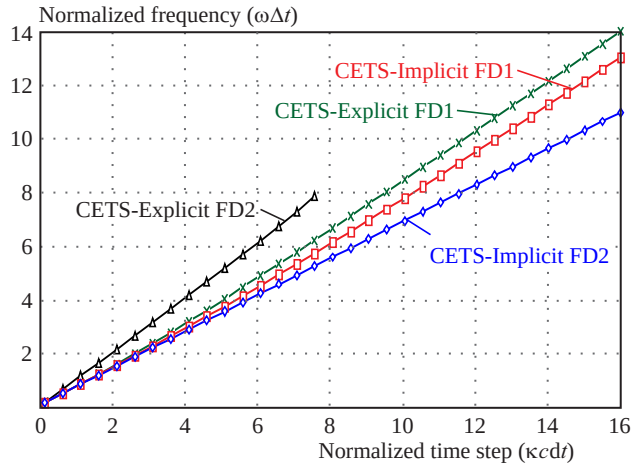

Fig. 3. The normalised numerical dispersion relation $\omega\Delta_t = \text{phase}(\theta) + \omega_0\Delta_t = (\Omega + \omega_0)\Delta_t$ versus $\kappa c\Delta_t$ for the explicit and implicit method, $\kappa\Delta_z = 2.51$. The values of ω_0/c and k_0 are 1) $\omega_0\Delta_z/c = 2.86$, $k_0\Delta_z = 1.43$ and 2) $\omega_0\Delta_z = 1.91$, $k_0\Delta_z = 0.955$

is met. Observe that for the limiting value $\omega_0 \to 0$, *ie* $\gamma \to 0$, (65) converges to (62).

The phase of θ in (51) for CETS FDTD provided the condition (57) is met, is given by

$$\text{phase}(\theta) = \Omega\Delta_t = \arctan\left\{2|\alpha|B\sqrt{|\eta|^2 - |\alpha|^2B^2}/\left(|\eta|^2 - 2|\alpha|^2B^2\right)\right\}, \quad (66)$$

with the pertaining numerical phase velocity $v_p(k) = \omega/k = (\Omega + \omega_0)/(k_0 + \kappa)$ and the group velocity $v_g(k) = \mathrm{d}\omega/\mathrm{d}k = \mathrm{d}\Omega/\mathrm{d}\kappa$, for any value of the ratio $b = c\Delta_t/\Delta_z$, provided, in case $b = c\Delta_t/\Delta_z > 1$, the condition (60) is met. Observe that for the limiting value $k_0 \to 0$, (66) again converges to (65).

6 POWER CONSERVATION AND THE NUMERICAL DISPERSION FOR THE IMPLICIT METHODS

The von Neumann stability analysis of the equations for implicit FDTD methods can be performed similarly as for explicit methods. After substituting for $f|_m^n$, $\phi|_m^n$ and ϕ_m^n from (41), (45) and (48) into (16), (28) and (40) respectively, one obtains for the standard implicit FDTD method the equation

$$\xi^2(1 + A^2) - 2\xi(1 - A^2) + (1 + A^2) = 0, \quad (67)$$

for the implicit CET FDTD method the equation

$$\zeta^2\{\eta^2 + A^2\} - 2\zeta\{|\eta|^2 - A^2\} + \{\eta^{*2} + A^2\} = 0, \quad (68)$$

and for the implicit CETS FDTD method the equation

$$\theta^2\{\eta^2|\alpha|^2B^2\} - 2\theta\{|\eta|^2 - |\alpha|^2B^2\} + \{\eta^{*2} + |\alpha|^2B^2\} = 0. \quad (69)$$

The solutions are

$$\xi = \{1 - A^2 + 2jA\}/\{1 + A^2\}, \quad (70)$$

$$\zeta = \{|\eta|^2 - A^2 + j2A\}/\{\eta^2 + A^2\}, \quad (71)$$

$$\theta = \{|\eta|^2 - |\alpha|^2B^2 + j2|\alpha|B\}/\{\eta^2 + |\alpha|^2B^2\}. \quad (72)$$

All three implicit FDTD methods are absolutely power conserving, *ie* $|\xi| = 1$, $|\zeta| = 1$ and $|\theta| = 1$ always holds.

The dispersion relation for the standard implicit FDTD method is

$$\text{phase}(\xi) = \omega\Delta_t = \arctan\frac{2A}{1 - A^2}, \quad (73)$$

or in the form analogous to (64)

$$\frac{\tan(\omega\Delta_t/2)}{c\Delta_t} = \frac{\sin(k\Delta_z/2)}{\Delta_z}. \quad (74)$$

For the implicit CET FDTD and CETS FDTD the following formulas hold

$$\text{phase}(\zeta) = \Omega\Delta_t = \arctan\frac{2A}{1 - A^2 + \gamma^2} - \arctan\frac{2\gamma}{1 - A^2 - \gamma^2}, \quad (75)$$

$$\text{phase}(\theta) = \Omega\Delta_t = \arctan\frac{2|\alpha|B}{1 - |\alpha|^2B^2 + \gamma^2} - \arctan\frac{2\gamma}{1 + |\alpha|^2B^2 - \gamma^2}. \quad (76)$$

7 CONCLUSIONS

For the explicit methods the power conservation conditions are given by (53), (54) and (55). When these conditions are fulfilled the numerical dispersion relations for the three explicit finite-differences formulations are given by (62), (65) and (66).

The implicit FDTD methods are unconditionally power conserving. The numerical dispersion relations for the three implicit finite-differences formulations are given by (73), (75) and (76).

The dependence of $\omega\Delta_t = \text{phase}(\xi)$ versus $kc\Delta_t$ for standard explicit and implicit FDTD method is shown in Fig. 1. The deviation of both curves from the dashed line $\omega = kc$ represents the numerical dispersion of the respective method. In both cases the numerical phase velocity $v_p(k) = \omega(k)/k$ as well as the numerical group velocity $v_g(k) = \mathrm{d}\omega(k)/\mathrm{d}k$ is smaller than the physical value $v_p = v_g = c$ as shown already in [4] in Figs. 5 and 6. Due to stability requirements the independent variable values are for the explicit method limited to $kc\Delta_t \leq \pi$.

As can be easily seen from Figs. 2 and 3 for the explicit and implicit CET FDTD and CETS FDTD methods the dependencies $\Omega\Delta_t = \text{phase}(\zeta)$ and $\Omega\Delta_t = \text{phase}(\theta)$ in Figs. 2 and 3 are much more linear, *ie* they are much less loaded by the numerical dispersion error, approximating more closely the true linear dependence of $\omega = kc$, particularly the curves "CET EXPLICIT FD 2" and "CETS EXPLICIT FD 2" that is practically identical with the dashed line in Fig. 1. However, for this case of conditional stability, and the value $\omega_0\Delta_z/c = 1.91$, the limit of stability of CET FDTD in accordance with (52) equals $kc\Delta_t \leq 10.41$. Similarly for the CETS FDTD the limit of stability accordingly (52) is reached for $\kappa c\Delta_t \leq 7.67$.

References

[1] YEE, K. S.: Numerical Solution of Initial Boundary Value Problems Involving Maxwell's Equations in Isotropic Media, IEEE Trans. Antennas & Propagation **AP-14** (1966), 302–307.

[2] TAFLOVE, A.—HAGNESS, S. C.: Computational Electrodynamics: The Finite-Difference Time-Domain Method, 2nd ed., Artech, Boston, MA, 2000.

[3] KUNZ, K. S.—LUEBBERS, R. J.: The Finite Difference Time Domain Method for Electromagnetics, CRC Press, Boca Raton, 1993.

[4] ŠUMICHRAST, Ľ.: On Some Strategies for Computer Simulation of the Wave Propagation using Finite Differences I., One-dimensional FDTD method, JEEEC **64** No. 4 (2013), 212–221.

[5] ŠUMICHRAST, Ľ.: On Some Strategies for Computer Simulation of the Wave Propagation using Finite Differences II. Multi-Dimensional FDTD Method, J. Electrical Engineering **64** No. 6 (2013), 337–345.

[6] CHANGNING MA—ZHIZHANG CHEN: Stability Analysis of the CE-FDTD Method, IEEE Microwave & Wireless Comp. Lett. **14** No. 5 (2004), 243.

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