Journal of Electrical Engineering, Vol. 75, No. 5, 2024, pp. 372-382

sciendo
PAPERS

Models and methods for RZ-signals distinction in non-Gaussian noise for information-measurement systems

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Polynomial Decision Rules (DR) for the problems of discrete RZ (Return-to-Zero) signal distinction in asymmetric-excess non-Gaussian noise are proposed. A new approach is proposed, which is based on the use of a modified moment quality criterion of statistical hypothesis testing and the application of higher-order statistics to describe the characteristics of non-Gaussian noise. Simulation of the DR with different parameters of the signal and noise was carried out. It is shown that taking into account the coefficients of skewness and kurtosis of the non-Gaussian noise the efficiency of signal distinction increases with non-linear processing DR compared to known results, which are optimal for the Gaussian noise model. The conducted studies demonstrate a reduction in false decisions in the processing of RZ signals when considering the coefficients of skewness and kurtosis of non-Gaussian noise. Such an increase in efficiency can exceed twofold, depending on the noise parameters. It is shown that the efficiency of the proposed approach is much higher for small SNR (Signal-to-Noise Ratio) values, for example, less than 1.

Keywords: RZ signal, moment quality criterion, higher order statistics, non-Gaussian noise, information-measurement systems

1 Introduction

Developing new models and methods for signal processing is crucial for advancing modern technical systems, as it enables enhanced performance, efficiency, and adaptability to diverse environments. Data transmission and reception systems are a defining part of modern diagnostic, control, management, and other systems. The development of such systems is characterized by growing requirements for the quality of processing received data. When designing these systems, one of the types of linear message encoding used is RZ encoding [1]. This type of coding has several advantages, namely:

- simple implementation compared to multi-level coding methods;
- elimination of the problem of constant offset by using opposite potential levels;
- self-synchronization of signals in communication systems.

Various destabilizing factors affect the functioning of these systems during data transmission. The influence of noise degrades the quality and efficiency of these systems. Destabilizing factors arise during the multipath propagation of radio signals, as they pass through heterogeneous media, fluctuations in communication channel parameters, etc. In most practical cases, such destabilizing factors are random non-Gaussian processes [2-4].

Traditionally, the design of systems for signal detection and distinction has depended on classical methods from statistical hypothesis testing theory. This theory typically does not restrict the type of probability density distribution for random variables [4-8]. However, the traditional approach to researching and developing discrete signal processing systems in non-Gaussian noise is characterized by significant limitations. These limitations are associated with the complexity of algorithmic implementation of classical methods and the increased demand for computing resources, leading to difficulties in creating high-quality software and hardware for signal processing.

Recent research suggests that an alternative approach to solving problems related to the processing of non-Gaussian processes can be quite effective. This method for describing the statistical properties of random variables does not rely on probability density functions (PDF) of the distribution of random processes. Instead, it utilizes other characteristics in the form of a sequence of moments and cumulants. This sequence forms higher-order statistics (HOS), allowing for an acceptable approximation of the statistical properties of non-Gaussian processes [9-12].

It is important to emphasize the special significance of characteristics such as cumulants and cumulant coefficients. Unlike moments, these parameters have independent statistical significance and allow for the description of the characteristics of the non-Gaussian

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https://doi.org/10.2478/jee-2024-0045, Print (till 2015) ISSN 1335-3632, On-line ISSN 1339-309X

distribution of random variables. This approach increases the accuracy of processing non-Gaussian signals compared to the traditional correlation method, despite constraints on complexity. Additionally, the complexity of signal detection and discrimination algorithms is reduced, and it becomes possible to account for the correlations of non-Gaussian random variables when using multidimensional moments and cumulants.

The moment-cumulant approach to describing random variables allows for an acceptable approximation of the statistical properties of non-Gaussian processes and improves the accuracy of processing these processes compared to traditional correlation methods [13-17]. Statistical methods of signal processing based on higher-order statistics are widely used in the development of data reception systems over noisy communication channels, where traditional methods are less effective [18-21].

The purpose of this work is to improve the efficiency of data reception systems for distinguishing RZ signals in non-Gaussian noise by applying the moment-cumulant representation of random variables. This includes developing a moment quality criterion for statistical hypothesis testing and polynomial decision rules [22-25].

Problem statement. Let random signals $\xi_i(t)$, $i = 0,1,2$ be observed on the observation interval $(0 - T)$ which are an additive mixture of constant useful signals a_1 and a_2 in asymmetric-excess non-Gaussian noise with zero mathematical expectation and variance

χ_2: $\xi_0(t) = \eta(t)$, $\xi_1(t) = a_i + \eta(t)$, $\xi_2(t) = -a_i + \eta(t), i = 1,2$.

From the random signals $\xi_i(t)$, $i = 0,1,2$ we obtain a vector of sample values $X = \{x_1, x_2, \dots x_n\}$. Based on the processing results of these values, it is necessary to make a decision on the implementation of the hypothesis H_1 or H_2. The implementation of these hypotheses corresponds to the reception of a constant useful signal a_1 or $(-a_2)$, respectively, Otherwise, a decision is made to implement the hypothesis H_0, which characterizes the presence of additive non-Gaussian noise.

Each received signal corresponds to a moment-cumulant description, presented in the form of a finite sequence of moments $m_i[\{0, \chi_{i2}, \gamma_{i3}, \dots \gamma_{ij}\}]$, where $\gamma_{i3}, \dots \gamma_{ij}$ are cumulant coefficients that describe the parameters of non-Gaussian noise η(t).

2 Models and methods of RZ – signals distinction in non-Gaussian noise

In line with the classical approach, the optimal Bayesian signal detection algorithm is characterized as the one that minimizes the average risk [5-8]. The fundamental statistic essential for hypothesis testing is identified as the likelihood ratio, which can be obtained from

$$\Lambda(\boldsymbol{X}) = \frac{P(\boldsymbol{X}|H_1)}{P(\boldsymbol{X}|H_0)}. \quad (1)$$

The solution to such problems is typically formulated under the assumption that the random variables adhere to the Gaussian probability density function (PDF). However, deriving solutions in the format of equation (1) for non-Gaussian PDFs poses challenges due to the inherent uncertainty in the PDF, parameters, and algorithmic implementation. Hence, alternative approaches are needed to circumvent these issues. One such alternative approach may entail expressing the likelihood ratio as a power-law polynomial function, as detailed in references [15, 22-25].

Assume the likelihood ratio in Eqn. (1) is a continuous function and is represented as a stochastic power polynomial of degree s for independent random samples x_v:

$$\Lambda(x_1, x_2, \dots, x_n) = k_0 + \sum_{i=1}^{\infty} \sum_{v=1}^{n} k_{iv} \phi_i(x_v), \quad (2)$$

where the functions $\phi_i(x_v)$ represent transformations of sample values x_v, which may include power or trigonometric functions. The coefficients k_{iv} and k_0 are unknown parameters chosen based on the relevant quality criterion. Furthermore, if the functions $\phi_i(x_v)$ are linearly independent and form a basis, then for a broad class of functions $\Lambda(\boldsymbol{X}) = f(x_1, x_2, \dots, x_n)$, a decomposition in the form of series (2) is feasible.

In practice, instead of infinite series (2), polynomials with a finite number of terms are used.

Then an expression of the form

$$\Lambda(\boldsymbol{X})_{sn} = k_0 + \sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv} \phi_i(x_v), \quad (3)$$

where x_v are random variables, this will be referred to as a generalized stochastic polynomial of degree s with dimension n.

This stochastic polynomial (3), as a function of the sample values, is a function of the likelihood ratio, which can be used in the form of DR to test statistical hypotheses. In the case of using power-law transformations of sample values, the DR for uniformly distributed random variables takes the form:

$$\Lambda(\boldsymbol{X})_{sn} = k_0 + \sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv} x_v^i \underset{H_0}{\overset{H_1}{\gtrless}} 0, \quad (4)$$

where unknown coefficients k_{iv} and k_0 (4) can be determined by minimizing a well-known probabilistic quality criterion such as Bayes or minimax criterion. However, this determination can be complex in general cases. Therefore, a new moment-based quality criterion for statistical hypothesis testing is proposed [15, 23-25]. Let us outline the development of such a quality criterion for testing statistical hypotheses.

Let us assume that there exists a DR

$$\Lambda(\boldsymbol{X}) = \gamma(\boldsymbol{X}) - k_0 \underset{H_0}{\overset{H_1}{\gtrless}} 0. \quad (5)$$

where $\gamma(\boldsymbol{X})$ – function from sample values $\boldsymbol{X}$ and k_0 is chosen so that

$$M_0 = E\left[\frac{\Lambda(\boldsymbol{X})}{H_0}\right] = \int_{-\infty}^{\infty} \Lambda(\boldsymbol{X}) p(\boldsymbol{X}/H_0) \Pi \; dx < 0,$$

$$M_1 = E\left[\frac{\Lambda(\boldsymbol{X})}{H_1}\right] = \int_{-\infty}^{\infty} \Lambda(\boldsymbol{X}) p(\boldsymbol{X}/H_1) \Pi \; dx \geq 0.$$

The probabilities of type I and II errors are defined according to the Chebyshev inequality:

$$\alpha = P\left[\Lambda(\boldsymbol{X}) \geq \frac{0}{H_0}\right] \leq G_0/M_0^2 = \alpha_0,$$

$$\beta = P\left[\Lambda(\boldsymbol{X}) < \frac{0}{H_1}\right] \leq G_1/M_1^2 = \beta_0,$$

where $G_i(\gamma) = \int_{-\infty}^{\infty} [f(\mathbf{X}) - M_i]^2 p(\mathbf{X}/H_i) \Pi \; dx$ represents the variance of the decision function $\gamma(\mathbf{X})$ for hypothesis H_i, $i = 0{,}1$.

Then the sum of error probabilities can be expressed as

$$F(\alpha, \beta) = \alpha + \beta \leq \alpha_0 + \beta_0 = \frac{G_0}{M_0^2} + \frac{G_1}{M_1^2} = \Phi(G, M). \quad (6)$$

Let us take that for expression M_0 and M_1 the coefficient k_0 is defined as:

$$k_0 = 0.5(E_0 + E_1), \quad (7)$$

where $E_i(\gamma) = E[\gamma(\mathbf{X}) | H_i]$ - the mean of the decision function $\gamma(\mathbf{X})$ (5) for hypothesis H_i, $i = 0{,}1$.

Then, the function $\Phi(G, M)$ (6) for such coefficient k_0 (7) has the form $\Phi(G, M) = 4Ku1(G, E)$, where

$$Ku1(E, G) = \frac{G_{0(sn)} + G_{0(sn)}}{\left(E_{1(sn)} - E_{0(sn)}\right)^2}, \quad (8)$$

where the mean $E_{i(sn)}$ and the variance $G_{i(sn)}$ of the DR (4) of power *S* for hypotheses H_i (*i=0,1*) for sample values of volume *n* are defined as

$$E_{0(sn)} = \sum_{v=1}^{n} \sum_{i=1}^{s} k_{iv} u_i, \quad (9)$$

$$E_{0(sn)} = \sum_{v=1}^{n} \sum_{i=1}^{s} k_{iv} m_i, \quad (10)$$

$$G_{0(sn)} = \sum_{v=1}^{n} \sum_{i=1}^{s} \sum_{j=1}^{s} k_{iv} k_{jv} F_{(i,j)v}(H_0), \quad (11)$$

$$G_{0(sn)} = \sum_{v=1}^{n} \sum_{i=1}^{s} \sum_{j=1}^{s} k_{iv} k_{jv} F_{(i,j)v}(H_1), \quad (12)$$

where m_i, u_i - the initial moments of the *i*-th order for hypotheses H_1 and H_0 respectively in (9-12), $F_{(i,j)v}(H_0) = u_{(i+j)v} - u_{iv} u_{jv}$, $F_{(i,j)v}(H_1) = m_{(i+j)v} - m_{iv} m_{jv}$.

Definition 1. Let us define the functional $Ku1(E, G)$ as the quality criterion for decision making. We assume that the optimal coefficients k_0 in the form of (7) and k_{iv}, which minimize the right-hand side of (8), form the basis of this criterion, referred to as the "Moment quality criterion of probability upper bound errors" for statistical hypothesis testing, or briefly the "Ku1 criterion".

The proposed quality criterion provides an upper bound on the probabilities of errors of the first and second kind for the DR (4), offering a clear interpretation. The minimum of the Ku1 criterion (8) corresponds to minimizing the error probabilities of the DR. This minimum is achieved when the variances (11, 12) of the decision rule are minimal, and the distance between the mean (9, 10) of the decision rule for the hypothesis and the alternative is maximal.

The optimal coefficients k_{iv} for the decision rule can be determined by minimizing the proposed quality criterion (8). Specifically, we use the following system of linear equations to find such coefficients:

$$\sum_{j=1}^{s} k_{jv} \left[F_{(i,j)v}(H_0) + F_{(i,j)v}(H_1)\right] = m_{iv} - u_{iv}, \quad (13)$$

i=1,s.

This approach has proven effective in solving various signal detection problems [22-25]. Let us adapt this method for multi-alternative statistical hypotheses testing, using the example of distinguishing RZ-signals in non-Gaussian nois. In this case the likelihood ratio for multi-alternative statistical hypotheses testing H_g and H_r for independent and unequally distributed random samples x_v will differ from (4) and defined as

$$\Lambda(\boldsymbol{X})_{sn}^{(gr)} = \sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(gr)} x_v^i + k_0^{(gr)} \underset{H_g}{\overset{H_r}{\gtrless}} 0, \quad (14)$$

where $g, r = \overline{0, N-1}$, $g \neq r$ and the unknown optimal coefficients $k_i^{(gr)}$ are determined by minimizing the adapted moment quality criterion for multi-alternative statistical hypotheses testing:

$$Ku2(E, G)^{(gr)} = \frac{G_g^{(gr)}[\gamma] + G_r^{(gr)}[\gamma]}{\left(E_g^{(gr)}[\gamma] - E_r^{(gr)}[\gamma]\right)^2}, \quad (15)$$

and $k_0^{(gr)}$ is chosen as the average of the mean functions $E_g^{(gr)}$ and $E_r^{(gr)}$ for hypothesis testing H_g and H_r in DR (14):

$$k_0^{(gr)} = -\frac{1}{2}\left(E_g^{(gr)} + E_r^{(gr)}\right) =$$

$$-\frac{1}{2} \sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(gr)} \left(m_{iv}^{(g)} + m_{iv}^{(r)}\right), \quad (16)$$

$$g, r = \overline{0, N-1}, \quad g \neq r.$$

The minimum value of criterion (15) also guarantees the minimization of the sum of the upper bounds of the probabilities of errors of the first and second kind for the DR (14). It is demonstrated that the optimal DR coefficients $k_{iv}^{(gr)}$ in (14) will be determined by the following system of equations

$$\sum_{j=1}^{s} k_{jv}^{(gr)} \left[F_{(i,j)v}^{(r)} + F_{(i,j)v}^{(g)}\right] = m_{iv}^{(g)} - m_{iv}^{(r)}, \quad (17)$$

$$v = \overline{1,n},\ i = \overline{1,s},\ g,r = \overline{0,N-1},\ \ g \neq r.$$

Definition 2. Let us define the functional $Ku2(E,G)$ as the moment quality criterion for decision making in the form of DR (14). We assume that the optimal DR coefficients $k_0^{(gr)}$ in (16) and $k_{iv}^{(gr)}$, which minimize the right-hand side of (15), constitute this criterion, termed the "Modified Moment Quality Criterion of Probability Upper Bound Errors for Multiple Statistical Hypothesis Testing" or briefly the MMQC criterion. A lower value of the criterion (15) indicates a reduced likelihood of errors in the DR (14).

The mean and variance of DR (14) for multiple hypotheses testing H_g and H_r are defined as follows:

$$E_g^{(gr)} = \sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(gr)} m_{iv}^{(g)}, \quad (18)$$

$$E_r^{(gr)} = \sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(gr)} m_{iv}^{(r)}, \quad (19)$$

$$G_g^{(gr)} = \sum_{i=s}^{s} \sum_{j=1}^{s} \sum_{v=1}^{n} k_{iv}^{(gr)} k_{jv}^{(gr)} F_{(i,j)v}^{(g)}, \quad (20)$$

$$G_r^{(gr)} = \sum_{i=s}^{s} \sum_{j=1}^{s} \sum_{v=1}^{n} k_{iv}^{(gr)} k_{jv}^{(gr)} F_{(i,j)v}^{(r)}, \quad (21)$$

where $m_{iv}^{(r)}$, $m_{iv}^{(g)}$ are the initial moments of the i –th order of the random variable ξ for hypotheses H_r and H_g respectively,

$$F_{(i,j)v}^{(g)} = m_{(i+j)v}^{(g)} - m_{iv}^{(g)} m_{jv}^{(g)},$$

$$F_{(i,j)v}^{(r)} = m_{(i+j)v}^{(r)} - m_{iv}^{(r)} m_{jv}^{(r)}.$$

Then, the general structure of the DR (14) for multiple statistical hypothesis testing H_g and H_r is defined:

$$H_g\!: \max_{g=\overline{1,N-1}} \left\{\sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(g0)} x_v^i + k_0^{(g0)}\right\} > 0;$$

$$H_0\!: \max_{g=\overline{1,N-1}} \left\{\sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(g0)} x_v^i + k_0^{(g0)}\right\} < 0.$$

$$\sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(g0)} x_v^i + k_0^{(g0)} > \sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(r0)} x_v^i + k_0^{(r0)},\ g,r = \overline{0,N-1},\ \ g \neq r,\ i = \overline{1,s}.$$

Property 1. If the optimal DR (14) coefficients are determined by solving the system of algebraic equations (17), then they satisfy the following condition

$$I_{sn}^{(gr)} = \sum_{v=1}^{n} \sum_{i=1}^{s} k_{jv}^{(gr)} k_{iv}^{(gr)} \left[F_{(i,j)v}^{(g)} + F_{(i,j)v}^{(r)}\right] = \sum_{v=1}^{n} \sum_{i=1}^{s} k_{iv}^{(gr)} \left(m_{iv}^{(g)} - m_{iv}^{(r)}\right), \quad (22)$$

$$g,r = \overline{0,N-1},\ g \neq r.$$

Definition 3. Let us determine the value $I_{Ku}{}_{sn}^{(gr)}$ (22), which we will refer to as the value of extracted information from a samples of size n regarding distinction between hypotheses H_r, H_g when using polynomial DR (14) of degree *s*.

Property 2. For the coefficients determined from the system of equations (17), the value of the quality criterion $Ku2(E,G)$ (15) is inversely proportional to the amount of extracted information from sample values of size *n* concerning the distinction between hypotheses H_r, H_g and expressed as follows:

$$I_{sn}^{(gr)} = \frac{1}{Ku(E,G)^{(mr)}} = \sum_{i=1}^{s} \sum_{v=1}^{n} k_{iv}^{(gr)} \left(m_{iv}^{(g)} - m_{iv}^{(r)}\right),$$

$$g,r = \overline{0,N-1},\ g \neq r. \quad (23)$$

3 Results

Let us carry out a synthesis of algorithms for RZ-signals distinction in non-Gaussian noise for the degree of the polynomial DR (14) S=1,2 on the basis of the proposed approach and the MMQC (15).

The initial moments up to the 4th order for the signal $\xi_0(t)$ when implementing the hypothesis H_0 are as follows:

$$m_1^{(0)} = 0,\ m_2^{(0)} = \chi_2,$$

$$m_3^{(0)} = \gamma_3 \chi_2^{\frac{3}{2}},\ m_4^{(0)} = (3+\gamma_4)\chi_2^2,$$

where γ_3, γ_4– coefficients of the third (kewness) and fourth (kurtosis) orders.

The initial moments up to the 4th order for the signal $\xi_1(t)$ when implementing the hypothesis H_1 are as follows:

$$m_1^{(1)} = a_1, m_2^{(1)} = a_1^2 + \chi_2,$$

$$m_3^{(1)} = a_1^3 + 3a_1\chi_2 + \gamma_3 \chi_2^{\frac{3}{2}},$$

$$m_4^{(1)} = a_1^4 + 6a_1^2\chi_2 + 4a_1\gamma_3\chi_2^{\frac{3}{2}} + 3\chi_2^2 + \gamma_4\chi_2^2.$$

The initial moments up to the 4th order for the signal $\xi_2(t)$ when implementing the hypothesis H_2 are as follows:

$$m_1^{(2)} = -a_2, m_2^{(2)} = a_2^2 + \chi_2,$$

$$m_3^{(2)} = -a_2^3 - 3a_2\chi_2 + \gamma_3\chi_2^{\frac{3}{2}},$$

$$m_4^{(2)} = a_2^4 + 6a_2^2\chi_2 - 4a_2\gamma_3\chi_2^{\frac{3}{2}} + 3\chi_2^2 + \gamma_4\chi_2^2.$$

Let us formulate the expressions for the centered correlations based on the provided equations

$$F_{(k,j)}^{(i)} = m_{(k+j)}^{(i)} - m_k^{(i)} m_j^{(i)}, i = \overline{0,2}.$$

Then, for hypotheses H_0, H_1, H_2 we will obtain the following expressions:

$$F_{(1,1)}^{(0)} = \chi_2,\ F_{(1,2)}^{(0)} = F_{(2,1)}^{(0)} = \chi_2^{3/2}\gamma_3,$$

$$F_{(2,2)}^{(0)} = 2\chi_2^2 + \gamma_4\chi_2^2,$$

$$F_{(1,1)}^{(1)} = \chi_2 F_{(1,2)}^{(1)} = F_{(2,1)}^{(1)} = 2\sqrt{p_1}\chi_2^{\frac{3}{2}} + \gamma_3\chi_2^{\frac{3}{2}}, F_{(2,2)}^{(1)} =$$
$$2\chi_2^2 + 4p_1\chi_2^2 + 4\sqrt{p_1}\gamma_3\chi_2^2 + \gamma_4\chi_2^2,$$

$$F_{(1,1)}^{(2)} = \chi_2, F_{(1,2)}^{(2)} = F_{(2,1)}^{(2)} = -2\sqrt{p_2}\chi_2^{\frac{3}{2}} + \gamma_3\chi_2^{\frac{3}{2}},$$
$$F_{(2,2)}^{(2)} = 2\chi_2^2 + 4p_2\chi_2^2 - 4\sqrt{p_2}\gamma_3\chi_2^2 + \gamma_4\chi_2^2,$$

where $p_i = \frac{a_i^2}{\chi_2}, i = 1{,}2$ – the ratio of the power of the useful signal a_i to the variance χ_2 of the additive non-Gaussian noise (SNR).

Let us give a synthesis of linear DR (14) rules at the degree of the polynomial S=1, which are as follows:

$$\Lambda(\boldsymbol{X})_{1n}^{(i0)} = k_1^{(i0)} \sum_{v=1}^{n} x_v + k_0^{(i0)} \underset{H_i}{\overset{H_0}{\gtrless}} 0, i = 1{,}2 \quad (24)$$

$$\Lambda(\boldsymbol{X})_{1n}^{(21)} = k_1^{(21)} \sum_{v=1}^{n} x_v + k_0^{(21)} \underset{H_2}{\overset{H_1}{\gtrless}} 0, \quad (25)$$

The general structure of the DR (14) for multiple statistical hypothesis testing will be presented as a system of equations for testing hypotheses (Table 1).

Table 1. Systems of equations for testing hypotheses

H_1	H_2	H_0
$\begin{cases}\Lambda(\boldsymbol{X})^{(10)} > 0\\ \Lambda(\boldsymbol{X})^{(21)} < 0\end{cases}$	$\begin{cases}\Lambda(\boldsymbol{X})^{(20)} > 0\\ \Lambda(\boldsymbol{X})^{(21)} > 0\end{cases}$	$\begin{cases}\Lambda(\boldsymbol{X})^{(10)} < 0\\ \Lambda(\boldsymbol{X})^{(20)} < 0\end{cases}$

Here, $\Lambda(\boldsymbol{X})^{(10)}$, $\Lambda(\boldsymbol{X})^{(20)}$ are decision functions for hypotheses testing H_1 and H_2 against H_0, respectively, $\Lambda(\boldsymbol{X})^{(21)}$ is the decision function for hypotheses testing H_2 and H_1.

The unknown coefficients $k_1^{(i0)}$ and $k_1^{(21)}$ for DR (24) are found from the solution of the system of Eqns. (17), where the thresholds $k_0^{(21)}$ and $k_0^{(i0)}$ are found according to (16):

$$k_0^{(i0)} = -n\frac{p_i}{4}, k_0^{(21)} = \frac{n}{4}(p_1 - p_2), k_1^{(10)} = \frac{\sqrt{p_1}}{2\sqrt{\chi_2}},$$
$$k_1^{(20)} = -\frac{\sqrt{p_2}}{2\sqrt{\chi_2}}; k_1^{(21)} = -\frac{\sqrt{p_1}+\sqrt{p_2}}{2\sqrt{\chi_2}}, E_0^{(i0)} = 0.$$

The mean and variance of DR (24, 25) will take the form from Eqns. (18-21):

$$E_1^{(i0)} = \frac{np_i}{2}, \qquad E_1^{(21)} = -\frac{1}{2}n\sqrt{p_1}(\sqrt{p_1} + \sqrt{p_2}),$$

$$E_2^{(21)} = \frac{1}{2}n(\sqrt{p_1}\sqrt{p_1} + \sqrt{p_2}),$$

$$G_0^{(i0)} = G_1^{(i0)} = G_2^{(i0)} = np_i/4\,,$$

$$G_1^{(21)} = G_2^{(21)} = \frac{n}{4}\left(\sqrt{p_1} + \sqrt{p_2}\right)^2, i = 1{,}2.$$

To evaluate the effectiveness of the synthesized linear algorithms, we will use the expression (22) or (23) that characterizes the value of extracted information from a samples of size n regarding distinction between hypotheses H_r, H_g and will take the form

$$I_{1n} = I_{sn}^{(10)} + I_{sn}^{(20)} + I_{sn}^{(21)} = \quad (26)$$
$$n\left(p_1 + \sqrt{p_1}\sqrt{p_2} + p_2\right).$$

The obtained set of systems of equations (Table 1) for the synthesis of decision rules are linear and align with well-known results for Gaussian models of random processes. However, these synthesized decision rules do not account for the parameters of the non-Gaussian distribution of the studied random processes. This limitation arises because only the initial moments of the first and second orders, which describe the mean and variance of the investigated process, were used to characterize the random variables. To incorporate additional parameters such as asymmetry and kurtosis of the random process, it is essential to utilize higher-order statistics (HOS). Therefore, increasing the degree of the polynomial decision rule to S=2 (12) allows for the incorporation of initial moments of the third and fourth orders.

According to expression (14), let's derive a nonlinear DR with a polynomial degree S=2. In the general case, for uniformly distributed sample values, it takes the form:

$$\Lambda(\boldsymbol{X})_{2n}^{(i0)} = k_1^{(i0)} \sum_{v=1}^{n} x_v + k_2^{(i0)} \sum_{v=1}^{n} x_v^2 + k_0^{(i0)} \underset{H_0}{\overset{H_i}{\gtrless}} 0, \quad (27)$$

$$\Lambda(\boldsymbol{X})_{2n}^{(21)} = k_1^{(21)} \sum_{v=1}^{n} x_v + k_2^{(21)} \sum_{v=1}^{n} x_v^2 + k_0^{(21)} \underset{H_1}{\overset{H_2}{\gtrless}} 0,$$

where i=1,2.

The unknown coefficients for DR (27) are determined by solving the system of equations (16), (17) and look like:

$$k_0^{(10)} = \frac{2\sqrt{p_1}\gamma_3 - p_1(2+p_1+\gamma_4)}{4(2+p_1-\gamma_3^2+\gamma_4)};$$

$$k_0^{(20)} = -\frac{2\sqrt{p_2}\gamma_3 + p_2(2+p_2+\gamma_4)}{4(2+p_2-\gamma_3^2+\gamma_4)};$$

$$k_0^{(21)} = \frac{\left(\sqrt{p_1}+\sqrt{p_2}\right)^2\left(2+\left(\sqrt{p_1}+\sqrt{p_2}\right)^2 - 2\left(\sqrt{p_1}+\sqrt{p_2}\right)\gamma_3+\gamma_4\right)}{4\left(2+\left(\sqrt{p_1}+\sqrt{p_2}\right)^2-\gamma_3^2+\gamma_4\right)};$$

$$k_1^{(10)} = \frac{\sqrt{p_1}\left(2+p_1+\sqrt{p_1}\gamma_3+\gamma_4\right)}{2\sqrt{\chi_2}(2+p_1-\gamma_3^2+\gamma_4)};$$

$$k_1^{(20)} = -\frac{\sqrt{p_2}(2+p_2-\sqrt{p_2}\gamma_3+\gamma_4)+p_2\gamma_3}{2\sqrt{\chi_2}(2+p_2-\gamma_3^2+\gamma_4)};$$

$$k_2^{(10)} = -\frac{\sqrt{p_1}\gamma_3}{2\chi_2(2+p_1-\gamma_3^2+\gamma_4)};$$

$$k_2^{(20)} = \frac{\sqrt{p_2}\gamma_3}{2\chi_2(2+p_2-\gamma_3^2+\gamma_4)};$$

$$k_1^{(21)} = \frac{(\sqrt{p_1}+\sqrt{p_2})\left(2+(\sqrt{p_1}+\sqrt{p_2})^2+(\sqrt{p_1}-\sqrt{p_2})\gamma_3+\gamma_4\right)}{2\sqrt{\chi_2}\left(2+(\sqrt{p_1}+\sqrt{p_2})^2-\gamma_3^2+\gamma_4\right)};$$

$$k_2^{(21)} = \frac{(\sqrt{p_1}+\sqrt{p_2})\gamma_3}{2\chi_2\left(2+\left((\sqrt{p_1}+\sqrt{p_2})^2-\gamma_3^2+\gamma_4\right)\right)}.$$

The mean and variances of DR (27) are defined according to expressions (18-21).

For evaluating the effectiveness of the synthesized non-linear DR (27) we will use expression (22) or (23). The maximum value (22, 23) will correspond to the minimum value of MMQC (15), which leads to the minimum values of the upper bound of the probabilities of errors of the first and second kind of DR (27):

$$I_{2n} = n\left\{\left(p_1+\sqrt{p_1}\sqrt{p_2}+p_2\right)+\frac{1}{2}\gamma_3\left(\frac{p_1}{2+p_1-\gamma_3^2+\gamma_4}+\frac{(\sqrt{p_1}+\sqrt{p_2})^2}{2+(\sqrt{p_1}+\sqrt{p_2})^2-\gamma_3^2+\gamma_4}+\frac{p_2}{2+p_2-\gamma_3^2+\gamma_4}\right)\right\}. \quad (28)$$

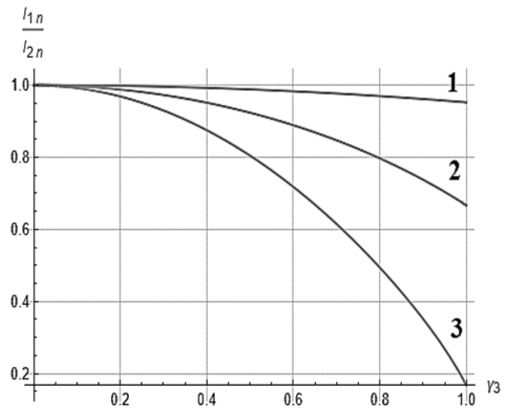
The synthesized DR (24, 25) with a polynomial degree of S=1 represents a system of equations for testing hypotheses H_{10}, H_{20}, H_{21}, which do not incorporate the non-Gaussian distribution characteristics of random processes. When the degree of the polynomial DR is increased to S=2 (27), it involves the use of initial moments of the 3rd and 4th orders. This expansion allows for the consideration of non-Gaussian parameters of random processes, specifically the coefficients of skewness γ_3 and kurtosis , γ_4.

4 Experiments

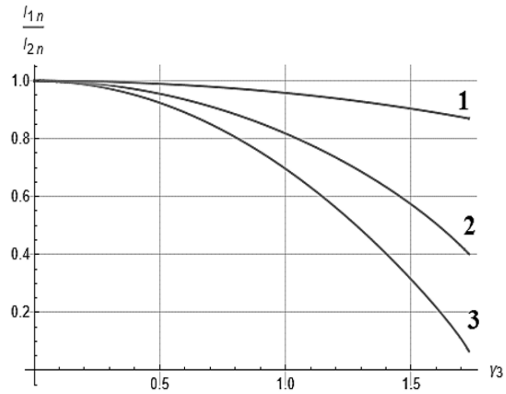
To evaluate the effectiveness of the obtained linear ($S=1$) (24, 25) and non-linear ($S=2$) (27) DR, we will use the modified criterion (15), which characterizes the sum of the probabilities of errors of the first and second kind of the DR. We also note that the inverse value of this criterion $Ku2(E,G)^{(gr)}$ is the extracted information value $I_{sn}^{(gr)}$ (28) from a samples for distinction between hypotheses H_r, H_g.

The ratio of the extracted information value I_{1n} (26) on the distinction among three hypotheses for the Gaussian noise model (DR S=1) to I_{2n} (28) for the non-Gaussian asymmetric-excess noise model (DR S=2) from the skewness coefficient γ_3 is shown in Fig. 1.

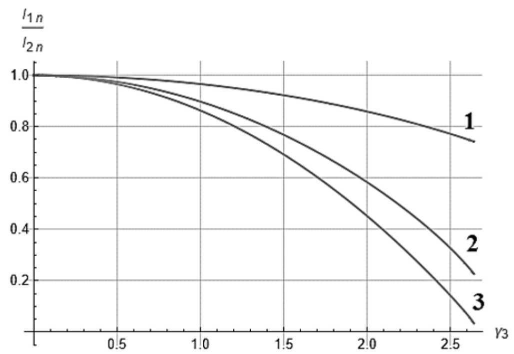
Research results show that linear DR (24, 25) for $S=1$ is equal to well-known results obtained from the probabilistic quality criterion by assuming the Gaussian noise model. In this case, only the first two initial moments are used to characterize the random variables, which describe the mean and variance of the random variables. The extracted information ratio I_{1n}/I_{2n} from samples about distinction hypotheses is equal to 1 for Gaussian model noise when the skewness coefficient is $\gamma_3=0$. Taking into account the asymmetry of random samples ($\gamma_3 \neq 0$), the extracted information I_{2n} for nonlinear DR ($S=2$) are more than I_{1n} for the well-known linear DR (24, 25). This tendency characterizes the lower values of MMQC (15) and accordingly probability the first and the second kind of errors of nonlinear DR (27) than linear DR (24, 25). For example (Fig. 1a), the probability errors value of nonlinear DR is decreased approximately twice for $\gamma_3=0.8$ and $\gamma_4=-1$ (at SNR $\mathrm{p}_1=\mathrm{p}_2=0.1$, curve -3) compared to linear DR. A similar positive effect in increasing the effectiveness of nonlinear processing of sample values is observed for other parameters of non-Gaussian noise, as demonstrated in Figs. 1b-1d. It should also be noted that the efficiency of signal processing improves with the increasing of the DR polynomial degree S.



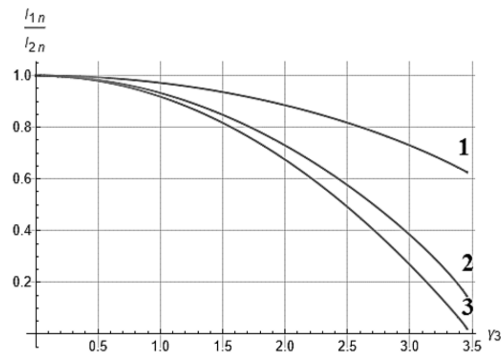
a) $\gamma_4 = -1$; 1)$p_1 = p_2 = 10$; 2) $p_1 = p_2 = 0.5$; 3)$p_1 = p_2 = 0.1$.



b) $\gamma_4 = 1$; 1)$p_1 = p_2 = 10$; 2) $p_1 = p_2 = 0.5$; 3)$p_1 = p_2 = 0.1$.



c) $\gamma_4 = 5$; 1)$p_1 = p_2 = 10$; 2) $p_1 = p_2 = 0.5$; 3)$p_1 = p_2 = 0.1$.



d) $\gamma_4 = 10$; 1)$p_1 = p_2 = 10$; 2) $p_1 = p_2 = 0.5$; 3)$p_1 = p_2 = 0.1$.

Fig. 1. Comparison of the extracted information ratio I_{1n}/I_{2n} from samples about distinction hypotheses from the skewness coefficient γ_3 using polynomial DR of order $S = 1,2$, *n*=100.

Simulation of the RZ-signal distinction in non-Gaussian noise was carried out. The simulation was implemented in the Matlab-Simulink environment, the results of which are shown in Fig. 2. The result of linear (a - *S*=1) and non-linear (b - *S*=2) signal processing DR for hypotheses testing H_1 and H_2 against H_0 is shown. Linear processing (a) is characterized by more chaotic DR emissions, which will lead to the appearance of errors of the first and second kind. Non-linear processing is characterized by less chaotic DR emissions, which improves the accuracy of the RZ-signal distinction in non-Gaussian.

The result of the noisy RZ-signals decoding is presented in Fig. 3. The simulation was carried out with different non-Gaussian noise parameters, namely with different skewness and kurtosis coefficients and SNR. Figure 3 shows a useful RZ-signals (a), a noisy RZ-signals in non-Gaussain Rayleigh noise (b) - $\gamma_3 = 0.631$, $\gamma_4 = 0.245$, the results of linear (c, e) and non-linear (d, f) signal processing. As can be seen from the results of the simulation, the number of RZ-signal distinction errors decreases with nonlinear processing of sample values (S=2 - d, f) compared to the processing results for Gaussian models of random processes (*S*=1 - c, e). It can be seen from the obtained oscillograms that taking into account the non-Gaussian characteristic of random processes in the form of the coefficient of asymmetry and kurtosis allows us to increase the efficiency of RZ-signal distinction in non-Gaussian noise by factors of 5 (c, d) and 3.5 (e, f) - in Fig. 3. This improvement in efficiency corresponds to the results obtained for theoretical studies (Fig. 1)

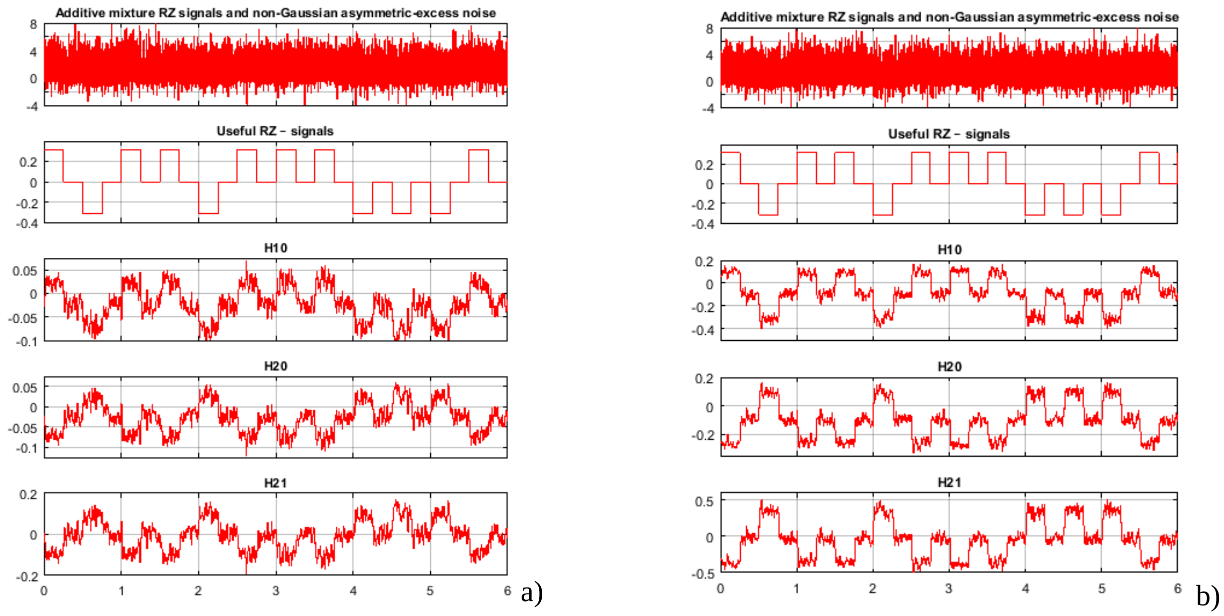


Fig. 2. Simulation RZ signals distinction between H_1, H_2 and H_0 hypotheses in non-Gaussian noise using linear $S = 1$ (a) and nonlinear $S = 2$ (b) DR for $p_1 = p_2 = 0.1$, $\gamma_3 = 3$, $\gamma_4 = 10$, n=100.

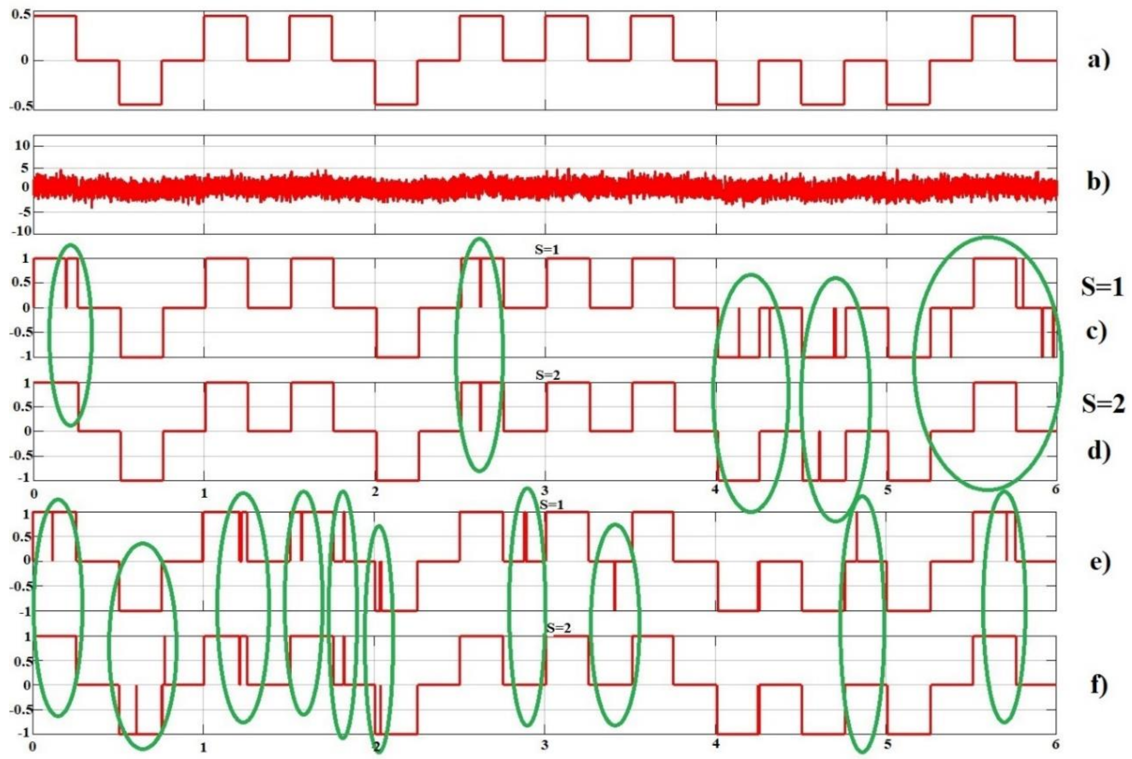


Fig. 3. Simulation RZ signals distinction in non-Gaussian noise: a) - useful RZ-signals; b) - additive mixture RZ signals and non-Gaussian asymmetric-excess noise; c), e) - linear signal processing DR (S=1); d), f) – non-linear signal processing DR (S=2), where n=100, $p_1 = p_2 = 0.1, \gamma_4 = -1, \gamma_3 = 1$ (c, d); $p_1 = p_2 = 0.2, \gamma_3 = 0.631$, $\gamma_4 = 0.245$ (e, f).

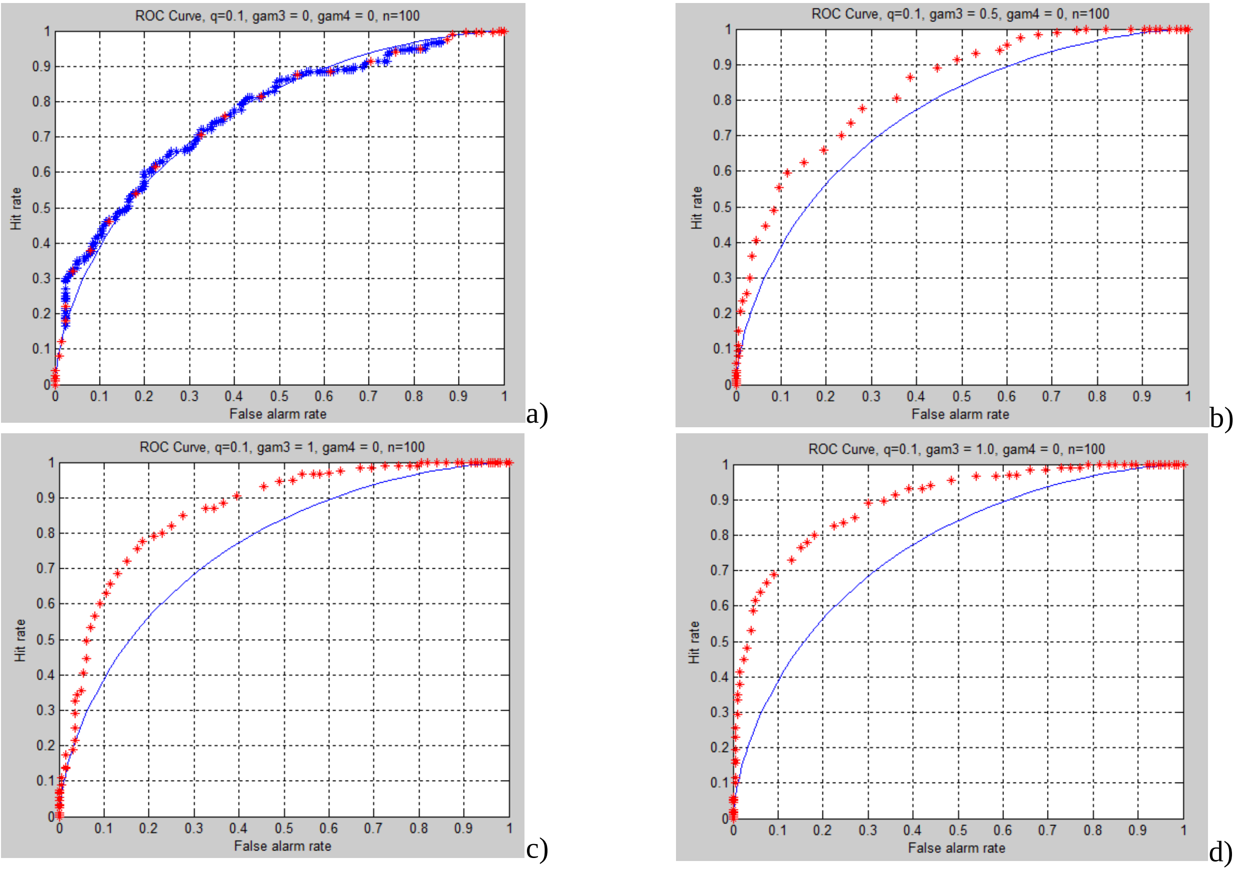


Fig. 4. Simulation of the ROC curve of a polynomial signal detector in non-Gaussian noise

Figure 4 shows the results of the simulation of the ROC curve (Receiver Operating Characteristic curve) for the polynomial signal detector in non-Gaussian noise.

It is shown that for the Gaussian noise model, when the coefficients of skewness (γ_3) and kurtosis (γ_4) are equal to zero, the characteristics of the linear (s=1) and nonlinear (s=2) polynomial DR are the same (Fig. 4 - a). It is important to note that the linear DR coincides with the classical DR, which is derived from the likelihood of the assumed Gaussian noise model. As the skewness coefficient increases (discrete values for γ_3 = 0.5, 1.0, 1.2 for n=100 are shown in Fig. 4b, c, d), the curve deviates to the upper left corner. Such a deviation of the curve indicates an increase in the quality of nonlinear signal processing (s=2) compared to the linear (s=1) processing under the assumption of non-Gaussian noise models

5 Discussions

Signal processing in noise is a significant statistical challenge for many practical applications. The foundation for addressing these problems lies in using the likelihood ratio, which involves the probability densities of random processes. However, applying this approach to non-Gaussian models of random processes presents practical difficulties related to determining the type of distribution density, its parameters, and the synthesis and analysis of algorithms.

The paper suggests an alternative approach to describing random processes. This approach is based on using moments and cumulants, an infinite sequence of which will accurately approximate the proposed description of random variables to a complete probabilistic description. However, to develop practical algorithms for signal processing in non-Gaussian noise, scientific research is required. This research must focus on the synthesis of decision rules and new moment-based quality criterion (Modified Moment Quality Criterion of Probability Upper Bound Errors for Multiple Statistical Hypothesis Testing) to determine optimal algorithms for signal distinction in non-Gaussian noise.

The use of polynomial decision rules is proposed, with optimal coefficients determined by the suggested criterion. Additionally, an adapted criterion for multiple hypothesis testing has been proposed, enabling the synthesis of polynomial decision rules for RZ signal distinction in non-Gaussian noise.

Based on the proposed approach, linear and nonlinear decision rules were derived for various polynomial orders. It is important to note that linear decision rules do not account for the non-Gaussian distribution of random variables processed by the proposed algorithm. This is because only the first two moments, representing the mean and variance of the random variables, are used to describe them. However, these linear decision rules exactly match those derived from the likelihood ratio for a Gaussian model of random variables.

Nonlinear processing of sample values and consideration of higher-order statistics of non-Gaussian processes lead to enhanced performance in RZ signal distinction systems compared to conventional approaches for Gaussian processes. In this case, four initial moments are used to describe random variables, which additionally characterize the properties of skewness and kurtosis of random processes. With such a description of random variables, the process can be non-Gaussian.

Studies have shown that this new approach to describing random variables and synthesizing decision rules can increase the accuracy of signal processing compared to the well-known results for widely used Gaussian models of random processes. The use of the proposed method and synthesized algorithms increases the noise immunity of the system for transmitting and receiving bipolar discrete RZ signals in information and measurement systems, as confirmed by simulation results in the SIMULINK environment.

Based on the DR (14) the structure generalized block diagram of polynomial RZ-signals distinction at the degree of DR $s = \overline{1,\ 6}$ is shown in Fig. 5. This structure is designed for statistical processing of input discrete independent values $\mathbf{X} = \{x_1,\ x_2, \dots, x_n\}$. Such a structure is not difficult to implement. It includes blocks such as sample value accumulators, multipliers, a threshold device and can be implemented on a modern element base, for example on an FPGA. The optimal DR coefficients $k_i^{(gr)}$, $k_0^{(gr)}$, which minimize the sum of error probabilities in accordance with the MMQC (15) for multiple hypothesis testing, are derived from solving the system of equations (17).

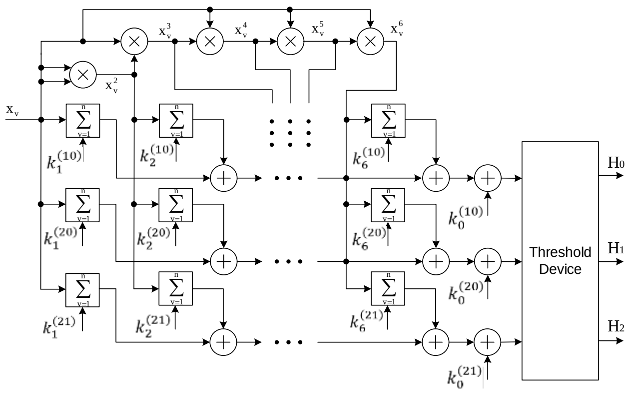


Fig. 5. Generalized block diagram of polynomial RZ-signals distinction signal distinction at the degree of DR $s = \overline{1,\ 6}$.

6 Conclusion

After analyzing the challenge of enhancing signal processing methods in non-Gaussian noise, a methodology for developing mathematical modeling techniques for signal detection and distinction was chosen and justified. This methodology considers the unique characteristics of random signals and involves utilizing a moment-cumulant representation of random processes along with polynomial DR. The coefficients of these decision rules are optimized to minimize the sum error probabilities, as dictated by the MMQC.

Methods for detecting and distinguishing signals in non-Gaussian noise have been devised. The synthesized polynomial signal processing algorithms demonstrate superior performance when contrasted with established outcomes for Gaussian noise models. This approach more accurately characterizes the traits of non-Gaussian stochastic processes by incorporating moments and cumulants of the third order and beyond. Considering higher-order statistics, such as third and higher order moments and cumulants like skewness and kurtosis coefficients, for additive non-Gaussian noise components can enhance the effectiveness of nonlinear signal processing. Such an increase in efficiency can exceed twofold, depending on the noise parameters. It is shown that the efficiency of the proposed approach is much higher for small SNR values, for example less than 1.

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Received 2 July 2024