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Precision of sinewave amplitude estimation in the presence of additive noise and quantization error

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This research paper delves into a comprehensive investigation concerning the impact of additive noise and quantization error on the precision of amplitude, offset, and phase estimates of a sine wave fitted to a set of data points acquired by a waveform digitizer. Simulation results are used to validate the expressions presented.

Keywords: sinewave, fitting, uncertainty, analog-to-digital conversion, ADC, quantization

1 Introduction

The procedure of fitting data points to a known function finds widespread application across diverse engineering fields. One notable context is dealing with sinusoidal signals, where the objective is to accurately fit the data points to a sine wave with unknown parameters, namely, amplitude, offset, initial phase, and frequency. This fitting process is particularly useful in testing Analog-to-Digital Converters (ADCs) [1], even when there is frequency error in the stimulus signal [5], voltage noise [6] or jitter [7]. By fitting the data to a sine wave, engineers can gain valuable insights into the performance and characteristics of these converters. This paper is an extended version of [8].

To achieve the best fitting results, a least squares approach is commonly employed [9], which minimizes the sum of squared residuals between the observed data points and the values predicted by the sine wave model. When the data is affected by additive white Gaussian noise, this fitting method delivers excellent outcomes. In fact, it is proven to be the best linear unbiased estimator of the sine wave coefficients according to the Gauss-Markov Theorem, provided that certain conditions are met. These conditions require that the errors affecting the data have zero expectation (*ie*, they are unbiased), are uncorrelated (though not necessarily independent), and have equal variances (though not necessarily identically distributed). There are, however, other non-idealities like phase noise and jitter [11] or power supply noise [12] that affect the estimation results. The effect of these is not the subject of this paper.

However, when dealing with digital data, such as that obtained from quantization by an analog-to-digital converter, new challenges arise. The data points to be fitted to the sine wave are affected by quantization error [13], which can visibly impact the performance of the fitting procedure. Quantization noise can introduce bias and alter the precision of the parameter estimates, as the errors are no longer uncorrelated due to the quantization process.

In this paper, the focus is on analyzing the specific effects of three critical factors: the number of samples (N), the quantization step (Q) and the standard deviation of the additive noise (σ_n). The goal is to evaluate the standard deviation of the estimates of amplitude, offset, and phase at the origin of time (initial phase) and determine the precision of the sine wave fitting results. Understanding the precision of the parameter estimates is vital for assessing the reliability and accuracy of the fitting process.

Furthermore, this paper presents an expression that helps determine the minimum number of samples required to achieve a desired precision for the sine wave parameters. This objective is essential for optimizing data acquisition strategies and minimizing data collection time while maintaining acceptable accuracy levels.

By thoroughly investigating the impact of sample size and noise standard deviation on parameter estimates, the paper aims to provide valuable insights for practitioners in the field. Understanding these effects allows engineers and researchers to make informed decisions regarding data collection, signal processing,

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and model fitting in various applications involving sinusoidal signals and digital data. Ultimately, the research contributes to enhancing the robustness and accuracy of parameter estimation techniques in the presence of quantization noise and limited data samples.

2 Sine wave fitting

Consider a sequence of N data points $y_1, y_2, \ldots, y_N$ to which we want to fit a sine wave given by $C + A \cdot \cos(\omega t_i + \varphi)$, where φ is the phase at the time origin, ω is the angular frequency ($2\pi f$), A is the sinewave amplitude and C is the sinewave average value. The model is

$$y_i = round\left(\frac{C + A\cdot\cos(\omega t_i + \varphi)}{Q}\right)Q + n_i\,, \tag{1}$$

where n is the additive white Gaussian noise with zero mean and Q the quantization step. In order to have a linear system, the cosine function is split and the following model is obtained:

$$y_i = round\left(\frac{C + A_1\cdot\cos(\omega t_i) + A_2\sin(\omega t_i)}{Q}\right)Q + n_i\,. \tag{2}$$

Parameters A_1 and A_2 are related with the sine wave amplitude (A) and initial phase (φ):

$$\begin{aligned} A_1 &= A\cos(\varphi) \\ A_2 &= A\sin(\varphi) \end{aligned} \tag{3}$$

The estimates of the sine wave are obtained, in a matrix form, with [10]

$$\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} = (\boldsymbol{D}^T\boldsymbol{D})^{-1}D^T\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_N \end{bmatrix}. \tag{4}$$

Matrix $\boldsymbol{D}$ is constructed with three columns and N rows:

$$\boldsymbol{D} = \begin{bmatrix} cos(\omega t_1) & sin(\omega t_1) & 1 \\ cos(\omega t_2) & sin(\omega t_2) & 1 \\ \cdots & \cdots & \cdots \\ cos(\omega t_N) & sin(\omega t_N) & 1 \end{bmatrix}. \tag{5}$$

Introducing (5) into (4) gives

$$\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} = \begin{bmatrix} W & S & O \\ S & R & P \\ O & P & N \end{bmatrix}^{-1}\begin{bmatrix} U \\ V \\ Y \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned} O &= \sum_i \cos(\omega t_i) & P &= \sum_i \sin(\omega t_i) \\ W &= \sum_i \cos^2(\omega t_i) & R &= \sum_i \sin^2(\omega t_i) \\ S &= \sum_i \cos(\omega t_i)\sin(\omega t_i) & Y &= \sum_i y_i \\ U &= \sum_i y_i\cos(\omega t_i) & V &= \sum_i y_i\sin(\omega t_i) \end{aligned} \tag{7}$$

Inverting the matrix in (6) leads to

$$\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} = \frac{\begin{bmatrix} NR-P^2 & OP-NS & PS-OR \\ OP-NS & NW-O^2 & OS-PW \\ PS-OR & OS-PW & WR-S^2 \end{bmatrix}}{NWR-P^2W+OPS-NS^2+OPS-O^2R}\begin{bmatrix} U \\ V \\ Y \end{bmatrix}. \tag{8}$$

Carrying out the multiplication leads to

$$\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} = \frac{\begin{bmatrix} NRU-P^2U+OPV-NSV+PSY-ORY \\ OPU-NSU+NWV-O^2V+OSY-PWY \\ PSU-ORU+OSV-PWV+WRY-S^2Y \end{bmatrix}}{NWR-P^2W+OPS-NS^2+OPS-O^2R}\,. \tag{9}$$

We thus see how the estimative of the three sinewave parameters is obtained from the summations in (7).

3 Precision of the estimates without quantization

Considering an additive noise with a standard deviation of σ_n, we can derive, from (9), the standard deviation of the estimate of the sine wave parameters in least significant bit units (LSB):

$$\begin{aligned} \sigma_{A_1}^2 &= \frac{NR\sigma_U^2 - P^2\sigma_U^2 + OP\sigma_V^2 - NS\sigma_V^2 + PS\sigma_Y^2 - OR\sigma_Y^2}{\left(NWR-P^2W+OPS-NS^2+OPS-O^2R\right)^2} \\ \sigma_{A_2}^2 &= \frac{OP\sigma_U^2 - NS\sigma_U^2 + NQ\sigma_V^2 - O^2\sigma_V^2 + OS\sigma_Y^2 - PQ\sigma_Y^2}{NWR-P^2W+OPS-NS^2+OPS-O^2R}\,. \\ \sigma_C^2 &= \frac{PS\sigma_U^2 - OR\sigma_U^2 + OS\sigma_V^2 - PW\sigma_V^2 + WR\sigma_Y^2 - S^2\sigma_Y^2}{NWR-P^2W+OPS-NS^2+OPS-O^2R} \end{aligned} \tag{10}$$

Considering (7) we have

$$\begin{aligned} \sigma_U^2 &= \sum_i \sigma_{y_i\cos(\omega t_i)}^2 = \sum_i \sigma_{y_i}^2\cos^2(\omega t_i) = W\sigma_n^2 \\ \sigma_V^2 &= \sum_i \sigma_{y_i\sin(\omega t_i)}^2 = \sum_i \sigma_{y_i}^2\sin^2(\omega t_i) = R\sigma_n^2 \\ \sigma_Y^2 &= \sum_i \sigma_{y_i}^2 = N\sigma_n^2 \end{aligned} \tag{11}$$

and

$$\begin{aligned}\sigma_{A_1}^2 &= \frac{NRW - P^2W + OPR - NSR + PSN - ORN}{\left(NWR - P^2W + OPS - NS^2 + OPS - O^2R\right)^2}\sigma_n^2 \\ \sigma_{A_2}^2 &= \frac{OPW - NSW + NWR - O^2R + OSN - PWN}{NWR - P^2W + OPS - NS^2 + OPS - O^2R}\sigma_n^2 . \\ \sigma_C^2 &= \frac{PSW - ORW + OSR - PWR + WRN - S^2N}{NWR - P^2W + OPS - NS^2 + OPS - O^2R}\sigma_n^2\end{aligned} \quad (12)$$

Considering the sampling instants equal to i/f_s, where i is the sample index and f_s is the sampling frequency, we can write

$$\sum_i \cos^2(\omega t_i) = \sum_i \cos^2\left(2\pi \frac{f}{f_s} i\right). \quad (13)$$

Considering the acquisition during one period of the sine wave, we have $f_s=N\times f$ and thus

$$\sum_i \cos^2(\omega t_i) = \sum_i \cos^2\left(2\pi \frac{i}{N}\right). \quad (14)$$

As the number of samples tends to ∞ the summations can be approximated by an integral.

$$\begin{aligned}&\sum_i \cos^2\left(2\pi\frac{i}{N}\right) = N\left[\frac{1}{N}\sum_i \cos^2\left(2\pi\frac{i}{N}\right)\right] \\ &\xrightarrow[N\to\infty]{} N\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2(x)\,dx\right] = \frac{N}{2}\end{aligned} . \quad (15)$$

Considering (7) and applying the same reasoning we have then

$$O = 0 \quad P = 0 \quad W = \frac{N}{2} \quad R = \frac{N}{2} \quad S = 0 . \quad (16)$$

Then, introducing into (12) , we obtain

$$\sigma_{A_1} = \frac{\sigma_n}{\sqrt{N/2}} \quad , \quad \sigma_{A_2} = \frac{\sigma_n}{\sqrt{N/2}} \quad , \quad \sigma_C = \frac{\sigma_n}{\sqrt{N}} . \quad (17)$$

To determine the variance of A and φ we can write (3) as

$$\begin{aligned}A &= \sqrt{A_1^2 + A_2^2} \\ \varphi &= \begin{cases} \operatorname{atan}\left(\dfrac{-A_2}{A_1}\right) & , \ A_1 \ge 0 \\ \operatorname{atan}\left(\dfrac{-A_2}{A_1}\right) + \pi & , \ A_1 < 0 \end{cases}\end{aligned} . \quad (18)$$

The variance of the estimated amplitude A will depend on the variances of A_1 and A_2. To determine this variance from (18) we can use the rule about the variance of a function of a random variable ([14], p. 113). It leads to:

$$\sigma_A^2 = \left(\frac{d\sqrt{A_1^2 + A_2^2}}{A_1^2 + A_2^2}\right)^2\Bigg|_{A_1^2 + A_2^2 = \mu_{A_1^2 + A_2^2}} \times \sigma_{A_1^2 + A_2^2}^2 . \quad (19)$$

Calculating the derivative leads to

$$\sigma_A^2 = \left(\frac{1}{2\sqrt{\mu_{A_1^2 + A_2^2}}}\right)^2 \times \sigma_{A_1^2 + A_2^2}^2 . \quad (20)$$

After some simplification it leads to

$$\sigma_A^2 = \frac{1}{4\left(\mu_{A_1^2 + A_2^2}\right)} \times \left(\sigma_{A_1^2}^2 + \sigma_{A_2^2}^2\right). \quad (21)$$

Again, using the same rule, we get

$$\sigma_A^2 = \sigma_{A_1}^2 . \quad (22)$$

Introducing (17) leads finally to

$$\sigma_A = \frac{\sigma_n}{\sqrt{N/2}} . \quad (23)$$

For the variance of the estimated phase we can use the same rule of the variance of a function of a random variable to write, from (18),

$$\sigma_\varphi^2 = \left(\frac{d\left(\operatorname{atan}\left(\dfrac{-A_2}{A_1}\right)\right)}{\dfrac{-A_2}{A_1}}\right)^2\Bigg|_{\frac{-A_2}{A_1} = \mu_{\frac{-A_2}{A_1}}} \times \sigma_{\frac{-A_2}{A_1}}^2 . \quad (24)$$

Calculating the derivative leads to

$$\sigma_\varphi^2 = \frac{\sigma_{\frac{-A_2}{A_1}}^2}{\left(1 + \left(\dfrac{\mu_{A_2}}{\mu_{A_1}}\right)^2\right)^2} . \quad (25)$$

On the other hand, from [15], we have

$$\sigma_{\frac{-A_2}{A_1}}^2 = \frac{\mu_{A_2}^2}{\mu_{A_1}^4}\sigma_{A_1}^2 + \frac{1}{\mu_{A_1}^2}\sigma_{A_2}^2 . \quad (26)$$

Since the variance of both A_1 and A_2 is the same, we can write

$$\sigma_{\frac{-A_2}{A_1}}^2 = \frac{\mu_{A_1}^2 + \mu_{A_2}^2}{\mu_{A_1}^4}\sigma_{A_1}^2 . \quad (27)$$

Introducing into (25) leads to

$$\sigma_\varphi = \frac{\sigma_{A_1}}{\mu_A}. \tag{28}$$

Using (17) and considering that the estimate of the amplitude is not biased ($\mu_A = A$), we have

$$\sigma_\varphi = \frac{\sigma_n}{\sqrt{N/2} \cdot A}. \tag{29}$$

Summarizing, we have, from expressions (17), (23) and (29):

$$\sigma_A = \frac{\sigma_n}{\sqrt{N/2}} \quad , \quad \sigma_C = \frac{\sigma_n}{\sqrt{N}} \quad , \quad \sigma_\varphi = \frac{\sigma_n}{A\sqrt{N/2}}. \tag{30}$$

We see that the standard deviation of the estimates is proportional to the standard deviation of the amount of additive noise present and inversely proportional to the square root of the number of samples.

4 Effect of quantization noise

When the data is digitalized through the quantization process, a new source of error is introduced, known as quantization error. This error arises from the fact that the analog signal is represented by discrete digital values, leading to a loss of information. The quantization error affects the estimates of the sine wave parameters obtained through the fitting method.

To understand and model the impact of quantization error on the parameter estimates, we treat it as a random variable with a null mean and a standard deviation of $Q/\sqrt{12}$ [11].

This assumption is a good approximation under certain conditions, particularly when there is also additive white Gaussian noise present in the data. The presence of this type of noise ensures that the digitalization errors introduced by quantization are uncorrelated.

The model, using this approximation, is

$$y_i = C + A_1 \cdot \cos(\omega t_i) + A_2 \sin(\omega t_i) + n_i + q_i, \tag{31}$$

where q is the quantization error. We can treat this as if the variance of the noise is now the sum of the variances of both additive noise and quantization noise, and so σ_n^2 becomes $\sigma_n^2 + Q^2/12$.

Using this in (30) leads to

$$\sigma_A = \frac{\sqrt{\sigma_n^2 + \frac{Q^2}{12}}}{\sqrt{N/2}} \quad \sigma_C = \frac{\sqrt{\sigma_n^2 + \frac{Q^2}{12}}}{\sqrt{N}} \quad \sigma_\varphi = \frac{\sqrt{\sigma_n^2 + \frac{Q^2}{12}}}{\sqrt{N/2} \cdot A}. \tag{32}$$

Note that, as expected, the precision of the estimates increases (standard deviation decreases) with increasing number of samples.

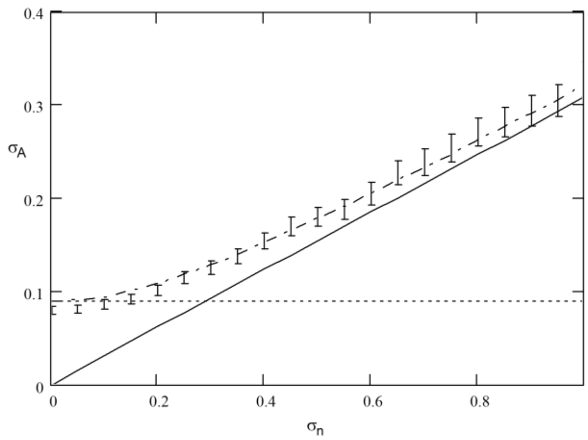

Fig. 1. Standard deviation of the estimated amplitude of a fitted sine wave as a function of additive noise standard deviation

Both quantities are normalized to the quantization step. The vertical bars correspond to a confidence interval of 99%. The dash-dot line represents the value given by (32), the dotted and solid lines represents the asymptotic approximation for low and high values, respectively, of additive noise standard deviation. The simulations were carried out with 21 points extracted from one period of a sine wave with A = 100 LSB, C = 0 and φ=π/4.

5 Validation

To validate the expressions derived, we performed some simulations of (1) with different values of additive noise standard deviation. The standard deviation of the sine wave parameters estimates was determined by repeating the fitting 1000 times with a random varying phase at the origin of time corresponding to asynchronous sample acquisition.

In Fig. 1, the relationship between the standard deviation of the estimated amplitude and the additive noise standard deviation is depicted. Both variables are normalized to the ideal quantization step Q, providing a convenient comparison and scaling of the results.

The simulated standard deviation of the estimated amplitude is represented by vertical bars, which correspond to a confidence interval of 99%. These bars indicate the range within which the estimated amplitude is likely to fall based on the simulation results.

The dash-dot line, represented by equation (32), shows the theoretical standard deviation of the estimated amplitude. This value is calculated based on the model that takes into account the additive noise and quantization error present.

Interestingly, in the graph, it can be observed that the simulated standard deviation is consistently lower than the theoretical value given by equation (32). This discrepancy is due to various factors, including the assumptions and approximations made in the theoretical model. Despite the theoretical prediction providing an overall understanding of the relationship between the variables, the simulation results reveal the practical limitations and uncertainties encountered in real data scenarios.

Furthermore, the graph features two additional lines for asymptotic approximations. The dotted line represents the asymptotic approximation for low values of the additive noise standard deviation. In this regime, the impact of the additive noise is relatively small compared to the quantization error, leading to a specific behavior in the standard deviation of the estimated amplitude.

On the other hand, the solid line represents the asymptotic approximation for high values of the additive noise standard deviation. In this regime, the additive noise dominates the overall noise characteristics, significantly influencing the standard deviation of the amplitude estimates.

By providing both asymptotic approximations, the graph captures the behavior of the standard deviation of the estimated amplitude in different noise regimes, allowing for a comprehensive understanding of the parameter estimation performance under various conditions.

Overall, Fig. 1 provides valuable insights into the relationship between the additive noise standard deviation, quantization error, and the standard deviation of the estimated amplitude. The comparison between theoretical predictions and simulation results highlights the practical challenges and limitations that engineers and researchers may encounter when estimating sine wave parameters in the presence of quantization and additive noise in real-world digital data scenarios.

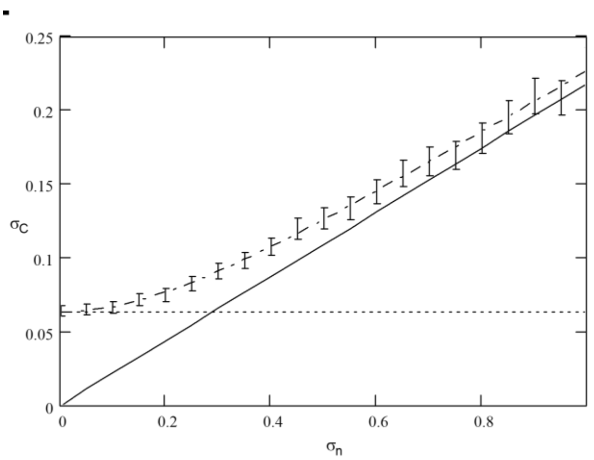

Fig. 2. Standard deviation of the estimated offset of a fitted sine wave as a function of additive noise standard deviation

Both quantities are normalized to the quantization step. The vertical bars correspond to a confidence interval of 99%. The dash-dot line represents the value given by (32), the dotted and solid lines represents the asymptotic approximation for low and high values, respectively, of additive noise standard deviation. The simulations were carried out with 21 points extracted from one period of a sine wave with $A = 100$ LSB, $C = 0$ and $\varphi=\pi/4$.

In Fig. 2 and 3, the simulated standard deviations of the phase at the origin of time and the offset, respectively, are presented. These graphs provide valuable insights into the precision and reliability of estimating these parameters in the presence of quantization and additive noise in digital data scenarios.

Fig. 2 shows the simulated standard deviation of the phase at the origin of time as a function of the additive noise standard deviation, normalized to the ideal quantization step Q. The vertical bars in the graph represent a confidence interval of 99%, indicating the range within which the estimated phase is likely to fall based on the simulation results.

Similarly, Fig. 3 illustrates the simulated standard deviation of the offset as a function of the additive noise standard deviation, also normalized to the ideal quantization step Q. The vertical bars in this graph represent the 99% confidence interval for the estimated offset.

The simulated standard deviations in Fig. 2 and 3 are essential in understanding the precision of estimating the phase and offset parameters in the presence of noise. As with the simulated standard deviation of the amplitude (shown in Fig. 1), the simulated standard deviations of the phase and offset provide practical insights that might differ from the theoretical predictions due to real-world complexities and uncertainties in the data.

Analyzing the trends and behaviors depicted in these graphs allows researchers and engineers to make informed decisions when designing and optimizing data acquisition systems and signal processing algorithms. It helps determine the necessary sample size and the acceptable levels of noise in practical applications to achieve the desired precision in estimating the phase and offset of the sine wave.

By presenting the simulated standard deviations alongside the theoretical predictions, Fig. 2 and 3 provide a comprehensive view of the estimation performance under different noise regimes. This comparison allows for a better understanding of the trade-offs between noise levels, quantization errors, and the achievable precision of phase and offset estimation.

In summary, Fig. 2 and 3 play a crucial role in evaluating and optimizing the performance of the sine wave fitting procedure in the presence of both quantization and additive noise. These graphs provide practical insights and empirical data that help bridge the gap between theoretical models and real-world data scenarios, enabling researchers and practitioners to make more accurate and informed decisions in various engineering applications.

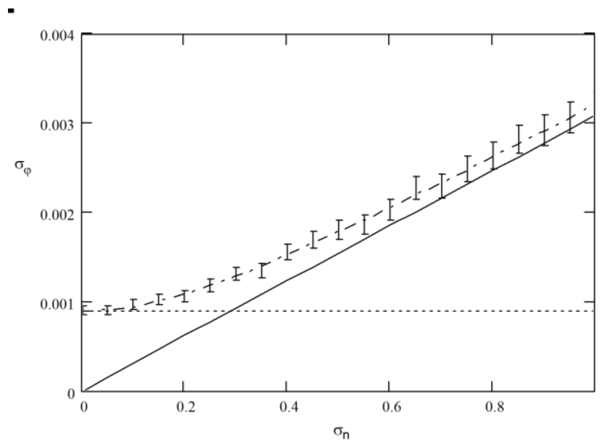

Fig. 3. Standard deviation of the estimated phase at the origin of time of a fitted sine wave as a function of additive noise standard deviation which is normalized to the quantization step

The vertical bars correspond to a confidence interval of 99%. The dash-dot line represents the value given by (32), the dotted and solid lines represents the asymptotic approximation for low and high values, respectively, of additive noise standard deviation. The simulations were carried out with 21 points extracted from one period of a sine wave with $A = 100$ LSB, $C = 0$ and $\varphi=\pi/4$.

These simulations validate expressions (32) derived here.

6 Minimum number of samples

Equations (32) provide valuable insights into determining the minimum number of samples required to ensure a certain confidence interval in the estimates of the sine wave parameters. These confidence intervals are obtained by multiplying the standard deviation of the estimators by a coverage factor (k), which depends on the type of distribution of the estimators [6].

For example, in the case of a normal distribution, a coverage factor of 3 corresponds to a 99% confidence level. This means that there is a 99% chance that the actual value of the parameter falls within the estimated interval.

To calculate the minimum number of samples needed for a desired confidence interval, we use equation (33), which is derived based on the standard deviation of the estimators:

$$N \geq k\left(\sigma_n^2 + \frac{Q^2}{12}\right) \cdot \max\left(\frac{2}{U_A^2}, \frac{1}{U_C^2}, \frac{2}{A \cdot U_\varphi^2}\right), \tag{33}$$

where U_A, U_C and U_φ are the half-width of the uncertainty intervals desired for the estimation of the amplitude, offset and phase at the origin of time of the sine wave.

The equation takes into account the desired precision (half-width of the uncertainty intervals) for each parameter estimation and relates it to the standard deviation of the noise and angular frequency (ω) of the sine wave.

By using equation (33), engineers and researchers can determine the minimum number of samples required to achieve a specific level of precision in estimating the parameters of the sine wave. This information is valuable in practical applications where data collection and processing resources are limited, as it allows for the optimization of the data acquisition process and the estimation algorithm to meet specific performance requirements.

In summary, equation (33) provides a practical tool for determining the minimum number of samples needed to achieve a desired confidence interval in the estimates of the amplitude, offset, and phase of a sine wave. This facilitates the design and implementation of accurate and reliable parameter estimation algorithms in various engineering applications, even in the presence of noise and quantization effects.

7 Conclusions

In this research paper, the focus was to investigate the influence of additive noise on digital data and its impact on the accuracy and precision of estimating the parameters of a sine wave. We employed a 3-parameter fitting method, as described in reference [10], to estimate the amplitude, offset, and phase at the origin of time for the sine wave model:

$$\boxed{\sigma_A = \frac{\sqrt{\sigma_n^2 + \frac{Q^2}{12}}}{\sqrt{N/2}} \quad \sigma_C = \frac{\sqrt{\sigma_n^2 + \frac{Q^2}{12}}}{\sqrt{N}} \quad \sigma_\varphi = \frac{\sqrt{\sigma_n^2 + \frac{Q^2}{12}}}{\sqrt{N/2} \cdot A}}$$

After conducting thorough analysis and simulations, we obtained valuable results in the form of expressions (32) that provide insight into the precision of the parameter estimates. These expressions can be utilized to compute the standard deviation of the estimated amplitude, offset, and phase of the sine wave in the presence of both quantization and additive noise in digital data scenarios.

Expressions (32) provide insight into how the standard deviation of the estimated amplitude, offset, and phase scales with the additive noise standard deviation, normalized to the ideal quantization step Q. The simulation results and theoretical predictions allowed us to gain a deeper understanding of the behavior of the estimation precision across different noise regimes.

Also, an expression for the minimum number of samples required to achieve a desired uncertainty on the sine wave parameters was presented:

$$\boxed{N \geq k\left(\sigma_n^2 + \frac{Q^2}{12}\right)\cdot \max\left(\frac{2}{U_A^2}, \frac{1}{U_C^2}, \frac{2}{A\cdot U_\varphi^2}\right)}.$$

These results are particularly valuable for engineering applications where precise parameter estimation is essential. By utilizing expressions (32), researchers and engineers can quantitatively evaluate the precision of their sine wave parameter estimates and make informed decisions regarding data acquisition strategies, noise reduction techniques, and optimization of the parameter estimation algorithm. There are, of course, other methods that are also affected by the presence of additive noise like the histogram test of ADCs [17]18[18] or even methods that are used to estimate the amount of noise itself [19], or other quantities not related to ADCs like geophysical exploration [20], liquid fluid velocity measurement [21].

In conclusion, this paper contributes to the field of parameter estimation for sine waves in digital data scenarios, by providing expressions (32) that offer insights into the precision of the estimated amplitude, offset, and phase. These results enhance our understanding of the impact of noise on parameter estimation and pave the way for further advancements in signal processing and data analysis in engineering applications.

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