

NON–SYMMETRIC FINITE NETWORKS: THE TWO–POINT RESISTANCE

Viera Čerňanová* — Juraj Brenkuš** — Viera Stopjaková**

An explicit formula for the resistance between two nodes in a network described by non-symmetric Laplacian matrix \mathbf{L} is obtained. This is of great advantage *eg* in electronic circuit fault analysis, where non-linear systems have to be solved repeatedly. Analysis time can be greatly reduced by utilization of the obtained formula. The presented approach is based on the “mutual orthogonality” of the full system of left and right-hand eigenvectors of a diagonalizable matrix \mathbf{L} . Simple examples are given to demonstrate the accuracy of this approach to circuit networks.

Key words: electronic circuit, fault analysis, non-symmetric Laplacian matrix, resistance computation

1 INTRODUCTION

The computation of the resistance between two nodes in a resistor network is a classical problem in electric circuit theory, indeed it was studied by numerous authors over many years. Since electrical potentials on a grid are governed by the same difference equations as those occurring in other problems, the computation of the resistances can be applied to a wide range of problems, such as lattice Green’s functions, random walks, the theory of harmonic functions and others. Due to variety of applications of the resistance problem and various methods of solution, this problem is still actual and intensively studied [1].

The resistance problem was often transformed in that of solving the difference equations, which is most conveniently carried out for infinite networks. Little attention was paid to finite networks, though these are occurring in real life. In 2004, an important result for finite lattices was published in [2]. He expressed the resistance between two arbitrary nodes in a resistor network in terms of eigenvalues and eigenvectors of the Laplacian matrix \mathbf{L} associated with the network. In [2], the Laplacian \mathbf{L} is a real symmetric $N \times N$ matrix with N real eigenvalues λ_i and an orthonormal basis of eigenvectors $\boldsymbol{\psi}_i = (\psi_{i1}, \dots, \psi_{iN})^*$. The resistance between the nodes α and β is given by

$$R_{\alpha\beta} = \sum_{i=2}^N \frac{1}{\lambda_i} |\psi_{i\alpha} - \psi_{i\beta}|^2 \quad (1)$$

However, if the Laplacian matrix of the network is non-symmetric, the relation (1) fails.

In this work, we study resistance networks with a real Laplacian matrix \mathbf{L} , symmetric or not. The Laplacian \mathbf{L} is a real diagonalizable $N \times N$ matrix with N real eigenvalues λ_i and two basis of “mutually orthogonal” left and right-hand eigenvectors $\boldsymbol{\varphi}_i^* = (\varphi_{i1}, \dots, \varphi_{iN})$ and $\boldsymbol{\psi}_i = (\psi_{i1}, \dots, \psi_{iN})^*$. To our best knowledge, this is the first work that presents an explicit formula for the total

resistance between two nodes in a resistor network with a real Laplacian matrix, in general non-symmetric. The obtained formula

$$R_{\alpha\beta} = \sum_{i=2}^N \frac{1}{\lambda_i \boldsymbol{\varphi}_i^* \boldsymbol{\psi}_i} (\varphi_{i\alpha} - \varphi_{i\beta})(\psi_{i\alpha} - \psi_{i\beta}) \quad (2)$$

is an extension of (1). The method used in this paper is similar to that of [2], where the orthonormal basis of eigenvectors of the matrix \mathbf{L} is replaced by two “mutually orthogonal” basis of left and right-hand eigenvectors of \mathbf{L} .

Our motivation is coming from the fault analysis of integrated circuits. The nodal conductance matrix of a circuit is presented, which is the Laplacian \mathbf{L} , and some of its properties are described. Theoretical framework is based on the theory of eigenvalues and eigenvectors of real matrices. The existence of “mutually orthogonal” basis $\boldsymbol{\varphi}_1^*, \dots, \boldsymbol{\varphi}_N^*$ and $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_N$ of left and right-hand eigenvectors of a real diagonalizable matrix \mathbf{A} is proved. Keeping the vectors $\boldsymbol{\varphi}_1^*, \dots, \boldsymbol{\varphi}_N^*$ intact, a convenient basis of right eigenvectors is constructed.

The main result is formulated as theorem which is then proved, and examples of calculating the two-point resistance are provided for two single circuit networks a simple bridge and an operational amplifier.

2 FAULT ANALYSIS — MOTIVATION

Today’s integrated circuits (ICs) represent complex systems realized on a small area using advanced fabrication technologies. At the early age of monolithic integration (early sixties), the fabrication process was dominated by yield as low as 10% due to poor process control [3]. As the fabrication process has evolved, the process control reached acceptable limits and the yield improved. On the other hand, with the increasing resolution, the fabrication

* Institute of Computer Science and Mathematics, vieracernanova@hotmail.com; ** Institute of Electronics and Photonics, juraj.brenkus@stuba.sk, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Bratislava, Slovakia

process became more sensitive to impurities of the environment and had to be moved to controlled areas, where the ambient impurity level was under control.

These days, a typical mixed-signal IC can consist of thousands to millions of transistors, occupy an area of several tens of square mm, while the fabrication process yield with resolution as high as tens of nanometers is well above 90%. Anyhow, even in a well controlled environment, the ambient still exhibits certain level of impurities. Besides, the fabrication process control is becoming even more complex and thus, resulting in a necessity to test the produced ICs.

During the test development process, so called *fault simulations* have to be conducted to determine the fault coverage of a given test. If performed in a traditional way, this step might be rather time consuming, since it requires a huge amount of simulations to be executed due to large amount of circuit nodes and possible faults taken into account. During the simulation, a set of nonlinear equations describing the circuit has to be solved using some iterative methods, *eg* in [4].

The numerical approach to DC fault simulations, presented in this paper, evaluates the resistance between two arbitrary nodes of the circuit in a more efficient way. This evaluation requires only one simulation for the given circuit configuration, providing information on the operating point, to be executed. It means that a system of nonlinear equations describing the circuit has to be solved only once for the given operating point. From this information, the circuit's nodal conductance matrix can be created and the resistance between the selected two nodes can be calculated. As has been already shown for symmetrical matrices, such an approach can lead to a significant analysis time reduction [5]. This resistance calculation can be used in the development of resistance-based tests, *eg* I_{DDQ} testing, where the resistance is indirectly evaluated by measuring the current flowing from the supply source [6, 7]. For a circuit network, where active elements such as MOS transistors are present, the nodal conductance matrix can become non-symmetrical, since voltage-controlled current sources (VCCS) are used to model such devices, as described below.

2.1 Nodal conductance matrix

Formulation of the nodal conductance matrix, which is the Laplacian, is based on the Kirchhoff current law

$$\sum_{j=1, j \neq i}^N c_{ij}(V_i - V_j) = I_i \quad i = 1, 2, \dots, N. \quad (3)$$

Here, c_{ij} are the conductances of resistor between the nodes i and j , V_i and V_j are the electric potentials of nodes i and j , and I_i is the current flowing into the network from sources connected directly to node i . As a consequence, the following constraint applies:

$$\sum_{i=1}^N I_i = 0. \quad (4)$$

In general, a circuit of N nodes can be described by a matrix equation of the form:

$$\mathbf{L}\mathbf{V} = \mathbf{I} \quad (5)$$

or

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1N} \\ c_{21} & c_{22} & \dots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \dots & c_{NN} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix}, \quad (6)$$

where \mathbf{L} is the Laplacian (nodal conductance matrix) with elements c_{ij} ($i, j \in \{1, \dots, N\}$), \mathbf{V} is a column vector of node potentials, and \mathbf{I} is a column vector of currents from independent external sources, [4].

To compute the resistance between two arbitrary nodes α and β of the network, an external source has to be connected to these nodes. Let the current flowing from the source be I and the potentials at the two nodes V_α and V_β . Then the desired resistance is determined by the Ohm's law

$$R_{\alpha\beta} = \frac{V_\alpha - V_\beta}{I}. \quad (7)$$

Now, the computation is reduced to solving (5) for V_α and V_β with the current given by

$$I_i = I(\delta_{i\alpha} - \delta_{i\beta}), \quad (8)$$

with δ_{ij} being the Konecker symbol.

As long as the network consists of resistances only, the Laplacian remains symmetric, since for a resistive element of a conductance c connected between the nodes i and j , the nodal conductance matrix \mathbf{L}_R can be expressed as

$$\mathbf{L}_R = \begin{matrix} & i & j \\ \begin{matrix} i \\ j \end{matrix} & \begin{pmatrix} c & -c \\ -c & c \end{pmatrix} \end{matrix}. \quad (9)$$

If this matrix is added to the Laplacian of the network, the symmetry of the Laplacian is maintained. On the other hand, if a VCCS is present in the network, the Laplacian can become non-symmetrical in cases when the input terminals are connected to different nodes as the output ones. In general, the nodal conductance matrix \mathbf{L}_{VCCS} of a VCCS, as depicted in Fig. 1, can be written as

$$\mathbf{L}_{VCCS} = \begin{matrix} & j & j' & k & k' \\ \begin{matrix} j \\ j' \\ k \\ k' \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g & -g & 0 & 0 \\ -g & g & 0 & 0 \end{pmatrix} \end{matrix}. \quad (10)$$

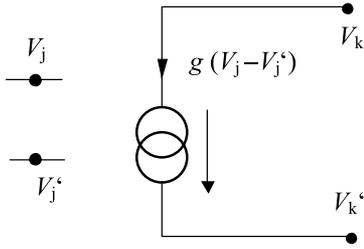


Fig. 1. Voltage controlled current source

Since most of the real life circuits contain active elements that are represented by VCCSs, an extension of the approach published in [2] to non-symmetrical Laplacians is of great importance.

3 THE TWO-POINT RESISTANCE THEOREM

In various applications based on matrix theory, a diagonalization of a square matrix is needed. A real $N \times N$ matrix \mathbf{A} is diagonalizable, if there exists an invertible matrix \mathbf{P} such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where \mathbf{D} is a diagonal matrix. Here, column vectors of \mathbf{P} are right-hand eigenvectors of \mathbf{A} and row vectors of \mathbf{P}^{-1} are left-hand eigenvectors.

When the matrix \mathbf{A} is symmetric, the eigenvectors can be chosen to form an orthonormal basis of \mathbb{R}^N . In this case, \mathbf{P} is an orthogonal matrix and \mathbf{P}^{-1} equals the transpose of \mathbf{P} . The diagonal elements of \mathbf{D} are eigenvalues of both \mathbf{A} as \mathbf{D} .

When a diagonalizable matrix \mathbf{A} is non-symmetric, the orthogonality of eigenvectors fails. Nevertheless, since neither the orthogonality of right eigenvectors nor that of left ones is necessary in our approach, this problem can be overpassed. What we need, is the “mutual orthogonality” of the left and right eigenvectors. Then, if Φ and Ψ denote matrices of left and right-hand eigenvectors of a square matrix \mathbf{A} , the matrix $\Phi\mathbf{A}\Psi$ is diagonal. Its diagonal elements are not necessarily the eigenvalues of \mathbf{A} .

We state our main result as a theorem. In its proof, we use a similar method as in the proof of Theorem in [2].

From Kirchhoff’s law we know that at least one eigenvalue of Laplacian matrix \mathbf{L} is zero. We consider this eigenvalue simple and put $\lambda_1 = 0$. The associated left and right-hand eigenvectors are $\varphi_1 = (1, 1, \dots, 1) = \psi_1^*$. The condition of orthogonality of left and right-hand eigenvectors ($\varphi_i^* \psi_j = 0 \Leftrightarrow (i \neq j)$) can be fulfilled for arbitrary diagonalizable matrix that satisfies any other condition of the following theorem [8].

THEOREM. Consider a resistance network with N nodes defined by a diagonalizable Laplacian matrix \mathbf{L} not necessarily symmetric. Suppose that \mathbf{L} has N real eigenvalues λ_i (including the multiplicity) with left and right-hand eigenvectors $\varphi_i^* = (\varphi_{i1}, \dots, \varphi_{iN})$ and $\psi_i = (\psi_{i1}, \dots, \psi_{iN})^*$, where φ_i and ψ_j are orthogonal if and

only if $i \neq j$. Denote λ_1 the zero eigenvalue. Then the resistance between nodes α and β is given by

$$R_{\alpha\beta} = \sum_{i=2}^N \frac{1}{\lambda_i \varphi_i^* \psi_i} (\varphi_{i\alpha} - \varphi_{i\beta})(\psi_{i\alpha} - \psi_{i\beta}). \quad (11)$$

Proof. Since one of eigenvalues of \mathbf{L} is zero, introducing of $\epsilon\mathbf{L}$ to the Laplacian \mathbf{L} will be helpful. Thus we will manipulate the Laplacian $\mathbf{L}(\epsilon) = \mathbf{L} + \epsilon\mathbf{L}$. The eigenvalues of $\mathbf{L}(\epsilon)$ are $\epsilon, \lambda_2 + \epsilon, \dots, \lambda_N + \epsilon$ and for a suitable value of ϵ , no one of them equals zero. The left and right-hand eigenvectors are those of \mathbf{L} . Similarly as in [2], we will compute the inverse function $\mathbf{G}(\epsilon)$ of $\mathbf{L}(\epsilon)$.

Naturally, $\mathbf{L}(\epsilon)$ is not symmetric, because \mathbf{L} is not. Non-symmetric matrices \mathbf{L} and $\mathbf{L}(\epsilon)$ are diagonalizable, ie

$$\Phi\mathbf{L}\Psi = \Lambda \quad \text{and} \quad \Phi\mathbf{L}(\epsilon)\Psi = \Lambda(\epsilon), \quad (12)$$

where Φ is the matrix of left-hand eigenvectors (rows) and Ψ that of right-hand eigenvectors (columns) of \mathbf{L} and $\mathbf{L}(\epsilon)$. Finally, $\Lambda(\epsilon)$ is a diagonal matrix whose diagonal elements are $(\lambda_i + \epsilon)\varphi_i^* \psi_i$.

Both Φ and Ψ are invertible, because $\{\varphi_1, \dots, \varphi_N\}$ as well as $\{\psi_1, \dots, \psi_N\}$ define a basis of \mathbb{R}^N . From (12) we obtain

$$\Psi^{-1}\mathbf{L}^{-1}(\epsilon)\Phi^{-1} = \Lambda^{-1}(\epsilon). \quad (13)$$

Hence

$$\mathbf{G}(\epsilon) = \Psi\Lambda^{-1}(\epsilon)\Phi. \quad (14)$$

Really, $\mathbf{G}(\epsilon)$ is an inverse matrix of $\mathbf{L}(\epsilon)$, because

$$\mathbf{G}(\epsilon)\mathbf{L}(\epsilon) = (\Psi\Lambda^{-1}(\epsilon)\Phi)(\Phi^{-1}\Lambda(\epsilon)\Psi^{-1}) = \mathbf{I}.$$

From (14) it follows that elements of the matrix $\mathbf{G}(\epsilon)$ are

$$\mathbf{G}_{\alpha\beta}(\epsilon) = \sum_{i=1}^N \psi_{i\alpha} \frac{1}{(\lambda_i + \epsilon)\varphi_i^* \psi_i} \varphi_{i\beta} = \frac{1}{N\epsilon} + \sum_{i=2}^N \frac{\psi_{i\alpha}\varphi_{i\beta}}{(\lambda_i + \epsilon)\varphi_i^* \psi_i}. \quad (15)$$

So, the function $\mathbf{G}(\epsilon)$ is explicitly expressed.

Now, return to the equation $\mathbf{L}(\epsilon)\mathbf{V}(\epsilon) = \mathbf{I}$ and multiply it from the left by $\mathbf{G}(\epsilon)$. We obtain $\mathbf{V}(\epsilon) = \mathbf{G}(\epsilon)\mathbf{I}$. Coordinates of the vector $\mathbf{V}(\epsilon)$ are

$$V_i(\epsilon) = \sum_{j=1}^N G_{ij}(\epsilon)I_j \quad \text{for} \quad i = 1, 2, \dots, N. \quad (16)$$

Denote in (15)

$$g_{\alpha\beta}(\epsilon) = \sum_{i=2}^N \frac{\psi_{i\alpha}\varphi_{i\beta}}{(\lambda_i + \epsilon)\varphi_i^* \psi_i}. \quad (17)$$

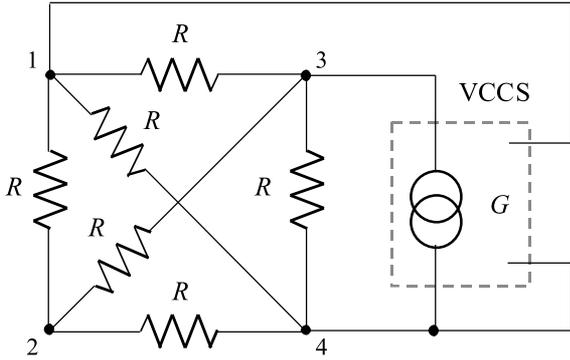


Fig. 2. A network of 4 nodes with VCCS

The substitution of $G_{ij}(\epsilon) = (N\epsilon)^{-1} + g_{ij}(\epsilon)$ in (16) yields

$$V_i(\epsilon) = \sum_{j=1}^N \left(\frac{1}{N\epsilon} + g_{ij}(\epsilon) \right) I_j = \frac{1}{N\epsilon} \sum_{j=1}^N I_j + \sum_{j=1}^N g_{ij}(\epsilon) I_j. \quad (18)$$

Since $\sum_{i=1}^N I_j = 0$, we have $V_i(\epsilon) = \sum_{i=1}^N g_{ij}(\epsilon) I_j$ and we can put $\epsilon = 0$. This gives

$$V_i = \sum_{i=1}^N g_{ij} I_j, \quad (19)$$

where V_i and g_{ij} replace $V_i(0)$ and $g_{ij}(0)$.

We use (19) in $R_{\alpha\beta} = I^{-1}(V_\alpha - V_\beta)$ which is (7)

$$R_{\alpha\beta} = \frac{1}{I} \left(\sum_{j=1}^N g_{\alpha j} I_j - \sum_{j=1}^N g_{\beta j} I_j \right) = \sum_{j=1}^N (g_{\alpha j} - g_{\beta j}) \frac{I_j}{I}. \quad (20)$$

Due to (8) we obtain

$$R_{\alpha\beta} = \sum_{j=1}^N (g_{\alpha j} - g_{\beta j}) (\delta_{j\alpha} - \delta_{j\beta}). \quad (21)$$

The last expression can be simplified as

$$R_{\alpha\beta} = (g_{\alpha\alpha} - g_{\beta\alpha})(1 - 0) + (g_{\alpha\beta} - g_{\beta\beta})(0 - 1). \quad (22)$$

We introduce (18) with $\epsilon = 0$ and (22) becomes

$$R_{\alpha\beta} = \sum_{i=2}^N \frac{\psi_{i\alpha} \varphi_{i\alpha}}{\lambda_i \varphi_i^* \psi_i} - \sum_{i=2}^N \frac{\psi_{i\beta} \varphi_{i\alpha}}{\lambda_i \varphi_i^* \psi_i} + \sum_{i=2}^N \frac{\psi_{i\beta} \varphi_{i\beta}}{\lambda_i \varphi_i^* \psi_i} - \sum_{i=2}^N \frac{\psi_{i\alpha} \varphi_{i\beta}}{\lambda_i \varphi_i^* \psi_i}. \quad (23)$$

A simple rearrangement of the last relation returns (2).

4 EXAMPLES

In the following examples, the proposed resistance calculation approach is demonstrated on simple networks (circuits). The calculations were conducted using *GNU Octave* language. The circuit analysis results were obtained from *Cadence IC* design environment.

EXAMPLE 1. Consider the network depicted in Fig. 2. Let $c_1 = 1/R$. For this network, the following non-symmetrical nodal conductance matrix, which is the Laplacian \mathbf{L} , can be created.

$$\mathbf{L} = \begin{pmatrix} 3c_1 & -c_1 & -c_1 & -c_1 \\ -c_1 & 3c_1 & -c_1 & -c_1 \\ -c_1 + G & -c_1 & 3c_1 & -c_1 - G \\ -c_1 - G & -c_1 & -c_1 & 3c_1 + G \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 4c_1$, $\lambda_4 = 4c_1 + G$. Left and right-hand eigenvectors φ_i and ψ_i of this matrix are

$$\Phi^T = \begin{pmatrix} \varphi_1^* & \varphi_2^* & \varphi_3^* & \varphi_4^* \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -3 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

As can be observed, the left and right-hand eigenvectors are independent of c_1 and G . An example calculation of the first row of the resistance matrix \mathbf{R}_c is given below. We can use substitution $c_1 = 1/R$ for this calculation.

$$\frac{1}{\lambda_2 \varphi_2^* \psi_2} = -\frac{R}{16}, \quad \frac{1}{\lambda_3 \varphi_3^* \psi_3} = \frac{R}{8}, \quad \frac{1}{\lambda_4 \varphi_4^* \psi_4} = \frac{R}{4 + GR},$$

$$R_{12} = -\frac{R}{16}(-1+1)(1-1) + \frac{R}{8}(1+1)(1+1) + \frac{R}{4+GR}(1-0)(0-0) = 0 + \frac{R}{8}4 + 0 = \frac{R}{2},$$

$$R_{13} = -\frac{R}{16}(-1-1)(1+3) + \frac{R}{8}(1-0)(1+1) + \frac{R}{4+GR}(1-0)(0-1) = \frac{R}{2} + \frac{R}{4} - \frac{R}{4+GR} = R \left(\frac{3}{4} - \frac{1}{4+GR} \right),$$

$$R_{14} = -\frac{R}{16}(-1-1)(1-1) + \frac{R}{8}(1-0)(1-1) + \frac{R}{4+GR}(1+1)(0+1) = 0 + 0 + \frac{2R}{4+GR}.$$

Now, set the values of parameters $R = 1\Omega$ and $G = 2S$. The values of resistances are $R_{12} = 1/2$, $R_{13} = 7/12$ and $R_{14} = 1/3$. The complete resulting matrices of calculated resistance \mathbf{R}_c and resistance from circuit analysis \mathbf{R}_a are shown below. One can observe that $\mathbf{R}_c = \mathbf{R}_a$.

$$\mathbf{R}_c = \begin{pmatrix} 0 & 1/2 & 7/12 & 1/3 \\ 0 & 0 & 1/2 & 5/12 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{R}_a = \begin{pmatrix} 0 & 5.000 \times 10^{-1} & 5.833 \times 10^{-1} & 3.333 \times 10^{-1} \\ 0 & 0 & 5.000 \times 10^{-1} & 4.167 \times 10^{-1} \\ 0 & 0 & 0 & 3.333 \times 10^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

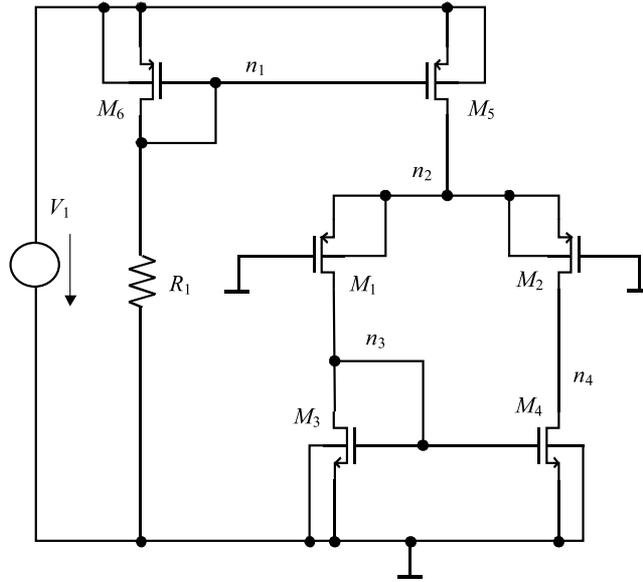


Fig. 3. Circuit schematic of an operational amplifier

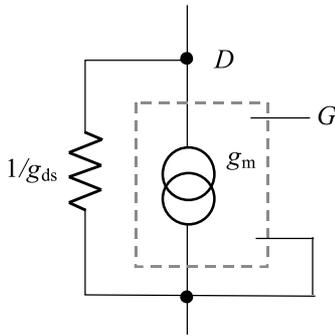


Fig. 4. Simplified electrical model of a MOS transistor

Table 1. Device parameters of the operational amplifier

	gm (S)	gds (S)	Ron(Ω)
M1	5.155032×10^{-4}	2.816268×10^{-3}	–
M2	5.155032×10^{-4}	2.816268×10^{-3}	–
M3	5.953932×10^{-4}	3.403570×10^{-6}	–
M4	5.953932×10^{-4}	3.403570×10^{-6}	–
M5	4.195651×10^{-3}	9.344114×10^{-6}	–
M6	5.247305×10^{-4}	2.065995×10^{-6}	–
R1	–	–	6.847545×10^4

EXAMPLE 2 (OPERATIONAL AMPLIFIER). The circuit depicted in Fig. 3 represents a typical topology of an operational amplifier that is one of the most frequently used analog building block.

This particular circuit has been designed for a standard $0.35 \mu\text{m}$ mixed-signal CMOS process. Parameters of the devices obtained from a typical .op (operating point) analysis are shown in Tab. 1.

These parameters are used to create an electrical model of the circuit, where only resistances and voltage-controlled current sources (VCCS) are used. MOS transistor is modeled by a VCCS, which is represented by a

transconductance gm and a resistor of value $1/g_{ds}$ connected between its output terminals, as depicted in Fig. 4.

Using the parameters from Tab. 1, the following non-symmetrical nodal admittance matrix \mathbf{L} of this circuit can be created for the internal nodes.

$$\mathbf{L} = \begin{pmatrix} 5.414 \times 10^{-4} & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 4.196 \times 10^{-3} & 6.673 \times 10^{-3} & -2.816 \times 10^{-3} & 0.000 \times 10^0 & -2.816 \times 10^{-3} & -5.236 \times 10^{-3} \\ 0.000 & -3.332 \times 10^{-3} & 3.415 \times 10^{-3} & 2.820 \times 10^{-3} & -8.329 \times 10^{-5} & -8.329 \times 10^{-5} \\ 0.000 & -3.332 \times 10^{-3} & 5.954 \times 10^{-4} & -3.404 \times 10^{-6} & 5.944 \times 10^{-3} & -3.404 \times 10^{-6} \\ -4.737 \times 10^{-3} & -9.344 \times 10^{-6} & -1.194 \times 10^{-3} & 0.000 & -5.414 \times 10^{-4} & -2.816 \times 10^{-3} \end{pmatrix}$$

The node V_{dd} is excluded from this matrix, since when assuming the impedance between two nodes, all the voltage and current sources have to be substituted: a voltage source is substituted by a short, while a current source is omitted (open circuit). In our case, it means that the node V_{dd} becomes shorted to node gnd . The order of network nodes assumed for this matrix is $n1, n2, n3, n4, gnd$.

For the nodal admittance matrix, which is the Laplacian \mathbf{L} , the eigenvalues obtained are $\lambda_1 = 0, \lambda_2 = 1.617872 \times 10^{-4}, \lambda_3 = 2.819672 \times 10^{-3}, \lambda_4 = 8.008586 \times 10^{-3}, \lambda_5 = 8.402968 \times 10^{-3}$.

The corresponding left and right-hand eigenvectors (in columns) of the matrix \mathbf{L} are as follows:

$$\Phi^T = \begin{pmatrix} \varphi_1^* & \varphi_2^* & \varphi_3^* & \varphi_4^* & \varphi_5^* \\ 0.447214 & 0.890916 & 0 & 0.606347 & -0.596298 \\ 0.447214 & -0.250715 & 0 & 0.233943 & -0.286327 \\ 0.447214 & -0.223688 & -0.707107 & 0.034793 & 0.002639 \\ 0.447214 & -0.265848 & 0.707107 & -0.126481 & 0.143977 \\ 0.447214 & -0.150665 & 0 & -0.748603 & 0.736009 \end{pmatrix}$$

$\Psi =$

$$\begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 \\ 0.447214 & -0.638056 & -0.038249 & 0.023987 & -0.017999 \\ 0.447214 & 0.363601 & 0.098610 & -0.654286 & 0.698457 \\ 0.447214 & 0.360920 & 0.574328 & 0.480565 & -0.470914 \\ 0.447214 & 0.360920 & -0.795646 & 0.480565 & -0.470914 \\ 0.447214 & -0.447385 & 0.160957 & -0.330831 & 0.261369 \end{pmatrix}$$

The resulting matrices of the numerically calculated resistance \mathbf{R}_c and the resistance from the circuit analysis \mathbf{R}_a are shown below. One can observe that

$\mathbf{R}_c = \mathbf{R}_a =$

$$\begin{pmatrix} 0 & 9.276302 \times 10^3 & 9.098426 \times 10^3 & 9.448754 \times 10^3 & 1.847062 \times 10^3 \\ 0 & 0 & 3.000575 \times 10^2 & 3.546762 \times 10^2 & 8.490922 \times 10^2 \\ 0 & 0 & 0 & 7.093024 \times 10^2 & 8.317060 \times 10^2 \\ 0 & 0 & 0 & 0 & 1.182034 \times 10^3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5 CONCLUSION

A formula for resistance computation between two arbitrary nodes of a network with non-symmetric Laplacian matrix \mathbf{L} has been formulated and proved in this paper. The calculation is based on eigenvalues and eigenvectors of the Laplacian matrix. The application of the formula is demonstrated on two examples of networks (circuits): a resistive bridges and an operational amplifier. The calculated results are compared with the results from a circuit simulator.

In electric circuit theory, resistor networks are a special case of impedance networks. In general, these are defined by a complex Laplacian matrix \mathbf{L} . Our numerical results indicate, that Theorem formulated in this paper holds also for systems with complex Laplacian matrix. Therefore, (2) should be applicable to computation of the complex impedance as well.

Acknowledgments

This work was supported in part by the Ministry of Education, Science, Research and Sport of the Slovak Republic under grant VEGA 1/0823/13 and by ENIAC JU under project E2SG (Grant Agr. No. 296131).

REFERENCES

- [1] OWAI DAT, M. Q.—HIJJAWI, R. S.—M QHALIFEH, J. M. Interstitial Single Resistor in a Network of Resistors Application of the Lattice Green's Vs Function: *J. Phys. A: Math. Theor.* **43**, 375204.
- [2] WU, F. Y.: Theory of Resistor Networks: the Two-Point Resistance, *J. Phys. A: Mathematical and General* **37** (2004), 6653–73.
- [3] KILBY, J. S.: Turning Potential into Realities: The Invention of the Integrated Circuit (Nobel lecture), *ChemPhysChem* **2** (2001), 482–489.
- [4] VLACH, J.—SINGHAL, K.: *Computer Methods for Circuits Analysis and Design*, Van Nostrand Reinhold press, 1994.

- [5] BRENKUŠ, J.—STOPJAKOVÁ, V.—GYEPES, G. Numerical Method for dc Fault Analysis Simplification and Simulation Time Reduction: in: *Design and Diagnostics of Electronic Circuits and Systems, DDECS 2013. Proceedings of the 16th IEEE International Symposium on*, 2013, pp. 170–174.
- [6] TSIATOUHAS, Y.—MOISIADIS, Y.—HANIOTAKIS, T.—NIKOLOS, D.—ARAPOYANNI, A.: A New Technique for {IDDQ}, Testing in Nanometer Technologies, Integration, the {VLSI}, *Journal* **31** No. 2 (2002), 183–194.
- [7] BERESINSKI, M.—BOREJKO, T.—PLESKACZ, W.—STOPJAKOVA, V.: Built-in Current Monitor for iddq Testing in cmos 90 nm technology, in: *Design and Diagnostics of Electronic Circuits and Systems, DDECS 2008, 11th IEEE Workshop on*, 2008, pp. 1–4.
- [8] ČERŇANOVÁ, V. On an Explicit Solution of a Finite Network Problem: in: *ODAM 2013*.

Received 12 December 2013

Viera Čerňanová obtained her degree of MS (1984, RNDr) in Mathematical Analysis from the Comenius University, Bratislava, and lectured at this University. In 1990 became a teacher of mathematics in French language at a bilingual high school. She was a teacher-specialist, member of international commissions for creating and testing programs, fun of technologies in teaching mathematics, specialist in creating tests, researcher. In 2006 returned into the academic milieu and obtained her degree of PhD (2009) in Theory of Teaching Mathematics from the Comenius University. Since 2009 she teaches Mathematical Analysis and Logic Systems at Slovak University of Technology in Bratislava, Faculty of Electrical Engineering. Her domains of research are Applied Mathematics and Theory of Teaching Mathematics.

Juraj Brenkuš received the MS and PhD degrees in Electronics from Slovak University of Technology (STU) in Bratislava, Slovakia, in 2006 and 2013, respectively. Since September 2013 he has been working at Institute of Electronics and Photonics of the same university as a researcher. He has been involved in EU funded research projects ENIAC-JU END and E2SG and several grants funded by the Slovak Ministry of Education. He is the author or co-author of more than 25 papers presented at international conferences or published in journals. His main research interests include analog and mixed IC design and testing, systems-on-chip, on-chip testing, test methodologies and test development, test development automation and wireless systems.

Viera Stopjaková received the MS degree and the PhD degree in Electronics from Slovak University of Technology in Bratislava, Slovakia in 1992 and 1997, respectively. From October 2003 to June 2009 she was an associated professor at Microelectronics Department, Faculty of Electrical Engineering and Information Technology of Slovak University of Technology in Bratislava. Currently, she is a professor at the Institute of Electronics and Photonics of the same university. She has been involved in several EU funded research projects under different funding schemes such as TEMPUS, ESPRIT, Copernicus, FP, ENIAC-JU, etc. She has published over 120 papers in various journals and conference proceedings; and she is a co-inventor of two US patents in the field of on-chip supply current testing. Her main research interests include IC design, VLSI & SoC testing, on-chip testing, design and test of mixed-signal circuits, e-health, biomonitoring and biomedical applications.