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DESIGN OF PID CONTROLLERS FOR DELAYED MIMO PLANTS USING MOMENTS BASED APPROACH

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This paper deals with the design and the optimization of robust PID controllers for delayed Multi-Inputs Multi-Outputs (MIMO) plants using moments approach. This methodology is based on a reference model used to specify both time and frequency closed loop requirements. The PID controller is obtained after a Non Linear Optimization procedure minimizing a quadratic cost between the reference model frequency moments and the closed loop ones.

K e y w o r d s: PID control, reference model, moments, delay

1 INTRODUCTION

Since the pioneering research works of the early sixties (see for example Horowitz [9]), the synthesis of multidimensional controllers has received more attention in the industrial field and more particularly in the domain of chemical engineering. It is still common practice, specially in industrial applications, to design a SISO (Single-Input Single-Output)controller for each (I/O) pair of a MIMO plant by simply ignoring the interaction between these pairs [16].

Such SISO controllers may work satisfactorily for some MIMO plants, but advances in performance can only be achieved through the use of MIMO controllers. There are several advanced synthesis techniques available in the literature for developing MIMO controllers. However these techniques, produce optimal high-order and unstructured controllers. Although the order of the controller can be reduced, the resulting controller is not necessarily optimal [7, 16, 17].

PID controllers are undoubtedly the most popular and commonly used controllers. Their simple and intuitive structure leads to easy and quick designs [7, 11, 13]. Furthermore, they are in use for many years now and practicing engineers have confidence in their operation. These advantages make PID controllers an attractive option for MIMO plants, more especially as several synthesis techniques treat MIMO systems such as that proposed by Wang [16] who formulated a general approach based on the multivariable Smith predictor treating the industrial systems with time delays.

We present in this paper a new approach to both design and reduce MIMO controllers in order to obtain a PID structure.

We consider more particularly the class of the multuivariable square systems, stable, aperiodic, where each transfer can be approached by a first order system including a time delay, this time delay can be close to the constant time, in this case, it is usage of saying generally that PID controllers are not adapted for this class of systems.

This technique is based on a reference model used to specify the closed loop performances in terms of relative overshoot, rise time, settling time, *etc*; the robustness against high frequencies uncertainties is ensured by introducing auxiliary poles to roll off the complementary sensitivity function [1].

The time delays are included in the reference model, mainly to make the controller realizable [14]. In the case of MIMO plants, coupling between transfer functions complicates the synthesis technique [18]; in order to perform decoupling, the reference model matrix is chosen diagonal $\big(ie\ T_{\text{ref}}(s) = \text{diag}(t_i(s))\big)$, with $\big(i = \overline{1, m}\big)$ where m is the number of the plant's outputs to be controlled and $t_i(s)$ are scalar functions.

There are several approaches to obtain a reduced order controller for implementation; in this paper, we present a technique based on the notion of the moments and the reference model, this approach has been the result of the work of our research group [13, 14].

Using time moments, it is easy to characterize the closed loop at low frequencies while frequency moments ensure its stability. The outline of this paper is: in Section 2, we present some definitions and a calculation method of frequency and time moments; Section 3 presents the class of systems concerned by this technique; the reference model and performances specifications are given in Section 4; in Section 5, we present the methodology to obtain the PID controller and the last section is devoted to a design example in which, we present a brief recapitulation of the technique suggested by Wang [16] in order to make a comparative study.

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2 MOMENTS

In this section we will present o brief survey of the moments technique, by giving some definitions, how to calculate them and their advantages. Let us consider a linear time invariant MIMO system, characterized by its transfer matrix $G(s)$ analytic in the RHP plan (*ie* $\mathrm{Re}(s) > 0$) and let $g(t)$ be its impulse response:

$$G(s) = \int_0^\infty g(t)e^{-st}\mathrm{d}t\,. \tag{1}$$

The transfer function is related to the following state space (not necessary minimal) realization:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D \tag{2}$$

where A, B, C and D are matrices of appropriate dimensions.

2.1 Time moments

By expanding e^{-st} in Taylor series in the vicinity of $s = 0$, we get:

$$G(s) = \int_0^\infty \sum_{n=0}^{\infty}(-1)^n s^n \frac{t^n}{n!} g(t)\mathrm{d}t \tag{3}$$

$$G(s) = \sum_{n=0}^{\infty}(-1)^n \mathcal{A}_n(g)s^n \tag{4}$$

where:

$$\mathcal{A}_n(g) = \int_0^\infty \frac{t^n}{n!} g(t)\mathrm{d}t \tag{5}$$

$\mathcal{A}_n$: represents the n^{th} order moment of $g(t)$.

R e m a r k 2.1 . The time moments $\mathcal{A}_n$ give a temporal description of the system.

- $\mathcal{A}_0(g)$ represents the area or the d.c gain of $g(t)$.
- $\mathcal{A}_1(g)$ defines mean time of $g(t)$.
- $\mathcal{A}_2(g)$ deals with the 'dispersion' of $g(t)$ around its mean time,*etc* [1, 13, 14].

2.2 Frequency moments

Let consider the variable $s = j\omega$. By expanding e^{-st} in Taylor series in the vicinity of $s_0 = j\omega_0$, we get:

$$G(j\omega) = \sum_{n=0}^{\infty}(-1)^n (j\omega - j\omega_0)^n \underline{\mathcal{A}}_{n,\omega_0}(g) \tag{6}$$

with:

$$\underline{\mathcal{A}}_{n,\omega_0}(g) = \int_0^\infty \frac{t^n}{n!} e^{-j\omega_0 t} g(t)\mathrm{d}t\,. \tag{7}$$

R e m a r k 2.2 . Like the time moments, the frequency moments describe the system around $\omega = \omega_0$:

- $\underline{\mathcal{A}}_{0,\omega_0}$ represents $G(j\omega)$ at $\omega = \omega_0$.
- $\underline{\mathcal{A}}_{0,\omega_0} - j(\omega-\omega_0)\underline{\mathcal{A}}_{1,\omega_0}$ permits to enlarge the previous approximation around $\omega = \omega_0$.
- *etc*

Notice that the moments $\underline{\mathcal{A}}_{n,\omega_0}$ are complex and considering $\omega_0 = 0$, we recover the time moments of the system (*ie* $\underline{\mathcal{A}}_{n,0} = \mathcal{A}_n$ [14]. That means moreover that the time moments characterize the system at low frequency.

2.3 Computing the moments using state space representation

It is interesting to develop a calculation method of the moments based on the state space representation in order to exploit it for the multivariable systems.

2.3.1 Time moments

¿From (2) and (4), we can write:

$$G(s) = -C\Big(\sum_{n=1}^{\infty} s^n A^{-(n+1)}\Big)B + \big(-CA^{-1}B + D\big)$$

so:

$$\mathcal{A}_0(g) = -CA^{-1}B + D \tag{8}$$

$$\mathcal{A}_n(g) = (-1)^{n+1}CA^{-(n+1)}B\,,\ (n = 1\dots\infty)\,. \tag{9}$$

2.3.2 Frequency moments

Realizing a variable change $\mu = j\omega - j\omega_0$, equation (6) becomes:

$$G(\mu) = \sum_{n=0}^{\infty}(-1)^n (\mu)^n \underline{\mathcal{A}}_{n,\omega_0}(g) \tag{10}$$

and (2):

$$G(\mu) = C\big(\mu I - (-j\omega_0 I + A)\big)^{-1}B + D \tag{11}$$

so, we get:

$$\underline{\mathcal{A}}_{0,\omega_0}(g) = -C\big(-j\omega_0 I + A\big)^{-1}B + D \tag{12}$$

$$\underline{\mathcal{A}}_{n,\omega_0}(g) = (-1)^{n+1}C(-j\omega_0 I + A)^{-(n+1)}B\,. \tag{13}$$

3 SYSTEM STRUCTURE

The multidimensional closed loop configuration is illustrated by Fig. 1.

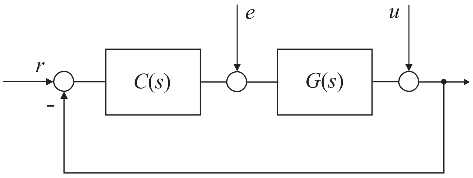


Fig. 1. Unitary feedback configuration

$G(s)$ represents the system transfer matrix and $C(s)$ the MIMO controller to be designed. $G(s)$ is supposed stable, square of dimension $(m \times m)$ and invertible.
$C(s)$ is an $(m \times m)$ matrix:

$$C(s) = \begin{bmatrix} c_{11}(s) & \dots & c_{1m}(s) \\ \vdots & \ddots & \vdots \\ c_{m1}(s) & \dots & c_{mm}(s) \end{bmatrix} \tag{14}$$

The $c_{ij}(s)$ are transfer functions incorporating an integral action. This integral action permits the rejection of constant disturbances and to have a zero steady state error between the system's outputs and the reference input signals [4, 12].

The different (I/O) transfers are given by:

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} T & SG & S \\ CS & -T & -CS \end{bmatrix} \begin{bmatrix} r \\ p_i \\ p_o \end{bmatrix} \tag{15}$$

where $S = (I + GC)^{-1}$ is the sensitivity function, $T = I - S$ is the complementary sensitivity.

3.1 The system model

The class of systems treated in this paper is given by the following transfer matrix:

$$\hat{G} = \begin{bmatrix} \hat{g}_{11}(s)e^{-L_{11}s} & \dots & \hat{g}_{1m}(s)e^{-L_{1m}s} \\ \vdots & \ddots & \vdots \\ \hat{g}_{m1}(s)e^{-L_{m1}s} & \dots & \hat{g}_{mm}(s)e^{-L_{mm}s} \end{bmatrix}$$

where $\hat{g}_{ij}(s)$ represents the estimated transfer function between the j^{th} input and the i^{th} output of the real system where L_{ij} is a time delay.

This kind of systems is difficult to control because of the time delays, more particularly when the L_{ij} are not neglictible compared to the time response of $g_{ij}(s)$ [16].

In practice, only an estimated model $\hat{G}(s)$ of $G(s)$ is available, where

$$G(s) = \hat{G}(s) + \Delta(s) \tag{16}$$

and

$$\Delta_r(s) = (G - \hat{G})\hat{G}^{-1} \tag{17}$$

where $\Delta_r(s)$ represents the relative modelling errors which must be considered during the design to guarantee the stability and performances robustness of $C(s)$.

3.2 The Internal Model Control

In order to control systems incorporating time delays, it is well known that it is preferable to use the internal model control technique presented by Fig. 2 or an equivalent Smith predictor [12, 16].

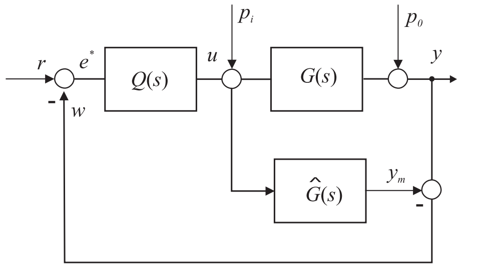


Fig. 2. Internal Model Control configuration

$Q(s)$ represents the internal model control controller and $\hat{G}(s)$ is the system's model.

THEOREM 3.1. *A multivariable system $G(s)$ including time delays characterized by its nominal model $\hat{G}(s) = G_{\text{nom}}(s)$ is controlled and stabilized by the IMC controller due to the open loop equality:*

$$G_{\text{nom}}(s)Q(s) = T_{\text{ref}}(s) \tag{18}$$

where $T_{\text{ref}}(s)$ is the reference model including time delays as defined in Section 4.

The equivalent feedback controller of Fig. 1 is given by:

$$C(s) = Q(s)\big(I - T_{\text{ref}}(s)\big)^{-1}. \tag{19}$$

This controller (19), is called the ideal controller *achieving the stability and the performances for the nominal model.*

P r o o f . if we consider in the IMC configuration:

$$\hat{G}(s) = G(s) = G_{\text{nom}}(s) \tag{20}$$

and $p_i = 0$ and $p_o = 0$ then $w = 0$ and the control loop will works in open loop and:

$$G_{\text{nom}}(s)Q(s) = T_{\text{ref}}(s)\,. \tag{21}$$

This open loop is stable if $G_{\text{nom}}(s)$ and $T_{\text{ref}}(s)$ are stable.

We define the equivalent feedback controller $C(s)$:

$$C(s) = Q(s)\big(I - T_{\text{ref}}(s)\big)^{-1} \tag{22}$$

where $Q(s)$ is given by (20).

Because the two control configurations are equivalent and the first one is unconditionally stable, then the feedback controller $C(s)$ stabilizes and controls unconditionally the closed loop for $\hat{G}(s) = G_{\text{nom}}(s)$.

Of course, when $\hat{G}(s) \neq G(s)$, performances and stability degradations will appear, so it is necessary to synthesize a robust controller.

$C(s)$ is defined as ideal feedback controller such that $T(s) = T_{\text{ref}}(s)$. Notice that $C(s)$ has a structure more complicated than the $Q(s)$ one, because it incorporates $T_{\text{ref}}(s)$ and $\hat{G}(s)$.

Finally, because $\hat{G}(s)$ includes time delays, $C(s)$ will be of infinite dimension.

4 THE REFERENCE MODEL

4.1 Introduction

Fundamentally, our approach is based on a reference model, which represents the ideal dynamical objective of the considered closed loop, (*ie*, $T(s) \to T_{\rm ref}(s)$.

Decoupling in the multidimensional case results from the fact that the corresponding reference matrix model $T_{\rm ref}(s)$ is diagonal:

$$T_{\rm ref}(s) = \operatorname{diag}\big(t_1(s), t_2(s), \dots, t_m(s)\big)\,. \qquad (23)$$

Then the dynamics of each decoupled loop is characterized by $t_{ij}(s)$.

When there are time delays in the system model $G(s)$, these cannot be compensated and they have to be included in $T_{\rm ref}(s)$, in order that $Q(s)$ and $C(s)$ are realizable controllers (with no prediction terms) [11, 16].

Finally, each $t_{iref}(s)$ is designed to characterize the transients performances of each loop and the robustness of the global controller.

4.2 Performance

The aim of any control device is to ensure transients performances:

- Settling time and relative overshoot of the outputs.
- Reduce the control input energy specially in high frequencies.
- Reduce the effect of the measurement noises.
- Robustness against uncertainties.

To face these specifications, we propose:

$$t_{iref}(s) = \frac{h_2(s)\Psi_i(s)}{d_i(s)} \qquad (24)$$

where:

$$h_2(s) = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2} \qquad (25)$$

With this second order model we can fix the dynamics of each loop with the help of the following relations for $h_2(s)$:

$$\text{settling time} : t_{r,5\%} \approx \frac{3}{\xi\omega_0} \qquad (26)$$

$$\text{relative overshoot} : D_r = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \qquad (27)$$

Notice that we can choose different dynamics for each loop.

$d_i(s)$ represent the auxiliary poles:

$$d_i(s) = \prod_{k=0}^{K} (1 + \tau_k s)\,. \qquad (28)$$

τ_k is chosen to roll off the complementary sensitivity function and so, confers robustness against high frequencies uncertainties.

Finally, $\Psi(s)$ represents the singularities of the system (*ie* time delays, and eventually RHP zeros ...); notice that time delays and non-minimum phase zeros are inherent characteristic of a process and can not be altered by feedback control. They are incorporated in the reference model by $\Psi_i(s)$ in order to ensure the stability and the realisability of IMC controller $Q(s)$ [1, 13, 14].

For example, when singularities are RHP zeros, $\Psi_i(s)$ avoids unstable pole-zero concellation during the synthesis phase [1, 13, 14].

More specifically, when singularities are time delays, we use $\Psi_i(s) = e^{-L_i s}$. The choice of $_i$ is made so that $Q(s)$ is realizable: in other words, the controller $Q(s)$ (and consequently $C(s)$) should not include a prediction term. When the system is SISO, the choice of $L_{\rm ref}$ is trivial: it is evident that $L_{\rm ref} = L_{\rm system}$. The problem is more complex for a MIMO system.

Let us consider the practical example $m = 2$ with two different reference models in the diagonal. Let $L_{\min} = \min\{(L_{11} + L_{22}), (L_{12} + L_{21})\}$. The computing of $Q(s) = G_{\rm nom}^{-1}(s)T_{\rm ref}(s)$ needs that each of its components is realisable (no prediction). Thus, the following inequalities have to be verified by L_1 and L_2.

$$\begin{aligned} L_{22} + L_1 &\ge L_{\min} \\ L_{21} + L_1 &\ge L_{\min} \\ L_{11} + L_2 &\ge L_{\min} \\ L_{12} + L_2 &\ge L_{\min} \end{aligned} \qquad (29)$$

4.3 Reference model and robustness

In the presence of uncertainties $\Delta(s)$, the closed loop dynamic will be modified with possibility of instability because:

$$\begin{aligned} T(s) &= T_{\rm ref}(s)\big(I + \Delta(s)Q(s)\big)^{-1} \\ &= T_{\rm ref}(s)\big(I + \Delta_r(s)\hat{G}(s)Q(s)\big)^{-1} \\ &= T_{\rm ref}(s)\big(I + \Delta_r(s)T_{\rm ref}(s)\big)^{-1} \end{aligned} \qquad (30)$$

where $\Delta_r(s)$, is defined by (17), represents the relative modelling errors. Let us consider a high frequency modelling error, which can result from three causes:

- nominal model is different from the real one because of neglected dynamics,
- practically, the nominal model is identified in a noisy experiment and the estimation error increases with frequency,
- system uncertainties and non-stationarities.

So to ensure $T(s) = T_{\rm ref}(s)$; in other words, the controller $C(s)$ ensures robustness of stability and performances [10] if:

$$\|\Delta_r T_{\rm ref}\|_\infty \le \delta_{RS}^{-1} \Leftrightarrow T \approx T_{\rm ref} \qquad (31)$$

where δ_{RS} represents the robust stability margin [7, 10, 18].

If $\delta_{SR} < 1$, we can show that at each frequency:

$$\sigma_i(T_{\text{ref}}) - \delta_{RS} < \sigma_i(T_{yr}) < \sigma_i(T_{\text{ref}}) + \delta_{RS}$$

where $\sigma_i(T_{\text{ref}})$ is the i^{th} singular value of T_{ref}.

Condition (31), shows that we have to choose the auxiliary poles ensuring the robustness condition.

5 REDUCED CONTROLLER

In the preceding parts, we presented the calculation method of the ideal controller $C(s)$ based on IMC approach and the reference model in order to ensure the robustness when the model changes in the presence of uncertainties $\Delta(s)$.

Notice that the implementation of either the IMC controller $Q(s)$ and the feedback configuration controller $C(s)$ is hard to do because of the presence of time delays.

The aim now, is to a reduced controller $C_r(s)$ of the ideal controller $C(s)$ for implementation. The reduced controller $C_r(s)$ must preserve as well as possible the same performances ensured by the ideal controller, so $C_r(s)$ must approach $C(s)$ in low frequencies for performances and intermediate frequencies for stability.

Let $C_r(s)$ be a transfer matrix of a reduced controller to be designed (for example PID).
Let:

$$\theta = \begin{bmatrix} \underline{\theta}_{11} & \cdots & \underline{\theta}_{1m} \\ \vdots & \ddots & \vdots \\ \underline{\theta}_{m1} & \cdots & \underline{\theta}_{mm} \end{bmatrix} \tag{32}$$

the matrix representing controller's parameters to be calculated. $\underline{\theta}_{ij}$ represents the parameters vector of numerator and denominator of $C_{r,ij}(s)$; *ie* the reduced controller between the j^{th} input and the i^{th} output.

Objectives

The aim of our technique is to ensure the approximation between the reference model and the closed loop:

$$T_{\text{ref}}(s) = T_{yr}(s) \tag{33}$$

since the reference model is diagonal; equation (33) is equivalent to:

$$C_r(s) \equiv C(s)\,. \tag{34}$$

This equality is linear in the controller which facilitates the optimization procedure. Let us define our cost function J as the 2 norm of errors between the different moments of the reference open loop $L_{\text{ref}}(s) = T_{ref}(s)\big(I - T_{\text{ref}}(s)\big)^{-1}$ and that of the open loop $G(s)C_r(s)$:

$$J = \sum_{n=0}^{N} \|\epsilon_n\|_2^2 \tag{35}$$

$$= \sum_{n=0}^{N} \|\underline{\mathcal{A}}_{n,\omega_0}(L_{\text{ref}}) - \underline{\mathcal{A}}_{n,\omega_0}(GC_r)\|_2^2 \tag{36}$$

where $\underline{\mathcal{A}}_{n,\omega_0}(GC_r)$ represents n^{th} order moments matrix, which is function of the parameters θ:

$$\underline{\mathcal{A}}_{n,\omega_0}(GC_r) = f_n(\theta)\,. \tag{37}$$

Let

$$J = \sum_{n=0}^{N} \|\underline{\mathcal{A}}_{n,\omega_0}(L_{\text{ref}}) - f_n(\underline{\theta})\|_2^2 \tag{38}$$

The objective is to determine the estimated parameters $\hat{\theta}$ minimizing J around ω_0.

This frequency ω_0 is chosen in order to preserve stability of the system, while we are face SISO systems or MIMO decoupled one, we choose ω_0 in the vicinity of the critical point $(-1, j0)$ of the decoupled open loop reference model. The aim of the optimization procedure is:

- to preserve the stability of the closed loop system, this means that the reduced controller must approach the ideal controller as well as possible in the vicinity of ω_0.
- to preserve the dynamic performances, in other words the reduced controller and the ideal one must be closer in low frequencies.

So, we try to find a reduced controller $C_r(s)$ around $\omega = \omega_0$ then, we impose an equality constraint to preserve the dynamic performances via the time moments.

It is difficult to realize an approximation around $\omega = \omega_0$ because the optimization problem is non linear. So, our technique is divided on two parts, in the first one we find a suboptimal linear approximation in order to initialize the global non linear optimization problem with the equality constraint in the second part.

5.1 Linear optimization

There are several PID controller structures used in practice, the one adopted in this paper is:

$$C_r^{ij}(s) = k_P^{ij} + \frac{k_I^{ij}}{s} + \frac{k_D^{ij} s}{\tau^{ij} s + 1} \tag{39}$$

where $k_P^{ij} \in \Re$ is the proportional, $k_I^{ij} \in \Re$ is the integral and $k_D^{ij} \in \Re$ is the derivative gain of $C_r^{ij}(s)$.

The first step consists on imposing the common denominator of the reduced controller (*ie* $(1+\tau s)$). So only the zeros have to be determined and the function $f_n(\theta)$ is linear; thus the minimization of J is obtained by Least Squares, let

$$C_r(s) = C_{LS}(s) \tag{40}$$

be the reduced controller which will be used to initialize the Non Linear Programming algorithm.

5.2 Non linear optimization

When we optimize poles and zeros of the PID controller, the function $f_n(\theta)$ is non linear; the estimation of θ is obtained by Non Linear Programming [14].

We use Marquardt's algorithm which is a good combination between rapidity and convergence [5].

5.3 Algorithm principle

Parameters estimation is performed using an iterative optimization procedure:

$$\hat{\theta}_{i+1} = \hat{\theta}_i - \{[J'' + \lambda_i I]^{-1} J'\}_{\hat{\theta}=\hat{\theta}_i} \tag{41}$$

$$\Big(\frac{\partial\partial J}{\partial\theta}\Big) = J' \ : \ \text{the Gradient vector} \tag{42}$$

$$\Big(\frac{\partial^2 J}{\partial\theta^2}\Big) = J'' \ : \ \text{the Hessian matrix} \tag{43}$$

$$\lambda_i \ : \ \text{coefficient to adjust} \tag{44}$$

We initialize this algorithm with the solution obtained by linear optimization:

$$\hat{\theta}_0 = \hat{\theta}_{LS} \tag{45}$$

5.4 Computing J' and J''

We use parametric sensitivity function to calculate J' and J'':

$$J' \approx -2\sum_{n=0}^{N} \varepsilon_n \Theta_n \tag{46}$$

$$J'' \approx 2\sum_{n=0}^{N} \Theta_n \Theta_n^{\top} \tag{47}$$

$$\Theta = \frac{\mathrm{d}\underline{\mathcal{A}}_{n,\omega_0}(GC_r)}{\mathrm{d}\theta} \tag{48}$$

where Θ represents the sensitivity functions of the moments in comparison to the parameters θ around ω_0; using (13):

$$\frac{\mathrm{d}\underline{\mathcal{A}}_{n,\omega_0}}{\mathrm{d}\underline{\theta}_r} = (-1)^{n+1}\Big(\frac{\mathrm{d}C}{\mathrm{d}\underline{\theta}_r} A_\mu^{-(n+1)} B - CA_\mu^{-(n+1)} \frac{\mathrm{d}A_\mu^{-(n+1)}}{\mathrm{d}\underline{\theta}_r} A_\mu^{-(n+1)} B + CA_\mu^{-(n+1)} \frac{\mathrm{d}B}{\mathrm{d}\underline{\theta}_r}\Big) \tag{49}$$

with

$$A_\mu = A - j\omega_0 I \tag{50}$$

where

$$\frac{\mathrm{d}A_\mu^{(n+1)}}{\mathrm{d}\theta} = \frac{\mathrm{d}A_\mu}{\mathrm{d}\theta} A_\mu^n + A_\mu \frac{\mathrm{d}A_\mu^n}{\mathrm{d}\theta}\,. \tag{51}$$

Remark 5.1. Notice that the previous relations to compute J' and J'', will be more simple if we use the controllable or the observable realization.

5.5 Optimization with equality constraints

The minimization of the cost function J suffices to approximate the open loop and the reference open loop in a range of frequencies ensuring stability of the closed loop. This approximation has also to preserve the low frequencies performances (*ie* time specifications).

In order to do that, we introduce the following constraint into the optimization problem:

$$F(\theta) = \|\mathcal{A}_1(L_{\mathrm{ref}}) - \mathcal{A}_1(GC_r)\| + \|\mathcal{A}_0(L_{\mathrm{ref}}) - \mathcal{A}_0(GC_r)\| \ \text{with}\ F(\theta) = 0\,. \tag{52}$$

The previous condition means:

- equality of 0^{th} order time moments
  $\Rightarrow$ equality of d.c gains,
- equality of 1^{st} order time moments
  $\Rightarrow$ equality of settling times

The optimization problem is reformulated as follows:

$$\min_\theta J \quad \text{subject to} \quad F(\theta) = 0 \tag{53}$$

or:

$$\min_\theta J_{\mathrm{const}} \quad \text{with} \quad J_{\mathrm{const}} = J + \gamma F(\theta) \tag{54}$$

which is equivalent to:

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}\theta} + \frac{\mathrm{d}F}{\mathrm{d}\theta}\gamma &= 0 \\ F(\theta) &= 0 \end{aligned} \tag{55}$$

γ represents the vector of Lagrange multipliers to be estimated.

The system (55) implies the verification of the equality constraint.

We can use the algorithm (41) to solve this optimization problem, by substituting J by J_{const}, with:

$$J'_{\mathrm{const}} = \begin{bmatrix} \frac{\partial J_{\mathrm{const}}}{\partial\theta} \\ \frac{\partial J_{\mathrm{const}}}{\partial\gamma} \end{bmatrix}, \tag{56}$$

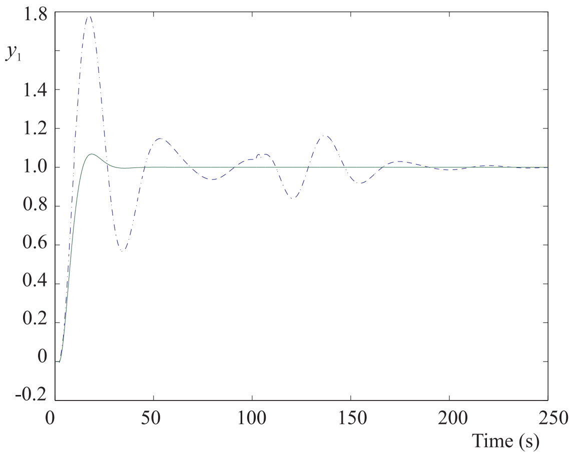
$$J''_{\mathrm{const}} = \begin{bmatrix} \frac{\partial^2 J_{\mathrm{const}}}{\partial\theta^2} & \frac{\partial^2 J_{\mathrm{const}}}{\partial\theta\partial\gamma} \\ \frac{\partial^2 J_{\mathrm{const}}}{\partial\theta\partial\gamma} & \frac{\partial^2 J_{\mathrm{const}}}{\partial\gamma^2} \end{bmatrix}. \tag{57}$$

6 DESIGN EXAMPLE

The example which will be treated in this part is a multivariable system with two inputs and two outputs,it represents the well-known binary distillation plant model, each transfer function of the system represents a first order with time delay [15]:

$$G(s) = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{18.9e^{-3s}}{21s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{19.4e^{-3s}}{14.4s+1} \end{bmatrix}. \tag{58}$$

It is a stable minimum phase system which has four distinct poles: $\{-599e^{-4}, -917e^{-4}, -476e^{-4}, -694e^{-4}\}$, and two distinct zeros $\{-.01079, -.0427\}$. This system exhibits strong interaction, and significant time delays in its I/O channels.



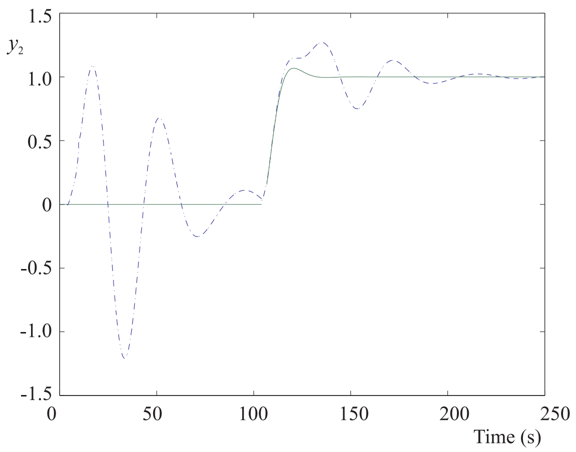


Fig. 3. Closed loop step responses, proposed method dashed line and $T_{\rm ref}$ solid line

6.1 Performances specification

The closed loop performances are gathered in the following reference model:

$$T_{\rm ref}(s) = \text{diag}\Big(\frac{h_1(s)\Psi_1(s)}{d_1(s)}, \frac{h_2(s)\Psi_2(s)}{d_2(s)}\Big)$$

where:

$$h_1(s) = h_2(s) = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2} \tag{59}$$

with:

$$\omega_0 = .25\,\text{rd/s} \ \text{ and } \ \xi = .65\,. \tag{60}$$

The auxiliary poles for each loop are:

$$d_1(s) = d_2(s) = (1 + .05s)^3. \tag{61}$$

6.1.1 The singularities

In this example, the singularities present in the system are given by the time delays; to choose the corresponding time delays of each output of the reference model, we present the following methodology:
let $L_{11} = 1s$, $L_{12} = 3s$, $L_{21} = 7s$ and $L_{22} = 3s$, so:

$$\begin{aligned} L_{11} + L_{22} &= 4 \\ L_{12} + L_{21} &= 10 \end{aligned} \tag{62}$$

let:

$$L_{\min} = \min\{(L_{11} + L_{22}), (L_{12} + L_{21})\} = 4\,. \tag{63}$$

After computing the IMC controller $Q(s)$, we must have:

$$\begin{aligned} L_{22} + L_1 \geq L_{\min} \quad &\Rightarrow \quad 3 + L_1 \geq 4 \\ L_{21} + L_1 \geq L_{\min} \quad &\Rightarrow \quad 7 + L_1 \geq 4 \\ L_{11} + L_2 \geq L_{\min} \quad &\Rightarrow \quad 1 + L_2 \geq 4 \\ L_{12} + L_2 \geq L_{\min} \quad &\Rightarrow \quad 3 + L_2 \geq 4 \end{aligned} \tag{64}$$

where L_1 and L_2 are the time delays of the different channels of the reference model. So from the previous inequalities and in order to have a realisable controller, we have:

$$L_1 = 1s \ \text{ and } \ L_2 = 3s \tag{65}$$

so:

$$\Psi_1(s) = e^{-s} \ \text{ and } \ \Psi_2(s) = e^{-3s}. \tag{66}$$

The performances with this reference model are recapitulated in the following table:

settling time without delays	24.2(sec)	(67)
rise time	7.93(sec)	
relative overshoot	6.8 %	

6.2 Linear Optimization

In this step, we fix the common denominator of the PID controller:

$$C_{LS}(s) = C_{PID}(s) = \frac{C(s)}{s(1 + \tau s)}\,, \tag{68}$$

$$\text{with} \qquad \tau = .15s\,. \tag{69}$$

Using the Least Squares optimization, we obtain the following controller:

$$C_{LS} = \frac{1}{s(s + 6.667)} \times \begin{bmatrix} -.1057s^2 + .3878s + .2189 & .1261s^2 + .2323s - .1652 \\ -.05916s^2 + .1997s + .06 & -.0906s^2 + .2308s + .1146 \end{bmatrix}. \tag{70}$$

The frequency ω_0 is chosen near to the critical point compared to the reference model, and in our case the precautionary range is located between $\omega_0 = 0.2$ rd/s and $\omega_0 = 1.2$ rd/s.

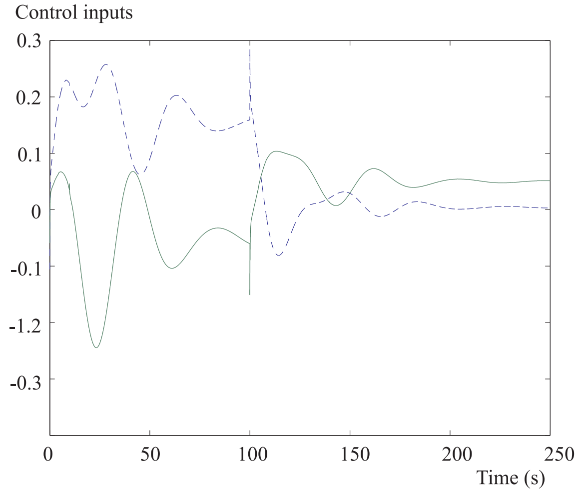


Fig. 4. The control inputs proposed method

For each controller reduction, we have used three moments and the frequencies ω_0 have been chosen as follows:

Controller	ω_0(rd/s)
$C_{11}(s)$	0.55rd/s
$C_{12}(s)$	0.43rd/s
$C_{21}(s)$	1rd/s
$C_{22}(s)$	0.6rd/s

(71)

The number of moments using for calculation is 3. The application of $C_{LS}(s)$ to $G(s)$ gives the results shown in Fig. 3. The corresponding control inputs are given by Fig. 4.

We observe the the controller obtained from the linear optimization can not ensure the performances objectives proposed by the reference model.

6.3 Non Linear Optimization

Using the algorithm (41) and introducing equality constraint between the first time moments, we optimize the poles and the zeros of the PID controller; notice that the initialization is given by the controller issued from the linear optimization (*ie* $C_0(s) = C_{LS}(s)$). The final PID controller is:

$$C_{PID} = \begin{bmatrix} \frac{.03423s^2+.8606s+.115}{s(s+4.65)} & \frac{-1.2s^2-163.2s-23.73}{s(s+2864)} \\ \frac{.1378s^2+.09104s-.4929}{s(s+27.06)} & \frac{-.1469s^2+.801s+.13}{s(s+10.48)} \end{bmatrix} \quad (72)$$

and the corresponding frequency for each controller is given in the following table:

Controller	ω_0(rd/s)
$C_{11}(s)$	0.65rd/s
$C_{12}(s)$	1(rd/s)
$C_{21}(s)$	0.37(rd/s)
$C_{22}(s)$	0.88(rd/s)

(73)

In this step the number of moments using for computing the controller is 4.

6.4 Wang's technique

Remark 6.1. In order to be able to make a comparative study, we will present the synthesis technique proposed by Wang that was originally applied to the system represented by (58) [16]. The idea is based on the multivariable decoupling Smith predictor whose the diagram is illustrated by Fig. 5 where $G(s)$ is the system to be controlled..

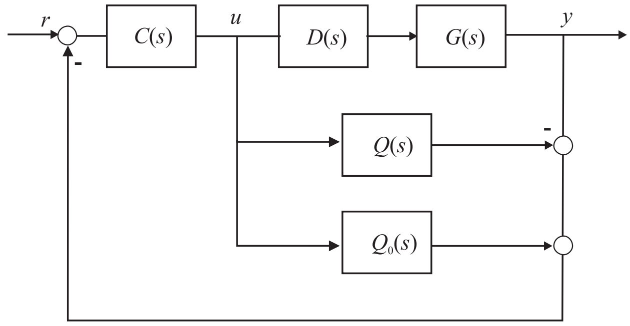


Fig. 5. Multivariable decoupling Smith predictor

6.4.1 Design (For more details, refer to [16])

For $G(s)D(s)$ to be decoupled, the i^{th} element of $D(s)$ should satisfy the conditions:

$$G(s)D(s) = \operatorname{diag}\Big\{\frac{|G|}{G^{ii}}d_{ii}\,, \quad i = 1,2,\dots,m\Big\}, \quad (74)$$

$$d_{ji} = \frac{G^{ij}}{G^{ii}}d_{ii}, \quad \forall i,j \in \{1,2,\dots,m, \quad j \neq i\} \quad (75)$$

where $|G|$ is the determinant of $G(s)$, and G^{ij} is the cofactor corresponding to g_{ij} in $G(s)$, where $\operatorname{adj} G = [G^{ji}]$. Thus, we wish to get a simplest decoupling $D(s)$ with minimum calculations. Let θ_{ji} as the smallest nonnegative number such that d_{ji} does not have no prediction $\forall j$. Let

$$\theta_i = \max_{\forall j \in \mathbf{m}, j\neq i} \theta_{ji} \quad (76)$$

where $\mathbf{m} = \{1,2,\dots,m\}$. By choosing $d_{ii} = e^{-\theta_i s}$, d_{ji} will have no prediction.

In the case of perfect decoupling, $Q(s)$ should be:

$$Q(s) = G(s)D(s) = \operatorname{diag}\Big\{\frac{|G|}{G^{ii}}d_{ii}, i = 1,2,\dots,m\Big\} = \operatorname{diag}\{q_{11}, q_{22}, \dots, q_{mm}\}\,. \quad (77)$$

Generally, q_{ii} is too complicated to implement. Thus, model reduction is applied to q_{ii} to obtain a simpler yet good approximation for implementation and $Q_0(s)$ is readily obtained as the delay free part of $Q(s)$.

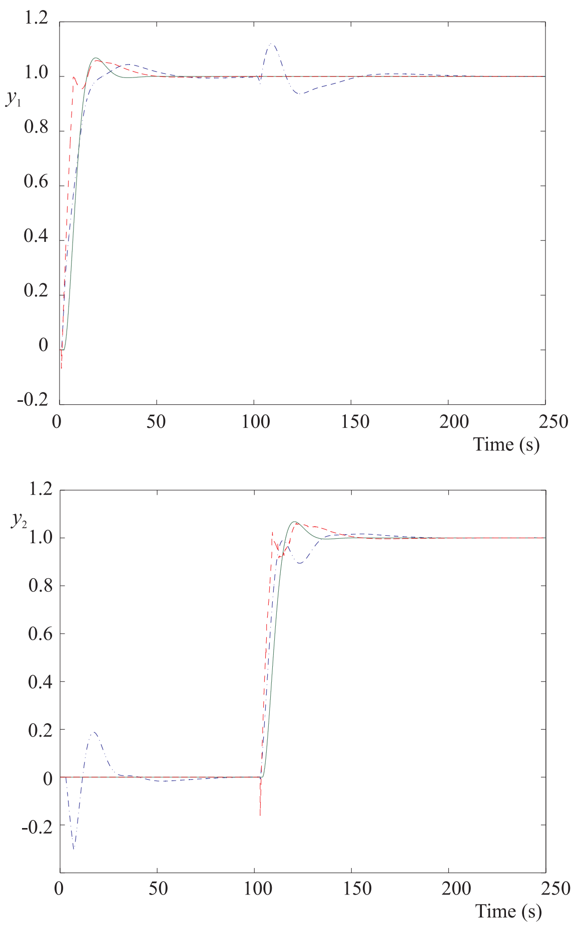


Fig. 6. Closed loop step responses, proposed method (-··-), Wang's method (- -) and $T_{\rm ref}$ (-)

With decoupling by $D(s)$, the multivariable Smith predictor design is now simplified to multiple single-loop Smith predictor control designs.

Let the primary controller be:

$$C(s) = \mathrm{diag}\{c_{11}(s), \ldots, c_{mm}(s)\}\,. \tag{78}$$

Each individual $c_{ii}(s)$ is designed with respect to the free delay part q_{ii0} of q_{ii} such tha the closed loop system formed by $c_{ii}(s)$ and q_{ii0} has the desired performance.

In our case, the diagonal elements of decoupler $D(s)$ are chosen as $d_{11}(s) = 1$ and $d_{22}(s) = 1$. Model reduction yields the off-diagonal elements of the decoupler as:

$$\begin{aligned} d_{12} &= \frac{24.66s + 1.48}{21s + 1} e^{-2s}\,, \\ d_{21} &= \frac{4.9s + 0.34}{10.9s + 1} e^{-4s} \end{aligned} \tag{79}$$

and the decoupler is formed as:

$$D(s) = \begin{bmatrix} 1 & \frac{24.66s+1.48}{21s+1} e^{-2s} \\ \frac{4.9s+0.34}{10.9s+1} e^{-4s} & 1 \end{bmatrix}. \tag{80}$$

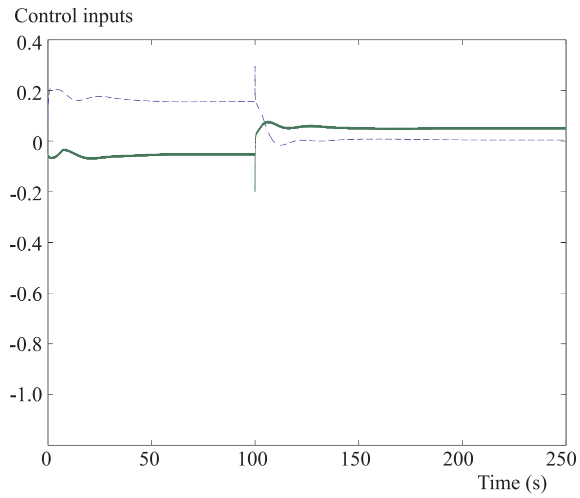


Fig. 7. The control inputs proposed method

The application of model reduction to $g_{ii}(s)$, the diagonal elements of $G(s)D(s)$, produces:

$$\begin{aligned} g_{11} &= \frac{6.37}{0.61s^2 + 5.45s + 1} e^{-0.92s} \\ g_{22} &= \frac{-9.65}{4.59s + 1} e^{-3.31s} \end{aligned} \tag{81}$$

giving:

$$Q(s) = \begin{bmatrix} \frac{6.37}{0.61s^2+5.45s+1} e^{-0.92s} & 0 \\ 0 & \frac{-9.65}{4.59s+1} e^{-3.31s} \end{bmatrix} \tag{82}$$

and

$$Q_0(s) = \begin{bmatrix} \frac{6.37}{0.61s^2+5.45s+1} & 0 \\ 0 & \frac{-9.65}{4.59s+1} \end{bmatrix}. \tag{83}$$

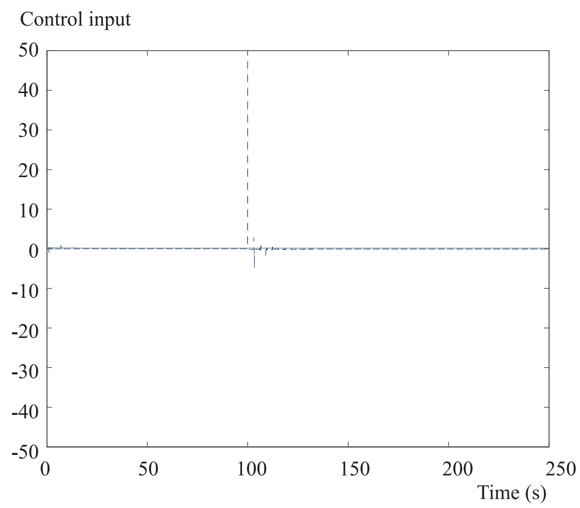


Fig. 8. The control inputs Wang's method

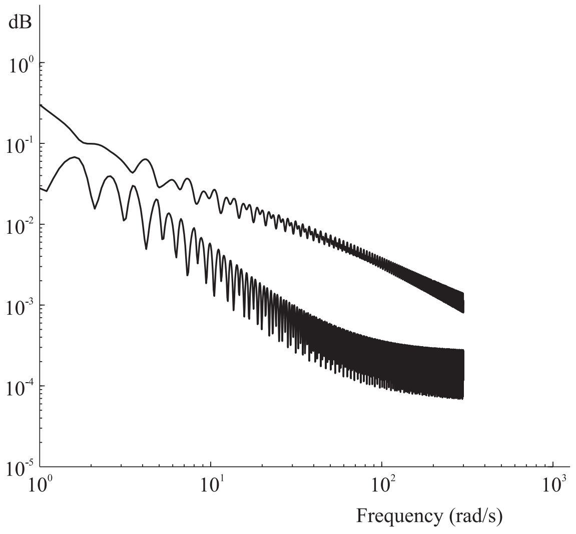


Fig. 9. The robustness condition $\|\Delta_r T_{ref}\|_\infty < 1$

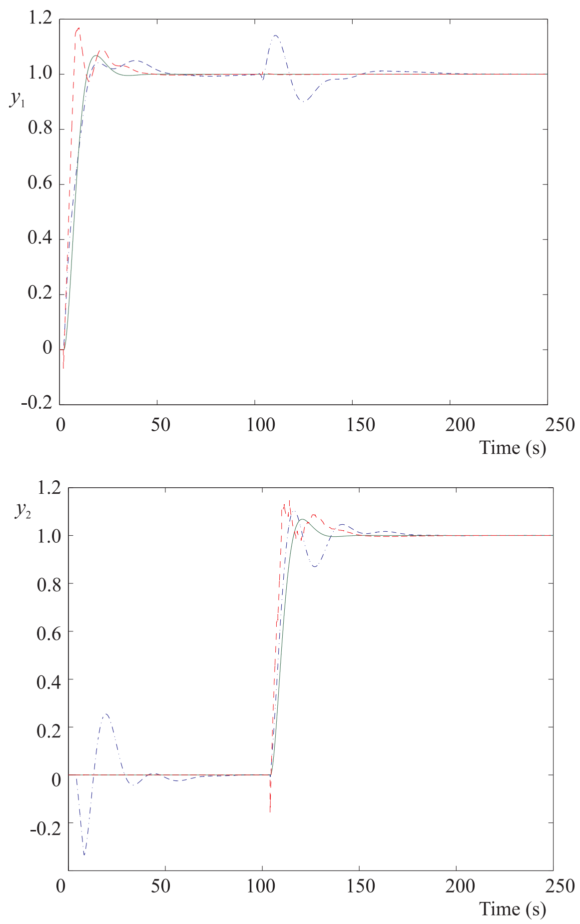


Fig. 10. Closed loop step responses for perturbed plant, proposed method(-··-), Wang's method (- -) and $T_{\rm ref}$ (-)

The primary controller is designed as simple PID:

$$C(s) = \begin{bmatrix} 0.23 + \frac{0.066}{s} - 0.09s & 0 \\ 0 & -0.14 - \frac{0.048}{s} + 0.097s \end{bmatrix}. \tag{84}$$

6.5 Simulation

The following result of closed loop responses are obtained in simulation thanks to MATLAB/Simulink software.

The closed loop step responses using the two synthesis methods are presented by Fig. 6. The control inputs are given by Figs. 7 and 8.

Conclusion: The Smith predictor gives excellent results, particularly with respect to decoupling. However, these results were obtained with a complex structure of infinite dimension because it incorporates time delays. On the other hand, the control device that we propose is of elementary structure and of low dimension because it acts of PID regulator. Its performances should be judged: in this reference frame: the tracking performances are as good as those of Wang, only decoupling is transitorily less powerful. In addition, it is noticed that our control signals are perfectly acceptable because the derived action is correctly filtered; this is not the case of the Smith predictor: it will be necessary to filter a posteriori the derived action, which will degrade the dynamic performances.

6.6 Robustness

To test the robustness of the two controllers, we have modified all the time delays of $G(s)$ have been augmented of $1s$ (the same robustness test in [16]) to get the following perturbed model:

$$G_P(s) = \begin{bmatrix} \frac{12.8e^{-2s}}{16.7s+1} & \frac{18.9e^{-4s}}{21s+1} \\ \frac{6.6e^{-8s}}{10.9s+1} & \frac{19.4e^{-4s}}{14.4s+1} \end{bmatrix} \tag{85}$$

Figure 9 illustrates the robustness condition $\|\Delta_r T_{\rm ref}\|_\infty < 1$, where $\Delta_r = \big(G_P - G\big)G^{-1}$.

R e m a r k 6.2. The good choice of the auxiliary poles of the reference model, allows to ensure the robustness condition by making a high frequency roll-off of the complementary sensitivity function. The step responses are illustrated by Fig. 10. The control inputs are given by Figs. 11 and 12.

Conclusion: From the results above, the proposed controller have excellent properties of robustness in tracking, and worse in decoupling but acceptable taking into account the simplicity of the controller. On the other hand, the robustness of the Wang's controller is worse in tracking, excellent in decoupling.

7 CONCLUSIONS

In this paper we have presented a new technique of synthesis and reduction to get a multivariable PID controller. The technique is based on the choice of a reference model gathering the desired performances of the closed loop as all the singularities which the system could have

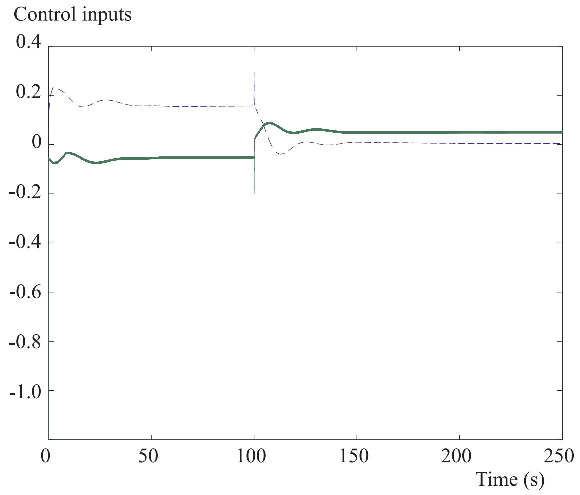


Fig. 11. Control Inputs for perturbed plant, proposed method

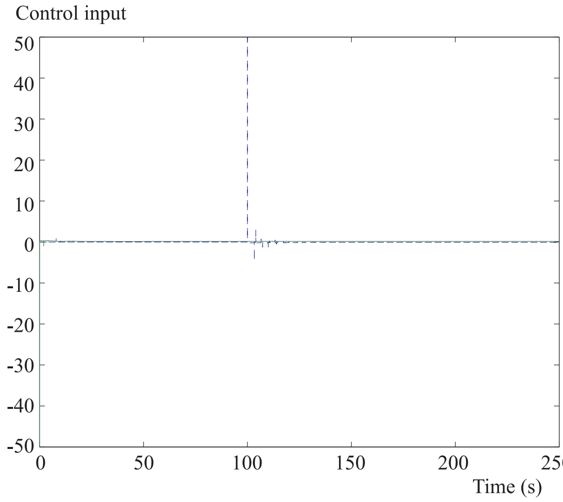


Fig. 12. Control Inputs for perturbed plant, Wang's method

such as the unstable zeros and/or time delays. The auxiliary poles are introduced in the reference model in order to ensure robustness face to high frequencies uncertainties by having a roll-off of the complementary sensitivity function. The ideal controller achieving all the performances and having a complex structure is ensured by the Internal Model Control. This ideal controller is then reduced through a nonlinear programming algorithm by using the frequency moments and an equality constraint between the two first time moments in order to ensure the low frequencies performances; the reduction technique leads to a PID controller which represent a simple structure for implementation adding to that, it ensures a good tracking and robustness properties. A comparative study with the technique suggested by Wang [16] is made in order to be able to show the advantages of our technique.

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