

# Robust PID controller design with $H_2$ performance: Descriptor systems approach

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The paper deals with the problem to obtain robust PID controller design procedure to linear time invariant descriptor uncertain polytopic systems using descriptor system stability theory and  $H_2$  criterion approach in the form of quadratic cost function. In the frame of Lyapunov function,  $H_2$  quadratic cost function and Bellman-Lyapunov equation the obtained designed novel procedure guarantees the robust properties of closed-loop system with parameter dependent quadratic stability/quadratic stability. In the obtained design procedure, the designer could use controller with different structure like as P, PI, PID, PI-D. For PI-D controllers D-part feedback the designer could choose any available output/state derivative variables of real systems. The effectiveness of the obtained results is demonstrated on the randomly generated examples.

**Key words:** robust PID controller, uncertain polytopic systems, Lyapunov function

## 1 Introduction

Robust stability and control are an important issue in the control theory for control of uncertain systems. In the book [1] the authors have introduced very different robust controller design procedures in time and frequency domain. In the time domain some of these design procedures are as follow: Robust static output feedback controller design using elimination approach, robust PID controller design using BMI approach, robust controller design using polynomial parameter dependent Lyapunov function, heuristic LMI design procedure, robust model predictive control design and so on. In the paper [2] the new robust controller design procedure is given, which belongs with [3] to the class of "classical" methods. To get around the conservatism of classical method arising from affine parameter dependent Lyapunov function two methods have been proposed. The less conservative results may be reached using polynomial Lyapunov function approach [4] or introducing to design procedure another auxiliary matrix [5]. Proposed method based on the enough condition stability theory of descriptor systems by Lyapunov function [6,7]. Observer based controller with enough and necessary condition to descriptor systems is given in [8].

In this paper the new approach to robust PID (PI-D) controller design is presented. The main idea of robust controller design procedure is based on the theory of descriptor systems, Bellman Lyapunov equation, and uncertain polytopic plant model.

## 2 Problem formulation and preliminaries

We are given a linear-time invariant (LTI) polytopic uncertain system in the form

$$\dot{x} = A(\xi)x + B(\xi)u, y = Cx, y_d = C_d x \quad (1)$$

where  $x$  is the plant state,  $u$  is the control input,  $y$  is the output vector to be controlled and  $y_d$  is the output derivative feedback variable for PID or PI-D controller; matrices  $A(\xi), B(\xi)$  belong to the convex set  $G$

$$\begin{aligned}
 G := \{A(\xi), B(\xi) : (A(\xi), B(\xi) = \\
 = \sum_{i=1}^N (A_i, B_i)\xi_i, \sum_{i=1}^N \xi_i = 1\} \\
 \xi_i \geq 0, \sum_{i=1}^N \dot{\xi}_i = 0
 \end{aligned} \quad (2)$$

$A_i, B_i, C, C_d$  are constant matrices of an appropriate dimensions,  $N$  is the number of polytopic vertices.

Because of PID controller, the plant model (1) for output feedback control algorithm

$$u = k_p y + k_i \int_0^t y(\tau) d\tau + k_d \dot{y}_d \quad (3)$$

is changes as follows. Integral term can be included into the state vector  $z = y$ , new state model is described as

$$\dot{x}_n = A_n(\xi)x_n + B_n(\xi)u, y_n = C_n x_n, y_{dn} = C_{dn} x_n \quad (4)$$

and

$$\begin{aligned}
 A_n &= \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, B_n = \begin{bmatrix} B \\ 0 \end{bmatrix}, x_n = \begin{bmatrix} x \\ z \end{bmatrix}, y_n = \begin{bmatrix} y \\ z \end{bmatrix} \\
 y_{dn} &= \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} x_n = \\
 &= C_{dn} x_n, y_n = \begin{bmatrix} C & 0 \\ 0 & I_m \end{bmatrix} x_n = C_n x_n
 \end{aligned}$$

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Control algorithm [3] for PID control could be rewritten to the form

$$u = k_p Cx + k_i z + k_d C_d \dot{x} = [k_p C \quad k_i] \begin{bmatrix} x \\ z \end{bmatrix} + [k_d C_d \quad 0] \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix}$$

$$u = Kx_n + K_D \dot{x}_n$$

where

$$K = [k_p C \quad k_i], \quad K_D = [k_d C_d \quad 0]$$

To obtain the performance of closed-loop system in the frame of  $H_2$ , the quadratic cost function has been used in the form

$$J = \int_0^\infty J(t) dt; J(t) = x_n^\top Q x_n + u^\top R u + \dot{x}_n^\top S \dot{x}_n \quad (6)$$

Due to (5) the closed-loop descriptor system is given as follows

$$E \dot{x}_d = A_d x_d + F_d E \dot{x}_d \quad (7)$$

where

$$A_d = \begin{bmatrix} A_n & B_n \\ K & -I_m \end{bmatrix} \in R^{n \times n}, \quad F_d = \begin{bmatrix} 0 & 0 \\ K_D & 0 \end{bmatrix} \in R^{n \times n},$$

$$x_d = \begin{bmatrix} x_n \\ u \end{bmatrix} \in R^n$$

Matrix  $E$  defines the structure of descriptor systems. Rewrite (7) to the form

$$(I_n - F_d) E \dot{x}_d - A_d x_d = 0 \quad (8)$$

Let us recall the Lyapunov stability condition of descriptor systems.

*Theorem 1* [Chen, Lin (1997)]

The equilibrium  $x_d = 0$  of a system (7) is asymptotically stable if an  $n \times n$  symmetric positive definite matrix  $P(\xi)$  exists, such that along the solution of the system (7) the first derivative of Lyapunov function  $V(Ex_d) = (Ex_d)^\top P(\xi) Ex_d$  is negative definite function for the variate of  $Ex_d$ .

*Theorem 2* Bellman-Lyapunov equation

Consider the system (7) with control algorithm (6). Control algorithm (6) is the guaranteed cost control law for the (7), if there is a Lyapunov function  $V(Ex_d)$  such that

$$B_e(\xi) = \max_u \left( \frac{dV(Ex_d)}{dt} + J(t) \right) < 0 \quad (9)$$

### 3 Main results

In this Section the new design procedure to design of robust PID (PI-D) controller is obtained. To split Lyapunov matrix  $P(\xi)$  from system matrices  $(A_d, F_d)$  and obtain less conservative results one could introduce the auxiliary (slack) matrices  $N_1, N_2 \in R^{n \times n}$  in the following way

$$v^\top \begin{bmatrix} 2N_1^\top \\ N_2^\top \end{bmatrix} [-A_d \quad (I_n - F_d)E] v = 0; v = \begin{bmatrix} x_d \\ \dot{x}_d \end{bmatrix} \quad (10)$$

Due to Theorem 1, Lyapunov function is

$V = (Ex_d)^\top P(\xi) Ex_d$  and its time derivative

$$\frac{dV(Ex_d)}{dt} = v^\top \begin{bmatrix} E^\top P(\xi) E & E^\top P(\xi) E \\ E^\top P(\xi) E & 0 \end{bmatrix} v \quad (11)$$

where

$$P(\xi) = \sum_{i=1}^N P_i \xi_i$$

If (11) with closed-loop system (10) is negative definite the closed-loop system is asymptotically stable for the variates of  $Ex_d$ . To guarantee the performance of closed-loop system in the frame of  $H_2$  and obtain the design procedure to design of robust PID controller one need to summarize the equations (11), (10) and (6)) and the obtained results substitute to (9). After some manipulation we have got

$$B_e = v^\top W(\xi) v < 0; W(\xi) = \{w_{ij}(\xi)\}_{2 \times 2} \quad (12)$$

Because, (12) is affine with respect to vector uncertainty  $\xi$ , one can split  $W(\xi)$  as

$$W(\xi) = \sum_{i=1}^N W_i \xi_i < 0; W_i = \{w_{ijk}\}_{2 \times 2} \quad (13)$$

where

$$w_{i11} = -N_1^\top A_{di} - A_{di}^\top N_1 + E^\top \sum_{j=1}^N P_j \xi_j E + Q_d$$

$$w_{i12} = N_1^\top (I_n - F_d) E - A_{di}^\top N_2 + E^\top P_i E$$

$$w_{i22} = N_2^\top (I_n - F_d) E + E^\top (I_n - F_d)^\top N_2 + E^\top S_d E$$

$$i = 1, 2, \dots, N$$

$$J(t) = x_n^\top Q x_n + u^\top R u + \dot{x}_n^\top S \dot{x}_n =$$

$$\begin{bmatrix} x_n^\top & u^\top \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} + (E \dot{x}_d)^\top \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} E \dot{x}_d$$

or

$$J(t) = v^\top Q_d v + \dot{x}_d^\top E^\top S_d E \dot{x}_d \quad (14)$$

From conditions (13) one can obtain the robust PID (PI-D) controller parameters which ensure the asymptotic stability of closed loop system (1)+(6), robust properties of closed-loop system and optimal value of performance (14).

#### 4 Numerical example

Consider for 3-rd to 6-th order systems randomly generated 100 linear continuous time invariant uncertain polytopic systems. The problem is to design PID controller which ensures robust properties of closed-loop systems and optimal value of quadratic SQR (state derivative, state and input) cost function. The following calculation parameters have been used: D-part controller output derivative matrix  $C_d = C$ , constraints on all Lyapunov matrices  $0 < P < \rho I_n$ ,  $\rho = 1000$ , parameters of quadratic cost function  $q = 10^{-6}$ ,  $s = 0$ ,  $r = 1$ , randomly generated systems are characterized by  $A1, B1, C, Cd = random(Normal', 0, 10, xx, xx)$  and uncertainty part of system  $Ai, B(i) = random(Normal, 0, 0.5, xx, xx)$ . The obtained results are summarized in the following table where the number of successful solved task are given as: *usd*– in the paper proposed method, *usq*–quadratic stability ([Peaucelle *et al* (2000)], [Vesely, Rosinova (2013)]), *uspd*–parameter dependent quadratic stability ([Peaucelle *et al* (2000)], [Vesely, Rosinova (2013)]).

**Table 1.** Robust PID controller results of calculation: Number of examples = 100

Order of system	Proposed PDQS-usd	QS -usq	PDQS -uspd
$N = 1$ , inputs = 1, outputs = 1			
3-rd	77	79	80
4-th	82	89	92
5-th	70	81	82
$N = 2$ , inputs = 1, outputs = 1			
3-rd	69	72	72
4-th	62	69	69
5-th	60	66	67
$N = 2$ , inputs = 2, outputs = 2			
3-rd	67	68	67
4-th	71	71	72
5-th	72	72	72

#### 5 Conclusion

Paper is devoted to obtain in the frame of  $H_2$  new robust PID (PI-D) controller design procedure. Obtained results ensure the robust properties of closed-loop system, optimal value of quadratic cost function (SQR) for real uncertain polytopic plant described by (1). The proposed method is based on the descriptor system stability theory and Bellman Lyapunov function which in the frame of  $H_2$  ensure guaranteed cost. The obtained results are twofold

at the first in the time domain transformation of "classical" systems to descriptor one and robust PID (PI-D) controller design procedure has been obtained for such descriptor systems. The solution of 900 examples show that the obtained results belong rather to "classical" that is the obtained robust PID controller to descriptor system have the same properties than methods of quadratic stability and parameter dependent quadratic stability. The conservativeness of both approaches is approximately the same. For the future there is an open question why the robust controller design procedure to descriptor system not decrease the conservativeness when the number of auxiliary variables is larger than used in "classical" approach. Maybe it is given by Theorem 1 which is only with sufficient stability conditions.

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